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Research Article

A Sharp Double Inequality for Trigonometric Functions and Its Applications

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We present the best possible parameters p and q such that the double inequality $((2/3)\cos^{2p}(t/2)+1/3)^{1/p} < \sin t/t < ((2/3)\cos^{2q}(t/2)+1/3)^{1/q}$ holds for any $t\in(0,\pi/2)$. As applications, some new analytic inequalities are established.

1. Introduction

It is well known that the double inequality

$$\cos^{1/3}t < \frac{\sin t}{t} < \frac{2+\cos t}{3} \tag{1}$$

holds for any $t \in (0,\pi/2)$. The first inequality in (1) was found by Mitrinović (see [1]), while the second inequality in (1) is due to Huygens (see [2]) and it is called Cusa inequality. Recently, the improvements, refinements, and generalizations for inequality (1) have attracted the attention of many mathematicians [3–8].

Qi et al. [9] proved that the inequality

$$\cos^2\frac{t}{2} < \frac{\sin t}{t} \tag{2}$$

holds for any $t\in(0,\pi/2)$. It is easy to verify that $\cos^{1/3}t$ and $\cos^2(t/2)$ cannot be compared on the interval $(0,\pi/2)$.

Neuman and Sándor [6] gave an improvement for the first inequality in (1) as follows:

$$\cos^{4/3}\frac{t}{2} = \left(\frac{1+\cos t}{2}\right)^{2/3} < \frac{\sin t}{t}, \quad t\in\left(0,\frac{\pi}{2}\right). \tag{3}$$

Inequality (3) was also proved by Lv et al. in [10]. In [11, 12], Neuman proved that the inequalities

$$\begin{aligned}\cos^{1/3}t &< \left(\frac{\sin t}{t}\cos t\right)^{1/4} < \left(\frac{\sin t}{\tanh^{-1}(\sin t)}\right)^{1/2}\\ &< \left(\frac{t\cos t+\sin t}{2t}\right)^{1/2} < \left(\frac{1+2\cos t}{3}\right)^{1/2}\\ &< \left(\frac{1+\cos t}{2}\right)^{2/3} < \frac{\sin t}{t}\end{aligned} \tag{4}$$

hold for any $t\in(0,\pi/2)$.

For the second inequality in (1), Klén et al. [13] established

$$\frac{\sin t}{t} \le \cos^3\frac{t}{3} \le \frac{2+\cos t}{3} \tag{5}$$

for $t\in(-\sqrt{135}/5, \sqrt{135}/5)$.

Inequality (5) was improved by Yang [14]. In [15], Yang further proved

$$\frac{\sin t}{t} < \left(\frac{2}{3}\cos\frac{t}{2}+\frac{1}{3}\right)^2 < \cos^3\frac{t}{3} < \frac{2+\cos t}{3}, \tag{6}$$

for $t\in(0,\pi/2)$.

Yang [16] proved that the inequalities

$$\cos^{1/3} t < \cos\frac{t}{\sqrt{3}} < \cos^{4/3}\frac{t}{2} < \frac{\sin t}{t} < \cos^3\frac{t}{3} < \cos^{16/3}\frac{t}{4} < e^{-t^2/6} < \frac{2+\cos t}{3} \quad (7)$$

hold for $t \in (0, \pi/2)$.

Zhu [8] and Yang [17] proved that $p = 4/5$ and $q = (\log 3 - \log 2)/(\log \pi - \log 2) = 0.8978\ldots$ are the best possible constants such that the double inequality

$$\left(\frac{2}{3} + \frac{1}{3}\cos^p t\right)^{1/p} < \frac{\sin t}{t} < \left(\frac{2}{3} + \frac{1}{3}\cos^q t\right)^{1/q} \quad (8)$$

holds for all $t \in (0, \pi/2)$.

More results involving inequality (1) can be found in the literature [18–22].

