Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2014, Article ID 592782, 10 pages
http://dx.doi.org/10.1155/2014/592782

Research Article

Gaussian Fibonacci Circulant Type Matrices

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Received 4 July 2014; Accepted 14 August 2014; Published 31 August 2014

Academic Editor: Huanhe Dong

Circulant matrices have become important tools in solving integrable system, Hamiltonian structure, and integral equations. In this paper, we prove that Gaussian Fibonacci circulant type matrices are invertible matrices for $n > 2$ and give the explicit determinants and the inverse matrices. Furthermore, the upper bounds for the spread on Gaussian Fibonacci circulant and left circulant matrices are presented, respectively.

1. Introduction

Circulant matrices have been used in solving integrable system [1], Hamiltonian structure [2, 3], and integral equations [4–8]. By using the KdV and Boussinesq systems, the circulant forward shift matrix, and the antisymmetric circulant matrix, Weiss in [1] constructed a cosymplectic form M_ξ and presents the factorization of the BLP equation by the periodic fixed points of its Bäcklund transformations. In [2], Kupershmidt and Wilson by Proposition 8.1 [2] showed that a "first" Hamiltonian structure for their modified equations, formed from circulant operators P [2], does not exist, since the relevant Hamiltonians H_{LP} [2] do not survive the specialization. However, the Hamiltonians H_P [2] do survive; then they verified that a "second" Hamiltonian structure exists. In [3], Kisisel solved the problem on the Hamiltonian structure of discrete KP equations by the properties of circulant matrices. Chan et al. considered solving potential equations by the boundary integral equation approach. The equations derived are Fredholm integral equations of the first kind and are known to be ill-conditioned. They proposed to solve the equations by the preconditioned conjugate gradient method with circulant integral operators as preconditioners in [4]. By minimizing the problem $\|W_{\mathrm{mn}} - A_{\mathrm{mn}}\|_F$ [5] obtained from the preconditioners of type block circulant with circulant blocks and in the case that the coefficient matrix of $C^{-1}A$ is positive definite, Maleknejad and Rabbani used CG method for solving system of the $C^{-1}Ax = C^{-1}b$ in [5]. Gohberg et al. stated that finite sections of a Wiener-Hopf integral operator can be approximated by circulant integral operators within a sum of a small and a finite rank operator in [6]. They gave two constructions of such circulant operators which can be used to accelerate convergence of the CG algorithm as applied to finite sections of a Wiener-Hopf equation. Cai developed a fast and direct Fourier spectral method for solving the Hilbert type singular integral equation. When the direct Fourier spectral method is used to solve (1.1) in [7], Cai observed that the matrix representation of operator A under the Fourier basis is a quasicirculant matrix. Abramyan proved the solvability of the system of (5) [8] for all $n \in N$, beginning with some n_0 via that any circulant matrix is a normal matrix and its spectral norm is equal to the maximal modulus of its eigenvalues.

Circulant type matrices have been put on the firm basis with the work in [9–13] and so on. There are discussions about the convergence in probability and in distribution of the spectral norm of circulant type matrices in [14]. Furthermore, the g-circulant matrices are focused on by many researchers; for more details please refer to [15–17] and the references therein.

Recently, some authors gave the explicit determinant and inverse of the circulant and skew-circulant involving famous numbers. Cambini presented an explicit form of the inverse of a particular circulant matrix in [18]. Jiang et al. [19] considered circulant type matrices with the k-Fibonacci and k-Lucas numbers and presented the explicit determinant and inverse matrix by constructing the transformation matrices. In [20], Jiang and Hong presented exact determinants of some special circulant matrices involving four kinds of famous numbers. Bozkurt and Tam gave determinants and inverses of circulant matrices with Jacobsthal and Jacobsthal-Lucas numbers in [21]. In [22], authors studied the nonsingularity of the skew circulant type matrices and presented explicit determinants and inverse matrices of these special matrices. Furthermore, four kinds of norms and bounds for the spread of these matrices are given separately. Shen et al. considered circulant matrices with Fibonacci and Lucas numbers and presented their explicit determinants and inverses in [23]. Jiang and Li [24] discussed the nonsingularity of the circulant type matrix and gave the explicit determinant and inverse matrices.

The Gaussian Fibonacci sequence [25, 26] is defined by the following recurrence relations:

$$G_{n+1} = G_n + G_{n-1}, \quad n \geq 1, \tag{1}$$

with the initial condition $G_0 = i$, $G_1 = 1$. $G_n = F_n + iF_{n-1}$, where F_n is the nth Fibonacci number, $i = \sqrt{-1}$.

The $\{G_n\}$ is given by the formula

$$G_n = \frac{(1 - i\beta)\,\alpha^n + (i\alpha - 1)\,\beta^n}{\alpha - \beta} = \frac{\alpha^n - \beta^n + \left(\alpha^{n-1} - \beta^{n-1}\right) i}{\alpha - \beta}, \tag{2}$$

where α and β are the roots of the characteristic equation $x^2 - x - 1 = 0$.

In this paper, circulant type matrices include the circulant, left circulant, and g-circulant matrices. Let r be a nonnegative integer. We define a Gaussian Fibonacci circulant matrix which is an $n \times n$ complex matrix with the following form:

$$\operatorname{Circ}\left(G_{r+1}, G_{r+2}, \ldots, G_{r+n}\right) = \begin{bmatrix} G_{r+1} & G_{r+2} & \cdots & G_{r+n} \\ G_{r+n} & G_{r+1} & \cdots & G_{r+n-1} \\ \vdots & \vdots & & \vdots \\ G_{r+2} & G_{r+3} & \cdots & G_{r+1} \end{bmatrix}. \tag{3}$$

Besides, a Gaussian Fibonacci left circulant matrix is given by

$$\operatorname{LCirc}\left(G_{r+1}, G_{r+2}, \ldots, G_{r+n}\right) = \begin{bmatrix} G_{r+1} & G_{r+2} & \cdots & G_{r+n} \\ G_{r+2} & G_{r+3} & \cdots & G_{r+1} \\ \vdots & \vdots & & \vdots \\ G_{r+n} & G_{r+1} & \cdots & G_{r+n-1} \end{bmatrix}, \tag{4}$$

where each row is a cyclic shift of the row above to the left.