Let $p \in \mathbb{R}$, $x > 0$, and $0 < \omega < 1$. Then $M_p(x, \omega)$ is defined by

$$M_p(x,\omega) = (\omega x^p + 1 - \omega)^{1/p} \ (p \neq 0), \qquad M_0(x,\omega) = \lim_{p\to 0} M_p(x,\omega) = x^\omega. \quad (9)$$

It is well known that $M_p(x, \omega)$ is strictly increasing with respect to $p \in \mathbb{R}$ for fixed $x > 0$ and $0 < \omega < 1$ (see [23]). If $0 < x < 1$, then it is easy to check that

$$M_{-\infty}(x,\omega) = \lim_{p\to-\infty} M_p(x,\omega) = x, \qquad M_{\infty}(x,\omega) = \lim_{p\to\infty} M_p(x,\omega) = 1. \quad (10)$$

It follows from (2) and (3) together with (6) that

$$\begin{aligned} M_{-\infty}\left(\cos^2\frac{t}{2}, \frac{2}{3}\right) &= \cos^2\frac{t}{2} < \cos^{4/3}\frac{t}{2} \\ &= M_0\left(\cos^2\frac{t}{2}, \frac{2}{3}\right) < \frac{\sin t}{t} \\ &< \left(\frac{2}{3}\cos\frac{t}{2} + \frac{1}{3}\right)^2 \\ &= M_{1/2}\left(\cos^2\frac{t}{2}, \frac{2}{3}\right) < \frac{2+\cos t}{3} \\ &= M_1\left(\cos^2\frac{t}{2}, \frac{2}{3}\right) < 1 \\ &= M_\infty\left(\cos^2\frac{t}{2}, \frac{2}{3}\right), \end{aligned} \quad (11)$$

for $t \in (0, \pi/2)$.

The main purpose of this paper is to present the best possible parameters p and q such that the double inequality

$$M_p\left(\cos^2\frac{t}{2}, \frac{2}{3}\right) < \frac{\sin t}{t} < M_q\left(\cos^2\frac{t}{2}, \frac{2}{3}\right) \quad (12)$$

holds for all $t \in (0, \pi/2)$. As applications, some new analytic inequalities are found. All numerical computations are carried out using MATHEMATICA software.

2. Lemmas

In order to prove our main results we need several lemmas, which we present in this section.

Lemma 1. *Let $p \in \mathbb{R}$ and the function g_p be defined on $(1/2, 1)$ by*

$$g_p(x) = 2px - x^{1-p} + 2x^p - (2p+1). \quad (13)$$

Then the following statements are true:

(i) *$g_p(x) < 0$ for all $x \in (1/2, 1)$ if and only if $p \geq 1/5$;*

(ii) *$g_p(x) > 0$ for all $x \in (1/2, 1)$ if and only if $p \leq p_2$, where $p_2 = 0.1872\ldots$ is the unique solution of equation*

$$g_p\left(\frac{1}{2}\right) = 2^{1-p} - 2^{p-1} - p - 1 = 0; \quad (14)$$

(iii) *if $p_2 < p < 1/5$, then there exists $x_1 = x_1(p) \in (1/2, 1)$ such that $g_p(x) < 0$ for $x \in (1/2, x_1)$ and $g_p(x) > 0$ for $x \in (x_1, 1)$.*

Proof. It follows from (13) and (14) that

$$\begin{aligned} g_{0.1872}\left(\frac{1}{2}\right) &= 0.000141\ldots > 0, \\ g_{0.1873}\left(\frac{1}{2}\right) &= -0.000119\ldots < 0, \\ \frac{\partial g_p(x)}{\partial p} &= \left(x^{1-p} + 2x^p\right)\log x - 2(1-x) < 0, \end{aligned} \quad (15)$$

for $x \in (0, 1)$.

Inequalities (15) lead to the conclusion that the function $g_p(x)$ is strictly decreasing with respect to $p \in \mathbb{R}$ for fixed $x \in (0, 1)$ and $p_2 = 0.1872\ldots$ is the unique solution of (14).

(i) If $x \in (1/2, 1)$ and $p \geq 1/5$, then from the monotonicity of the function $p \to g_p(x)$ we clearly see that

$$\begin{aligned} g_p(x) \leq g_{1/5}(x) &= \frac{2}{5}x - x^{4/5} + 2x^{1/5} - \frac{7}{5} \\ &= -\frac{1}{5}\left(1 - x^{1/5}\right)^2 \\ &\quad \times \left(-2x^{3/5} + x^{2/5} + 4x^{1/5} + 7\right) < 0. \end{aligned} \quad (16)$$

If $g_p(x) < 0$ for all $x \in (1/2, 1)$, then (13) leads to

$$\lim_{x\to 1^-} \frac{g_p(x)}{1-x} = 1 - 5p \leq 0. \quad (17)$$

(ii) If $x \in (1/2, 1)$ and $p \leq 0$, then the monotonicity of the function $p \to g_p(x)$ leads to the conclusion that $g_p(x) \geq g_0(x) = 1 - x > 0$.

If $x \in (1/2, 1)$ and $0 < p \leq p_2$, then (13) and the monotonicity of the function $p \to g_p(x)$ lead to

$$g_p\left(\frac{1}{2}\right) \geq g_{p_2}\left(\frac{1}{2}\right) = 0, \quad g_p(1) = 0, \quad (18)$$

$$\frac{\partial^2 g_p(x)}{\partial x^2} = p(p-1)x^{p-2}\left(2 - x^{1-2p}\right) < 0. \quad (19)$$

Inequality (19) implies that the function $g_p(x)$ is concave with respect to x on the interval $(1/2, 1)$. Therefore, $g_p(x) > 0$ follows from (18) and the concavity of $g_p(x)$.

If $g_p(x) > 0$ for all $x \in (1/2, 1)$, then $p \le p_2$ follows easily from the monotonicity of the function $p \to g_p(1/2)$ and $g_p(1/2) \ge 0$ together with the fact that $g_{p_2}(1/2) = 0$.

(iii) If $x \in (1/2, 1)$ and $p_2 < p < 1/5$, then from (13) and (19) together with the monotonicity of the function $p \to g_p(1/2)$ we get

$$g_p(1) = 0, \qquad g_p\left(\frac{1}{2}\right) < g_{p_2}\left(\frac{1}{2}\right) = 0, \tag{20}$$

$$g'_p(1) = 5p - 1 < 0, \tag{21}$$

$$\begin{aligned} g'_p\left(\frac{1}{2}\right) &= 2p - 2^p + p2^p + 2p2^{1-p} \\ &> 2 \times 0.1872 - 2^{0.1873} \\ &\quad + 0.1872 \times 2^{0.1872} \\ &\quad + 2 \times 0.1872 \times 2^{0.8127} \\ &= 0.1065\ldots > 0, \end{aligned} \tag{22}$$

and $g'_p(x)$ is strictly decreasing on $(1/2, 1)$.

It follows from (21) and (22) together with the monotonicity of $g'_p(x)$ that there exists $x_0 = x_0(p) \in (1/2, 1)$ such that $g_p(x)$ is strictly increasing on $(1/2, x_0]$ and strictly decreasing on $[x_0, 1)$. Therefore, Lemma 1 (iii) follows from (20) and the piecewise monotonicity of $g_p(x)$.

Let $p \in \mathbb{R}$ and the function f_p be defined on $(0, \pi/2)$ by

$$f_p(t) = t - \frac{2\cos^{2p}(t/2) + 1}{\cos t + 2\cos^{2p}(t/2)} \sin t. \tag{23}$$

Then elaborated computations lead to

$$f'_p(t) = \frac{4\left(1 - \cos^2(t/2)\right)\cos^{2p}(t/2)}{\left(2\cos^{2p}(t/2) + 2\cos^2(t/2) - 1\right)^2} g_p\left(\cos^2\frac{t}{2}\right), \tag{24}$$

where $g_p(x)$ is defined by (13). □

From Lemma 1 and (24) we get the following Lemma 2 immediately.