A Gaussian Fibonacci g-circulant matrix is an $n \times n$ complex matrix with the following form:

$$A_{g,r,n} = \begin{pmatrix} G_{r+1} & G_{r+2} & \cdots & G_{r+n} \\ G_{n+r-g+1} & G_{n+r-g+2} & \cdots & G_{n+r-g} \\ G_{n+r-2g+1} & G_{n+r-2g+2} & \cdots & G_{n+r-2g} \\ \vdots & \vdots & \ddots & \vdots \\ G_{r+g+1} & G_{r+g+2} & \cdots & G_{r+g} \end{pmatrix}, \tag{5}$$

where g is a nonnegative integer and each of the subscripts is understood to be reduced modulo n.

The first row of $A_{g,n}$ is $(G_{r+1}, G_{r+2}, \ldots, G_{r+n})$ and its $(j+1)$th row is obtained by giving its jth row a right circular shift by g positions (equivalently, g mod n positions). Note that $g = 1$ or $g = n + 1$ yields the Gaussian Fibonacci circulant matrix. If $g = n - 1$, then we obtain the Gaussian Fibonacci left circulant matrix.

2. Determinant, Inverse, and Spread of Gaussian Fibonacci Circulant Matrices

In this section, let $A_{r,n} = \operatorname{Circ}(G_{r+1}, \ldots, G_{r+n})$ be a Gaussian Fibonacci circulant matrix. First, we give the determinant equation of the matrix $A_{r,n}$. Afterwards, we prove that $A_{r,n}$ is an invertible matrix for $n > 2$, and then we find the inverse of the matrix $A_{r,n}$. Obviously, when $n = 2$, $r \neq 0$, or $n = 1$, $A_{r,n}$ is also an invertible matrix.

Theorem 1. *Let $A_{r,n} = \operatorname{Circ}(G_{r+1}, \ldots, G_{r+n})$ be a Gaussian Fibonacci circulant matrix. Then we have*

$$\begin{aligned} \det A_{r,n} = G_{r+1} \cdot \Bigg[& \left(G_{r+1} - \frac{G_{r+2}}{G_{r+1}} G_{r+n} \right) \\ & + \sum_{k=1}^{n-2} \left(G_{r+k+2} - \frac{G_{r+2}}{G_{r+1}} G_{r+k+1} \right) \\ & \times \left(\frac{G_{r+n} - G_r}{G_{r+1} - G_{r+n+1}} \right)^{n-(k+1)} \Bigg] \\ & \times \left(G_{r+1} - G_{r+n+1} \right)^{n-2}, \end{aligned} \tag{6}$$

where G_{r+n} is the $(r+n)$th Gaussian Fibonacci number.

Proof. In the case $n > 1$, let

$$\Delta = \begin{pmatrix} 1 & & & & & & & \\ -\dfrac{G_{r+2}}{G_{r+1}} & & & & & & & 1 \\ -1 & & & & & & 1 & -1 \\ 0 & & & 0 & & 1 & -1 & -1 \\ \vdots & & & & \iddots & \iddots & \iddots & \\ 0 & & & 1 & \iddots & \iddots & & \\ 0 & & 1 & -1 & \iddots & & 0 & \\ 0 & 1 & -1 & -1 & & & & \end{pmatrix},$$

$$\Theta_1 = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & \left(\dfrac{G_{r+n} - G_r}{G_{r+1} - G_{r+n+1}} \right)^{n-2} & 0 & \cdots & 0 & 1 \\ 0 & \left(\dfrac{G_{r+n} - G_r}{G_{r+1} - G_{r+n+1}} \right)^{n-3} & 0 & \cdots & 1 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \dfrac{G_{r+n} - G_r}{G_{r+1} - G_{r+n+1}} & 1 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \end{pmatrix} \tag{7}$$

be two $n \times n$ matrices; then we have

$$\Delta A_{r,n}\Theta_1 = \begin{pmatrix} G_{r+1} & f'_{r,n} & G_{r+n-1} & \cdots & G_{r+3} & G_{r+2} \\ 0 & f_{r,n} & a_n & \cdots & a_4 & a_3 \\ 0 & 0 & b & & 0 & 0 \\ 0 & 0 & c & & 0 & 0 \\ \vdots & \vdots & & \ddots & & \\ 0 & 0 & 0 & & b & 0 \\ 0 & 0 & 0 & & c & b \end{pmatrix}, \quad (8)$$

where

$$\begin{aligned} f'_{r,n} &= \sum_{k=1}^{n-1} G_{r+k+1}\left(\frac{G_{r+n}-G_r}{G_{r+1}-G_{r+n+1}}\right)^{n-(k+1)}, \\ f_{r,n} &= \left(G_{r+1} - \frac{G_{r+2}}{G_{r+1}}G_{r+n}\right) \\ &\quad + \sum_{k=1}^{n-2}\left(G_{r+k+2} - \frac{G_{r+2}}{G_{r+1}}G_{r+k+1}\right) \\ &\qquad \times \left(\frac{G_{r+n}-G_r}{G_{r+1}-G_{r+n+1}}\right)^{n-(k+1)}, \\ a_n &= G_{r+n} - \frac{G_{r+2}}{G_{r+1}}G_{r+n-1}, \\ a_4 &= G_{r+4} - \frac{G_{r+2}}{G_{r+1}}G_{r+3}, \\ a_3 &= G_{r+3} - \frac{G_{r+2}}{G_{r+1}}G_{r+2}, \\ b &= G_{r+1} - G_{r+n+1}, \qquad c = G_r - G_{r+n}. \end{aligned} \quad (9)$$

We obtain

$$\begin{aligned} &\det\Delta \det A_{r,n} \det\Theta_1 \\ &\quad = G_{r+1} f_{r,n} (G_{r+1} - G_{r+n+1})^{n-2} \\ &\quad = G_{r+1}\cdot\left[\left(G_{r+1} - \frac{G_{r+2}}{G_{r+1}}G_{r+n}\right)\right. \\ &\qquad + \sum_{k=1}^{n-2}\left(G_{r+k+2} - \frac{G_{r+2}}{G_{r+1}}G_{r+k+1}\right) \\ &\qquad \left.\times\left(\frac{G_{r+n}-G_r}{G_{r+1}-G_{r+n+1}}\right)^{n-(k+1)}\right] \\ &\qquad \times (G_{r+1} - G_{r+n+1})^{n-2}, \end{aligned} \quad (10)$$

while

$$\begin{aligned} \det\Delta &= (-1)^{(n-1)(n-2)/2}, \\ \det\Theta_1 &= (-1)^{(n-1)(n-2)/2}. \end{aligned} \quad (11)$$