Lemma 2. *Let $p \in \mathbb{R}$ and f_p be defined on $(0, \pi/2)$ by (23). Then*

(i) *$f_p(t)$ is strictly decreasing on $(0, \pi/2)$ if and only if $p \ge 1/5$;*

(ii) *$f_p(t)$ is strictly increasing on $(0, \pi/2)$ if and only if $p \le p_2$, where $p_2 = 0.1872\ldots$ is the unique solution of (14);*

(iii) *if $p_2 < p < 1/5$, then there exists $t_1 = t_1(p) \in (0, \pi/2)$ such that $f_p(t)$ is strictly increasing on $(0, t_1]$ and strictly decreasing on $[t_1, \pi/2)$.*

Lemma 3. *Let $p \in \mathbb{R}$ and f_p be defined on $(0, \pi/2)$ by (23). Then*

(i) *$f_p(t) < 0$ for all $t \in (0, \pi/2)$ if and only if $p \ge 1/5$;*

(ii) *$f_p(t) > 0$ for all $t \in (0, \pi/2)$ if and only if $p \le p_1 = \log(\pi - 2)/\log 2 = 0.1910\ldots$;*

(iii) *if $p_1 < p < 1/5$, then there exists $t_0 = t_0(p) \in (0, \pi/2)$ such that $f_p(t) > 0$ for $t \in (0, t_0)$ and $f_p(t) < 0$ for $t \in (t_0, \pi/2)$.*

Proof. (i) If $t \in (0, \pi/2)$ and $p \ge 1/5$, then from (23) and Lemma 2 (i) we clearly see that

$$f_p(t) < f_p(0^+) = 0. \tag{25}$$

If $f_p(t) < 0$ for all $t \in (0, \pi/2)$, then (23) leads to

$$\begin{aligned} 0 \ge \lim_{t \to 0^+} \frac{f_p(t)}{t^5} &= \lim_{t \to 0^+} \frac{(1/180)(1 - 5p)t^5 + o(t^5)}{t^5} \\ &= \frac{1 - 5p}{180}. \end{aligned} \tag{26}$$

(ii) If $f_p(t) > 0$ for all $t \in (0, \pi/2)$, then from (23) we get

$$0 \le f_p\left(\frac{\pi^-}{2}\right) = \frac{\pi - 2 - 2^p}{2}. \tag{27}$$

Inequality (27) leads to the conclusion that $p \le \log(\pi - 2)/\log 2$.

If $t \in (0, \pi/2)$ and $p \le p_1 = \log(\pi - 2)/\log 2$, then we divide the proof into two cases.

Case 1. Consider $p \le p_2$, where p_2 is the unique solution of (14). Then from Lemma 2 (ii) and (23) we clearly see that

$$f_p(t) > f_p(0^+) = 0. \tag{28}$$

Case 2. Consider $p_2 < p \le p_1$. Then (23) and Lemma 2 (iii) lead to

$$\begin{aligned} &f_p(0^+) = 0, \\ &f_p\left(\frac{\pi^-}{2}\right) = \frac{\pi - 2 - 2^p}{2} \ge \frac{\pi - 2 - 2^{p_1}}{2} = 0, \end{aligned} \tag{29}$$

and there exists $t_1 = t_1(p)$ such that $f_p(t)$ is strictly increasing on $(0, t_1]$ and strictly decreasing on $[t_1, \pi/2)$. Therefore, $f_p(t) > 0$ for all $t \in (0, \pi/2)$ follows from (29) and the piecewise monotonicity of $f_p(t)$.