We have

$$\begin{aligned} \det A_{r,n} &= G_{r+1}\cdot\left[\left(G_{r+1} - \frac{G_{r+2}}{G_{r+1}}G_{r+n}\right)\right. \\ &\quad + \sum_{k=1}^{n-2}\left(G_{r+k+2} - \frac{G_{r+2}}{G_{r+1}}G_{r+k+1}\right) \\ &\quad \left.\times\left(\frac{G_{r+n}-G_r}{G_{r+1}-G_{r+n+1}}\right)^{n-(k+1)}\right] \\ &\quad \times (G_{r+1} - G_{r+n+1})^{n-2}. \end{aligned} \quad (12)$$

□

Theorem 2. *Let $A_{r,n} = \mathrm{Circ}(G_{r+1}, \dots, G_{r+n})$ be a Gaussian Fibonacci circulant matrix. If $n > 2$, then $A_{r,n}$ is an invertible matrix.*

Proof. We discuss the singularity of the matrix $A_{r,n}$. When $n = 3$ in Theorem 1, we have $\det A_{r,n} = (G_{r+1} + G_{r+2} + G_{r+3})(G_{r+1}^2 + G_r G_{r+2}) \neq 0$; hence $A_{r,n}$ is invertible. In the case $n > 3$, since $G_{r+n} = ((1-i\beta)\alpha^{r+n} + (i\alpha-1)\beta^{r+n})/(\alpha-\beta)$, where $\alpha+\beta = 1$, $\alpha\beta = -1$, let $\varepsilon = \exp(2\pi i/n)$; we can get that the eigenvalues of $A_{r,n}$

$$\begin{aligned} f(\varepsilon^k) &= \sum_{j=1}^{n} G_{r+j}(\varepsilon^k)^{j-1} \\ &= \frac{1}{\alpha-\beta}\sum_{j=1}^{n}\left[(1-i\beta)\alpha^{r+j} + (i\alpha-1)\beta^{r+j}\right](\varepsilon^k)^{j-1} \\ &= \frac{1}{\alpha-\beta}\left[\frac{(1-i\beta)(1-\alpha^n)\alpha^{r+1}}{1-\alpha\varepsilon^k}\right. \\ &\quad \left.+\frac{(i\alpha-1)(1-\beta^n)\beta^{r+1}}{1-\beta\varepsilon^k}\right] \\ &= \frac{1}{\alpha-\beta}\left[\frac{\alpha^{r+1}-\beta^{r+1}+(\alpha^r-\beta^r)i}{1-\varepsilon^k-\varepsilon^{2k}}\right. \\ &\quad - \frac{\alpha^{r+n+1}-\beta^{r+n+1}+(\alpha^{r+n}-\beta^{r+n})i}{1-\varepsilon^k-\varepsilon^{2k}} \\ &\quad + \frac{\alpha^r-\beta^r+(\alpha^{r-1}-\beta^{r-1})i}{1-\varepsilon^k-\varepsilon^{2k}}\varepsilon^k \\ &\quad \left.- \frac{\alpha^{r+n}-\beta^{r+n}+(\alpha^{r+n-1}-\beta^{r+n-1})i}{1-\varepsilon^k-\varepsilon^{2k}}\varepsilon^k\right] \\ &= \frac{G_{r+1}-G_{r+n+1}+(G_r-G_{r+n})\varepsilon^k}{1-\varepsilon^k-\varepsilon^{2k}}, \\ &\qquad (k = 1, 2, \dots, n-1). \end{aligned} \quad (13)$$

Since $G_n = F_n + iF_{n-1}$, $n \geq 1$, let $r \neq 0$, $\varepsilon^k = \cos\theta + i\sin\theta$, where $\theta = 2k\pi/n$ and $0 < \theta < 2\pi$. Then

$$\begin{aligned} x &= G_{r+1} - G_{r+n+1} + (G_r - G_{r+n})\,\varepsilon^k \\ &= [F_{r+1} - F_{r+n+1} + (F_r - F_{r+n})\cos\theta \\ &\quad + (F_{r+n-1} - F_{r-1})\sin\theta] \\ &\quad + [F_r - F_{r+n} + (F_r - F_{r+n})\sin\theta \\ &\quad + (F_{r-1} - F_{r+n-1})\cos\theta]\, i. \end{aligned} \tag{14}$$

We assume that $\mathrm{Re}(x) = F_{r+1} - F_{r+n+1} + (F_r - F_{r+n})\cos\theta + (F_{r+n-1} - F_{r-1})\sin\theta$ and $\mathrm{Im}(x) = F_r - F_{r+n} + (F_r - F_{r+n})\sin\theta + (F_{r-1} - F_{r+n-1})\cos\theta$.

Now, we prove that $\mathrm{Re}(x) \neq 0$ or $\mathrm{Im}(x) \neq 0$ for $1 - \varepsilon^k - \varepsilon^{2k} \neq 0$. For the Fibonacci sequence $\{F_n\}$, when $n > 1$, F_n is an increasing sequence and $|F_{r+1} - F_{r+n+1}| \geq |F_r - F_{r+n}| \geq |F_{r-1} - F_{r+n-1}|$.

$$\begin{aligned} &\text{If } \sin\theta > 0, \quad \cos\theta > 0, \quad \mathrm{Im}(x) < 0; \\ &\text{If } \sin\theta < 0, \quad \cos\theta < 0, \quad \mathrm{Re}(x) < 0; \\ &\text{If } \sin\theta > 0, \quad \cos\theta < 0, \quad \mathrm{Im}(x) < 0; \\ &\text{If } \sin\theta < 0, \quad \cos\theta > 0, \quad \mathrm{Re}(x) < 0. \end{aligned} \tag{15}$$

It is verified that when $\sin\theta = 0$ or $\cos\theta = 0$, $x \neq 0$. When $r = 0$, the arguments for $G_{r+1} - G_{r+n+1} + (G_r - G_{r+n})\varepsilon^k \neq 0$ are similar.

Hence $G_{r+1} - G_{r+n+1} + (G_r - G_{r+n})\varepsilon^k \neq 0$ for any ε^k $(k = 1, 2, \ldots, n-1)$; that is, $f(\varepsilon^k) \neq 0$, $(k = 1, 2, \ldots, n-1)$, while $f(1) = -G_{r+1} + G_{r+n+1} - (G_r - G_{r+n}) = G_{r+n+2} - G_{r+2} \neq 0$. By Lemma 1 in [19], the proof is completed. □

Lemma 3. *Let the matrix $B = [b_{i,j}]_{i,j=1}^{n-2}$ be of the form*

$$b_{i,j} = \begin{cases} G_{r+1} - G_{r+n+1}, & i = j, \\ G_r - G_{r+n}, & i = j+1, \\ 0, & \text{otherwise}; \end{cases} \tag{16}$$

then the inverse $B^{-1} = [b'_{i,j}]_{i,j=1}^{n-2}$ of the matrix B is equal to

$$b'_{i,j} = \begin{cases} \dfrac{(G_{r+n} - G_r)^{i-j}}{(G_{r+1} - G_{r+n+1})^{i-j+1}}, & i \geq j, \\ 0, & i < j. \end{cases} \tag{17}$$

Proof. Let $c_{i,j} = \sum_{k=1}^{n-2} b_{i,k} b'_{k,j}$. Obviously, $c_{i,j} = 0$ for $i < j$. In the case $i = j$, we obtain

$$c_{i,i} = b_{i,i} b'_{i,i} = (G_{r+1} - G_{r+n+1}) \cdot \frac{1}{G_{r+1} - G_{r+n+1}} = 1. \tag{18}$$

For $i \geq j+1$, we have

$$\begin{aligned} c_{i,j} &= \sum_{k=1}^{n-2} b_{i,k} b'_{k,j} \\ &= b_{i,i-1} b'_{i-1,j} + b_{i,i} b'_{i,j} \\ &= (G_r - G_{r+n}) \cdot \frac{(G_{r+n} - G_r)^{i-j-1}}{(G_{r+1} - G_{r+n+1})^{i-j}} \\ &\quad + (G_{r+1} - G_{r+n+1}) \cdot \frac{(G_{r+n} - G_r)^{i-j}}{(G_{r+1} - G_{r+n+1})^{i-j+1}} = 0. \end{aligned} \tag{19}$$

Hence, we verify $BB^{-1} = I_{n-2}$, where I_{n-2} is an $(n-2) \times (n-2)$ identity matrix. Similarly, we can verify $B^{-1}B = I_{n-2}$. Thus, the proof is completed. □

Theorem 4. *Let $A_{r,n} = \mathrm{Circ}(G_{r+1}, \ldots, G_{r+n})$ $(n > 2)$ be a Gaussian Fibonacci circulant matrix. Then we have*

$$\begin{aligned} A_{r,n}^{-1} &= \frac{1}{f_{r,n}} \\ &\times \mathrm{Circ}\Bigg(1 + \sum_{i=1}^{n-2} \frac{(G_{r+n+2-i} - hG_{r+n+1-i})(G_{r+n} - G_r)^{i-1}}{(G_{r+1} - G_{r+n+1})^i}, \\ &\qquad - h + \sum_{i=1}^{n-2} \frac{(G_{r+n+1-i} - hG_{r+n-i})(G_{r+n} - G_r)^{i-1}}{(G_{r+1} - G_{r+n+1})^i}, \\ &\qquad - \frac{G_{r+3} - hG_{r+2}}{G_{r+1} - G_{r+n+1}}, \\ &\qquad - \frac{(G_{r+3} - hG_{r+2})(G_{r+n} - G_r)}{(G_{r+1} - G_{r+n+1})^2}, \\ &\qquad - \frac{(G_{r+3} - hG_{r+2})(G_{r+n} - G_r)^2}{(G_{r+1} - G_{r+n+1})^3}, \ldots, \\ &\qquad - \frac{(G_{r+3} - hG_{r+2})(G_{r+n} - G_r)^{n-3}}{(G_{r+1} - G_{r+n+1})^{n-2}}\Bigg), \end{aligned} \tag{20}$$

where

$$h = \frac{G_{r+2}}{G_{r+1}}, \tag{21}$$

$$\begin{aligned} f_{r,n} &= \left(G_{r+1} - \frac{G_{r+2}}{G_{r+1}} G_{r+n}\right) \\ &\quad + \sum_{k=1}^{n-2} \left(G_{r+k+2} - \frac{G_{r+2}}{G_{r+1}} G_{r+k+1}\right) \\ &\quad \times \left(\frac{G_{r+n} - G_r}{G_{r+1} - G_{r+n+1}}\right)^{n-(k+1)}. \end{aligned} \tag{22}$$

Proof. Let

$$\Theta_2 = \begin{pmatrix} 1 & -\dfrac{f'_{r,n}}{G_{r+1}} & x_3 & x_4 & \cdots & x_n \\ 0 & 1 & y_3 & y_4 & \cdots & y_n \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \end{pmatrix}, \tag{23}$$

where

$$\begin{aligned} x_i &= \frac{f'_{r,n}}{f_{r,n}} \frac{G_{r+n+3-i} - (G_{r+2}/G_{r+1}) G_{r+n+2-i}}{G_{r+1}} - \frac{G_{r+n+2-i}}{G_{r+1}}, \\ &\qquad (i = 3, 4, \ldots, n), \\ y_i &= -\frac{G_{r+n+3-i} - (G_{r+2}/G_{r+1}) G_{r+n+2-i}}{f_{r,n}}, \quad (i = 3, 4, \ldots, n), \\ f'_{r,n} &= \sum_{k=1}^{n-1} G_{r+k+1} \left(\frac{G_{r+n} - G_r}{G_{r+1} - G_{r+n+1}} \right)^{n-(k+1)}, \\ f_{r,n} &= \left(G_{r+1} - \frac{G_{r+2}}{G_{r+1}} G_{r+n} \right) \\ &\quad + \sum_{k=1}^{n-2} \left(G_{r+k+2} - \frac{G_{r+2}}{G_{r+1}} G_{r+k+1} \right) \\ &\quad \times \left(\frac{G_{r+n} - G_r}{G_{r+1} - G_{r+n+1}} \right)^{n-(k+1)}. \end{aligned} \tag{24}$$

We have

$$\Delta A_{r,n} \Theta_1 \Theta_2 = \Lambda \oplus B, \tag{25}$$

where $\Lambda = \operatorname{diag}(G_{r+1}, f_{r,n})$ is a diagonal matrix and $\Lambda \oplus B$ is the direct sum of Λ and B. If we denote $\Theta = \Theta_1 \Theta_2$, then we obtain

$$A_{r,n}^{-1} = \Theta \left(\Lambda^{-1} \oplus B^{-1} \right) \Delta. \tag{26}$$

Since the last row elements of the matrix Θ are $0, 1, y_3, y_4, \ldots, y_{n-1}, y_n$. By Lemma 3, if $A_{r,n}^{-1} = \operatorname{Circ}(u_1, u_2, \ldots, u_n)$, then its last row elements are given by the following:

$$\begin{aligned} u_2 &= -\frac{1}{f_{r,n}} \frac{G_{r+2}}{G_{r+1}} + \frac{1}{f_{r,n}} C_n^{(n-2)}, \\ u_3 &= -\frac{1}{f_{r,n}} C_n^{(1)}, \\ u_4 &= -\frac{1}{f_{r,n}} C_n^{(2)} + \frac{1}{f_{r,n}} C_n^{(1)}, \\ u_5 &= -\frac{1}{f_{r,n}} C_n^{(3)} + \frac{1}{f_{r,n}} C_n^{(2)} + \frac{1}{f_{r,n}} C_n^{(1)}, \\ &\vdots \\ u_n &= -\frac{1}{f_{r,n}} C_n^{(n-2)} + \frac{1}{f_{r,n}} C_n^{(n-3)} + \frac{1}{f_{r,n}} C_n^{(n-4)}, \\ u_1 &= \frac{1}{f_{r,n}} + \frac{1}{f_{r,n}} C_n^{(n-2)} + \frac{1}{f_{r,n}} C_n^{(n-3)}. \end{aligned} \tag{27}$$

Let

$$\begin{aligned} C_n^{(j)} &= \sum_{i=1}^{j} \frac{\left(G_{r+3+j-i} - (G_{r+2}/G_{r+1}) G_{r+2+j-i} \right) \left(G_{r+n} - G_r \right)^{i-1}}{\left(G_{r+1} - G_{r+n+1} \right)^i} \\ &= \sum_{i=1}^{j} \frac{\delta_{j,r}}{(\mu_{2,r})^i} (\mu_{1,r})^{i-1}, \quad (j = 1, 2, \ldots, n-2), \end{aligned} \tag{28}$$

we have

$$\begin{aligned} & C_n^{(2)} - C_n^{(1)} \\ &\quad = \sum_{i=1}^{2} \frac{\delta_{2,r} (\mu_{1,r})^{i-1}}{(\mu_{2,r})^i} - \frac{\delta_{1,r}}{\mu_{2,r}} \\ &\quad = \frac{G_{r+3} - (G_{r+2}/G_{r+1}) G_{r+2}}{\left(G_{r+1} - G_{r+n+1} \right)^2} \left(G_{r+n} - G_r \right) \\ &\quad = \frac{\delta_{1,r}}{(\mu_{2,r})^2} \mu_{1,r}, \\ & C_n^{(n-2)} + C_n^{(n-3)} \\ &\quad = \sum_{i=1}^{n-2} \frac{\delta_{n-2,r} (\mu_{1,r})^{i-1}}{(\mu_{2,r})^i} + \sum_{i=1}^{n-3} \frac{\delta_{n-3,r} (\mu_{1,r})^{i-1}}{(\mu_{2,r})^i} \\ &\quad = \sum_{i=1}^{n-3} \frac{\left(G_{r+n+2-i} - (G_{r+2}/G_{r+1}) G_{r+n+1-i} \right) (\mu_{1,r})^{i-1}}{(\mu_{2,r})^i} \\ &\qquad + \frac{\delta_{1,r} (\mu_{1,r})^{n-3}}{(\mu_{2,r})^{n-2}} \\ &\quad = \sum_{i=1}^{n-2} \frac{\left(G_{r+n+2-i} - (G_{r+2}/G_{r+1}) G_{r+n+1-i} \right) (\mu_{1,r})^{i-1}}{(\mu_{2,r})^i}, \\ & C_n^{(j+2)} - C_n^{(j+1)} - C_n^{(j)} \\ &\quad = \sum_{i=1}^{j+2} \frac{\delta_{j+2,r} (\mu_{1,r})^{i-1}}{(\mu_{2,r})^i} - \sum_{i=1}^{j+1} \frac{\delta_{j+1,r} (\mu_{1,r})^{i-1}}{(\mu_{2,r})^i} \\ &\qquad - \sum_{i=1}^{j} \frac{\delta_{j,r} (\mu_{1,r})^{i-1}}{(\mu_{2,r})^i} \\ &\quad = \frac{\left(G_{r+4} - (G_{r+2}/G_{r+1}) G_{r+3} \right) (\mu_{1,r})^j}{(\mu_{2,r})^{j+1}} \\ &\qquad + \frac{\left(G_{r+3} - (G_{r+2}/G_{r+1}) G_{r+2} \right) (\mu_{1,r})^{j+1}}{(\mu_{2,r})^{j+2}} \end{aligned}$$

$$
\begin{aligned}
&- \frac{\left(G_{r+3} - \left(G_{r+2}/G_{r+1}\right) G_{r+2}\right) \left(\mu_{1,r}\right)^{j}}{\left(\mu_{2,r}\right)^{j+1}} \\
&= \frac{\left(G_{r+3} - \left(G_{r+2}/G_{r+1}\right) G_{r+2}\right) \left(\mu_{1,r}\right)^{j+1}}{\left(\mu_{2,r}\right)^{j+2}} \\
&\qquad (j = 1, 2, \ldots, n-4).
\end{aligned} \tag{29}
$$

We can get

$$
\begin{aligned}
&A_{r,n}^{-1} \\
&= \operatorname{Circ}\Bigg(\frac{1 + C_n^{(n-2)} + C_n^{(n-3)}}{f_{r,n}}, \\
&\quad \frac{C_n^{(n-2)} - \left(G_{r+2}/G_{r+1}\right)}{f_{r,n}}, -\frac{C_n^{(1)}}{f_{r,n}}, -\frac{C_n^{(2)} - C_n^{(1)}}{f_{r,n}}, \\
&\quad -\frac{C_n^{(3)} - C_n^{(2)} - C_n^{(1)}}{f_{r,n}}, \ldots, \\
&\quad -\frac{C_n^{(n-2)} - C_n^{(n-3)} - C_n^{(n-4)}}{f_{r,n}} \Bigg) \\
&= \frac{1}{f_{r,n}} \\
&\quad \times \operatorname{Circ}\Bigg(1 \\
&\quad + \sum_{i=1}^{n-2} \frac{\left(G_{r+n+2-i} - \left(G_{r+2}/G_{r+1}\right) G_{r+n+1-i}\right) \left(\mu_{1,r}\right)^{i-1}}{\left(\mu_{2,r}\right)^{i}}, \\
&\quad -\frac{G_{r+2}}{G_{r+1}} + \sum_{i=1}^{n-2} \frac{\delta_{n-2,r} \left(\mu_{1,r}\right)^{i-1}}{\left(\mu_{2,r}\right)^{i}}, -\frac{\delta_{1,r}}{\mu_{2,r}}, \\
&\quad -\frac{\delta_{1,r}}{\left(\mu_{2,r}\right)^{2}} \mu_{1,r}, -\frac{\delta_{1,r}}{\left(\mu_{2,r}\right)^{3}} \left(\mu_{1,r}\right)^{2}, \ldots, \\
&\quad -\frac{\delta_{1,r}}{\left(\mu_{2,r}\right)^{n-2}} \left(\mu_{1,r}\right)^{n-3} \Bigg) \\
&= \frac{1}{f_{r,n}} \\
&\quad \times \operatorname{Circ}\Bigg(1 \\
&\quad + \sum_{i=1}^{n-2} \frac{\left(G_{r+n+2-i} - h G_{r+n+1-i}\right) \left(G_{r+n} - G_{r}\right)^{i-1}}{\left(G_{r+1} - G_{r+n+1}\right)^{i}}, \\
&\qquad - h \\
&\quad + \sum_{i=1}^{n-2} \frac{\left(G_{r+n+1-i} - h G_{r+n-i}\right) \left(G_{r+n} - G_{r}\right)^{i-1}}{\left(G_{r+1} - G_{r+n+1}\right)^{i}}, \\
&\quad - \frac{G_{r+3} - h G_{r+2}}{G_{r+1} - G_{r+n+1}}, \\
&\quad - \frac{\left(G_{r+3} - h G_{r+2}\right) \left(G_{r+n} - G_{r}\right)}{\left(G_{r+1} - G_{r+n+1}\right)^{2}}, \\
&\quad - \frac{\left(G_{r+3} - h G_{r+2}\right) \left(G_{r+n} - G_{r}\right)^{2}}{\left(G_{r+1} - G_{r+n+1}\right)^{3}}, \ldots, \\
&\quad - \frac{\left(G_{r+3} - h G_{r+2}\right) \left(G_{r+n} - G_{r}\right)^{n-3}}{\left(G_{r+1} - G_{r+n+1}\right)^{n-2}} \Bigg),
\end{aligned} \tag{30}
$$

where

$$
\begin{aligned}
h &= \frac{G_{r+2}}{G_{r+1}}, \\
f_{r,n} &= \left(G_{r+1} - \frac{G_{r+2}}{G_{r+1}} G_{r+n} \right) \\
&\quad + \sum_{k=1}^{n-2} \left(G_{r+k+2} - \frac{G_{r+2}}{G_{r+1}} G_{r+k+1} \right) \\
&\qquad \times \left(\frac{G_{r+n} - G_{r}}{G_{r+1} - G_{r+n+1}} \right)^{n-(k+1)}.
\end{aligned} \tag{31}
$$

□

Lemma 5 (see [23]). *Let $\{F_n\}$ be the Fibonacci sequence, we can have*

(i) $\sum_{j=1}^{n} F_{r+j} = F_{r+n+2} - F_{r+2}$,

(ii) $\sum_{j=1}^{n} F_{r+j}^{2} = F_{r+n} F_{r+n+1} - F_{r} F_{r+1}$.

Lemma 6. *Let $\{G_n\}$ be the Gaussian Fibonacci sequence, we can have*

(i) $\sum_{j=1}^{n} G_{r+j} = G_{r+n+2} - G_{r+2}$,

(ii) $\sum_{j=1}^{n} G_{r+j}^{2} = G_{r+n} G_{r+n+1} - G_{r} G_{r+1}$.

Proof. By Lemma 5, we can obtain

$$
\begin{aligned}
\sum_{j=1}^{n} G_{r+j} &= G_{r+1} + G_{r+2} + \cdots + G_{r+n} \\
&= \left(F_{r+1} + i F_{r}\right) + \left(F_{r+2} + i F_{r+1}\right) + \cdots \\
&\quad + \left(F_{r+n} + i F_{r+n-1}\right) \\
&= \sum_{j=1}^{n} F_{r+j} + i \left(\sum_{j=1}^{n} F_{r+j} + F_{r} - F_{r+n} \right) \\
&= \left(F_{r+n+2} - F_{r+2}\right) + i \left(F_{r+n+1} - F_{r+1}\right) \\
&= \left(F_{r+n+2} + i F_{r+n+1}\right) - \left(F_{r+2} + i F_{r+1}\right) \\
&= G_{r+n+2} - G_{r+2}.
\end{aligned} \tag{32}
$$

According to $G_{n+1} = G_n + G_{n-1}$, $n \geq 1$, we have

$$\begin{aligned}\sum_{j=1}^{n} G_{r+j}^2 &= G_{r+1}^2 + G_{r+2}^2 + G_{r+3}^2 + \cdots + G_{r+n}^2 \\ &= G_{r+1}(G_{r+2} - G_r) + G_{r+2}(G_{r+3} - G_{r+1}) \\ &\quad + G_{r+3}(G_{r+4} - G_{r+2}) + \cdots \\ &\quad + G_{r+n}(G_{r+n+1} - G_{r+n-1}) \\ &= G_{r+n}G_{r+n+1} - G_r G_{r+1}.\end{aligned} \tag{33}$$

□

Theorem 7. *Let $A_{r,n} =$ Circ $(G_{r+1}, \ldots, G_{r+n})$ be a Gaussian Fibonacci circulant matrix; then*

$$s(A_{r,n}) \leq (2n[F_{r+n}(F_{r+n+1} + F_{r+n-1}) - F_{r+1}(F_r + F_{r+2})])^{1/2}, \tag{34}$$

where F_{r+n} is the $(r+n)$th Fibonacci number and $s(A_{r,n})$ is the spread (see [13]) of $A_{r,n}$.

Proof. From Definition 4 in [13] and Lemma 5, we acquire

$$\begin{aligned}&\|A_{r,n}\|_F \\ &= \left(n\left(|G_{r+1}|^2 + |G_{r+2}|^2 + \cdots + |G_{r+n}|^2\right)\right)^{1/2} \\ &= \left(\frac{n}{(\alpha-\beta)^2}\left[\left(\alpha^{r+1} - \beta^{r+1}\right)^2 + (\alpha^r - \beta^r)^2\right.\right. \\ &\qquad + \left(\alpha^{r+2} - \beta^{r+2}\right)^2 + \left(\alpha^{r+1} - \beta^{r+1}\right)^2 + \cdots \\ &\qquad \left.\left. + (\alpha^{r+n} - \beta^{r+n})^2 + \left(\alpha^{r+n-1} - \beta^{r+n-1}\right)^2\right]\right)^{1/2} \\ &= \left(\frac{n}{(\alpha-\beta)^2}\left[2\sum_{j=1}^{n}\left(\alpha^{r+j} - \beta^{r+j}\right)^2 - 2(\alpha^{r+n} - \beta^{r+n})^2\right.\right. \\ &\qquad \left.\left. + (\alpha^r - \beta^r)^2 + (\alpha^{r+n} - \beta^{r+n})^2\right]\right)^{1/2} \\ &= \left(2n(F_{r+n}F_{r+n+1} - F_rF_{r+1}) + n\left(F_r^2 - F_{r+n}^2\right)\right)^{1/2}.\end{aligned} \tag{35}$$

From the elements in $A_{r,n}$, we get $a_{ii} = G_{r+1}$, so $\operatorname{tr} A_{r,n} = nG_{r+1} = n(F_{r+1} + iF_r)$; then

$$\begin{aligned}&2\|A_{r,n}\|_F^2 - \frac{2}{n}|\operatorname{tr} A_{r,n}|^2 \\ &= 4n(F_{r+n}F_{r+n+1} - F_rF_{r+1}) \\ &\quad + 2n\left(F_r^2 - F_{r+n}^2\right) - 2n\left(F_{r+1}^2 + F_r^2\right) \\ &= 4n(F_{r+n}F_{r+n+1} - F_rF_{r+1}) - 2n\left(F_{r+1}^2 + F_{r+n}^2\right) \\ &= 2n[F_{r+n}(F_{r+n+1} + F_{r+n-1}) - F_{r+1}(F_r + F_{r+2})].\end{aligned} \tag{36}$$

By (16) in [13], we have

$$s(A_{r,n}) \leq (2n[F_{r+n}(F_{r+n+1} + F_{r+n-1}) - F_{r+1}(F_r + F_{r+2})])^{1/2}, \tag{37}$$

where F_{r+n} is the $(r+n)$th Fibonacci number and $\alpha = (1 + \sqrt{5})/2$, $\beta = (1 - \sqrt{5})/2$. □

3. Determinant, Inverse, and Spread of Gaussian Fibonacci Left Circulant Matrices

In this section, let $A'_{r,n} = \operatorname{LCirc}(G_{r+1}, G_{r+2}, \ldots, G_{r+n})$ be a Gaussian Fibonacci left circulant matrices. By using the obtained conclusions, we give a determinant formula for the matrix $A'_{r,n}$ and prove that $A'_{r,n}$ is an invertible matrix for $n > 2$ for any positive integer n. The inverse and the upper bound for spread of the matrix $A'_{r,n}$ are also presented.

According to Lemma 2 in [19] and Theorems 1, 2, and 4, we can obtain the following theorems.

Theorem 8. *Let $A'_{r,n} =$ LCirc $(G_{r+1}, \ldots, G_{r+n})$ be a Gaussian Fibonacci left circulant matrix; then we have*

$$\begin{aligned}\det A'_{r,n} &= (-1)^{(n-1)(n-2)/2} \cdot G_{r+1} \\ &\quad \cdot \left[\left(G_{r+1} - \frac{G_{r+2}}{G_{r+1}}G_{r+n}\right)\right. \\ &\quad + \sum_{k=1}^{n-2}\left(G_{r+k+2} - \frac{G_{r+2}}{G_{r+1}}G_{r+k+1}\right) \\ &\quad \left.\times\left(\frac{G_{r+n} - G_r}{G_{r+1} - G_{r+n+1}}\right)^{n-(k+1)}\right] \\ &\quad \times (G_{r+1} - G_{r+n+1})^{n-2},\end{aligned} \tag{38}$$

where G_{r+n} is the $(r+n)$th Gaussian Fibonacci number.

Theorem 9. *Let $A'_{r,n} =$ LCirc $(G_{r+1}, \ldots, G_{r+n})$ be a Gaussian Fibonacci left circulant matrix; if $n > 2$, then $A'_{r,n}$ is an invertible matrix.*

Theorem 10. *Let $A'_{r,n} =$ LCirc $(G_{r+1}, \ldots, G_{r+n})$ $(n > 2)$ be a Gaussian Fibonacci left circulant matrix; then we have*

$$\begin{aligned}&A'^{-1}_{r,n} \\ &= \frac{1}{f_{r,n}} \\ &\quad \times \operatorname{LCirc}\left(1 + \sum_{i=1}^{n-2}\frac{(G_{r+n+2-i} - hG_{r+n+1-i})(G_{r+n} - G_r)^{i-1}}{(G_{r+1} - G_{r+n+1})^i},\right. \\ &\qquad -\frac{(G_{r+3} - hG_{r+2})(G_{r+n} - G_r)^{n-3}}{(G_{r+1} - G_{r+n+1})^{n-2}}, \ldots, \\ &\qquad -\frac{(G_{r+3} - hG_{r+2})(G_{r+n} - G_r)^2}{(G_{r+1} - G_{r+n+1})^3},\end{aligned}$$

$$-\frac{(G_{r+3}-hG_{r+2})(G_{r+n}-G_r)}{(G_{r+1}-G_{r+n+1})^2},$$
$$-\frac{G_{r+3}-hG_{r+2}}{G_{r+1}-G_{r+n+1}},$$
$$-h$$
$$\left.+\sum_{i=1}^{n-2}\frac{(G_{r+n+1-i}-hG_{r+n-i})(G_{r+n}-G_r)^{i-1}}{(G_{r+1}-G_{r+n+1})^i}\right), \tag{39}$$

where

$$h=\frac{G_{r+2}}{G_{r+1}},$$
$$f_{r,n}=\left(G_{r+1}-\frac{G_{r+2}}{G_{r+1}}G_{r+n}\right)+\sum_{k=1}^{n-2}\left(G_{r+k+2}-\frac{G_{r+2}}{G_{r+1}}G_{r+k+1}\right)\times\left(\frac{G_{r+n}-G_r}{G_{r+1}-G_{r+n+1}}\right)^{n-(k+1)}. \tag{40}$$

Theorem 11. *Let* $A'_{r,n}=\mathrm{LCirc}(G_{r+1},\dots,G_{r+n})$ $(n>2)$ *be a Gaussian Fibonacci left circulant matrix; then the upper bounds for the spread of* $A'_{r,n}$ *are*

$$s(A'_{r,n})\le\left(2n\left[F_{r+n}(F_{r+n+1}+F_{r+n-1})-F_r(F_{r+1}+F_{r-1})\right]-\frac{2}{n}\left[(F_{r+n+2}-F_{r+2})^2+(F_{r+n+1}-F_{r+1})^2\right]\right)^{1/2},\quad (n\text{ is odd}),$$
$$s(A'_{r,n})\le\left(2n\left[F_{r+n}(F_{r+n+1}+F_{r+n-1})-F_r(F_{r+1}+F_{r-1})\right]-\frac{8}{n}\left[(F_{r+n}-F_r)^2+(F_{r+n-1}-F_{r-1})^2\right]\right)^{1/2},\quad (n\text{ is even}). \tag{41}$$

Proof. From the elements in $A'_{r,n}$ and Lemma 6, if n is odd, the trace of $A'_{r,n}$ is $\operatorname{tr}A'_{r,n}=\sum_{j=1}^{n}G_{r+j}=G_{r+n+2}-G_{r+2}$; if n is even,

$$\operatorname{tr}A'_{r,n}=2(G_{r+1}+G_{r+3}+G_{r+5}+\cdots+G_{r+n-1})=2\left[G_{r+1}+\sum_{j=4}^{n}(G_{r+j-3}+G_{r+j-2})\right]=2(G_{r+n}-G_r). \tag{42}$$

Since $G_n=F_n+iF_{n-1}$ and

$$\|A'_{r,n}\|_F=\left(2n(F_{r+n}F_{r+n+1}-F_rF_{r+1})+n(F_r^2-F_{r+n}^2)\right)^{1/2}=\left(n\left[F_{r+n}(F_{r+n+1}+F_{r+n-1})-F_r(F_{r+1}+F_{r-1})\right]\right)^{1/2}, \tag{43}$$

by (16) in [13], we can get the upper bounds for $A'_{r,n}$ as the above easily. □

4. Determinant, Inverse of Gaussian Fibonacci g-Circulant Matrices

In this section, let $A_{g,r,n}=g\text{-Circ}(G_{r+1},\dots,G_{r+n})$ be a Gaussian Fibonacci g-circulant matrices. A determinant formula for the matrix $A_{g,r,n}$ and the inverse $A_{g,r,n}$ for $n>2$ when $(n,g)=1$ are obtained as follows.

From Lemmas 3 and 4 in [19] and Theorems 1, 2, and 4, we deduce the following results.

Theorem 12. *Let* $A_{g,r,n}=g\text{-}\mathrm{Circ}(G_{r+1},\dots,G_{r+n})$ *be a Gaussian Fibonacci g-circulant matrix; then we have*

$$\det A_{g,r,n}=\det\mathbb{Q}_g\cdot G_{r+1}\cdot\left[\left(G_{r+1}-\frac{G_{r+2}}{G_{r+1}}G_{r+n}\right)+\sum_{k=1}^{n-2}\left(G_{r+k+2}-\frac{G_{r+2}}{G_{r+1}}G_{r+k+1}\right)\times\left(\frac{G_{r+n}-G_r}{G_{r+1}-G_{r+n+1}}\right)^{n-(k+1)}\right]\times(G_{r+1}-G_{r+n+1})^{n-2}, \tag{44}$$

where G_{r+n} *is the* $(r+n)$*th Gaussian Fibonacci number.*

Theorem 13. *Let* $A_{g,r,n}=g\text{-}\mathrm{Circ}(G_{r+1},\dots,G_{r+n})$ *be a Gaussian Fibonacci g-circulant matrix and* $(g,n)=1$*; if* $n>2$*, then* $A_{g,r,n}$ *is an invertible matrix.*

Theorem 14. *Let* $A_{g,r,n}=g\text{-}\mathrm{Circ}(G_{r+1},\dots,G_{r+n})$ $(n>2)$ *be a Gaussian Fibonacci g-circulant matrix and* $(g,n)=1$*; then*

$$A^{-1}_{g,r,n}=\left[\frac{1}{f_{r,n}}\times\mathrm{Circ}\left(1+\sum_{i=1}^{n-2}\frac{(G_{r+n+2-i}-hG_{r+n+1-i})(G_{r+n}-G_r)^{i-1}}{(G_{r+1}-G_{r+n+1})^i},\right.\right.$$
$$-h$$
$$+\sum_{i=1}^{n-2}\frac{(G_{r+n+1-i}-hG_{r+n-i})(G_{r+n}-G_r)^{i-1}}{(G_{r+1}-G_{r+n+1})^i},$$

$$
\begin{aligned}
&- \frac{G_{r+3} - hG_{r+2}}{G_{r+1} - G_{r+n+1}}, \\
&- \frac{\left(G_{r+3} - hG_{r+2}\right)\left(G_{r+n} - G_r\right)}{\left(G_{r+1} - G_{r+n+1}\right)^2}, \\
&- \frac{\left(G_{r+3} - hG_{r+2}\right)\left(G_{r+n} - G_r\right)^2}{\left(G_{r+1} - G_{r+n+1}\right)^3}, \dots, \\
&- \frac{\left(G_{r+3} - hG_{r+2}\right)\left(G_{r+n} - G_r\right)^{n-3}}{\left(G_{r+1} - G_{r+n+1}\right)^{n-2}} \Bigg) \Bigg] \mathbb{Q}_g^T,
\end{aligned} \tag{45}
$$

where

$$
\begin{aligned}
h &= \frac{G_{r+2}}{G_{r+1}}, \\
f_{r,n} &= \left(G_{r+1} - \frac{G_{r+2}}{G_{r+1}} G_{r+n}\right) \\
&\quad + \sum_{k=1}^{n-2} \left(G_{r+k+2} - \frac{G_{r+2}}{G_{r+1}} G_{r+k+1}\right) \\
&\quad \times \left(\frac{G_{r+n} - G_r}{G_{r+1} - G_{r+n+1}}\right)^{n-(k+1)}.
\end{aligned} \tag{46}
$$

5. Conclusion

In this paper, the explicit determinants and the inverse matrices of Gaussian Fibonacci circulant type matrices are presented. Furthermore, we give the upper bounds for the spread on Gaussian Fibonacci circulant and left circulant matrices, respectively. The reason why we focus our attentions on circulant type matrices is to explore the application of it in the related field. On the basis of existing application situation [1–8], we will develop solving integrable system, Hamiltonian structure, and integral equations.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

The research is supported by the Development Project of Science & Technology of Shandong Province (Grant no. 2012GGX10115) and the AMEP of Linyi University, China.

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