(iii) If $p_1 < p < 1/5$, then $p_2 < p < 1/5$. It follows from (23) and Lemma 2 (iii) that

$$\begin{aligned} &f_p(0^+) = 0, \\ &f_p\left(\frac{\pi^-}{2}\right) = \frac{\pi - 2 - 2^p}{2} < \frac{\pi - 2 - 2^{p_1}}{2} = 0, \end{aligned} \tag{30}$$

and there exists $t_1 = t_1(p)$ such that $f_p(t)$ is strictly increasing on $(0, t_1]$ and strictly decreasing on $[t_1, \pi/2)$. Therefore, Lemma 3 (iii) follows from (30) and the piecewise monotonicity of $f_p(t)$.

Let $p \in \mathbb{R}$ and F_p be defined on $(0, \pi/2)$ by

$$F_p(t) = \log\frac{\sin t}{t} - \frac{1}{p}\log\left(\frac{2}{3}\cos^{2p}\frac{t}{2} + \frac{1}{3}\right) \quad (p \ne 0), \tag{31}$$

$$F_0(t) = \lim_{p \to 0} F_p(t) = \log\frac{\sin t}{t} - \frac{4}{3}\log\left(\cos\frac{t}{2}\right). \tag{32}$$

Then elaborated computations give

$$F_p'(t) = \frac{\cos t + 2\cos^{2p}(t/2)}{t(1 + 2\cos^{2p}(t/2))\sin t} f_p(t), \tag{33}$$

where $f_p(t)$ is defined by (23). □

From Lemma 3 and (33) we get Lemma 4 immediately.

Lemma 4. *Let $p \in \mathbb{R}$ and F_p be defined on $(0, \pi/2)$ by (31) and (32). Then*

(i) *$F_p(t)$ is strictly decreasing on $(0, \pi/2)$ if and only if $p \geq 1/5$;*

(ii) *$F_p(t)$ is strictly increasing on $(0, \pi/2)$ if and only if $p \leq p_1 = \log(\pi - 2)/\log 2 = 0.1910\ldots$;*

(iii) *if $p_1 < p < 1/5$, then there exists $t_0 = t_0(p) \in (0, \pi/2)$ such that $F_p(t)$ is strictly increasing on $(0, t_0]$ and strictly decreasing on $[t_0, \pi/2)$.*

Lemma 5. *Let $p \in \mathbb{R}$ and F_p be defined on $(0, \pi/2)$ by (31) and (32). Then the following statements are true:*

(i) *if $F_p(t) < 0$ for all $t \in (0, \pi/2)$, then $p \geq 1/5$;*

(ii) *if $F_p(t) > 0$ for all $t \in (0, \pi/2)$, then $p \leq p_0$, where $p_0 = 0.1941\ldots$ is the unique solution of the equation*

$$p \log \frac{2}{\pi} - \log\left(1 + 2^{1-p}\right) + \log 3 = 0, \tag{34}$$

on the interval $(0.1, \infty)$.

Proof. (i) If $F_p(t) < 0$ for all $t \in (0, \pi/2)$, then from (31) and (32) we have

$$\begin{aligned} 0 \geq \lim_{t\to 0^+} \frac{F_p(t)}{t^4} &= \lim_{t\to 0^+} \frac{(1/720)(1-5p)t^4 + o\left(t^4\right)}{t^4} \\ &= \frac{1-5p}{720}. \end{aligned} \tag{35}$$

(ii) We first prove that $p_0 = 0.1941\ldots$ is the unique solution of (34) on the interval $(0.1, \infty)$. Let $p \in (0.1, \infty)$ and

$$H(p) = p \log \frac{2}{\pi} - \log\left(1 + 2^{1-p}\right) + \log 3. \tag{36}$$

Then numerical computations show that

$$\begin{aligned} H(0.1941) &= 8.13\ldots \times 10^{-7} > 0, \\ H(0.1942) &= -2.52\ldots \times 10^{-7} < 0, \end{aligned} \tag{37}$$

$$\begin{aligned} H'(p) &= \log \frac{2}{\pi} + \frac{\log 4}{2 + 2^p} < \log \frac{2}{\pi} + \frac{\log 4}{2 + 2^{0.1}} \\ &= -2.81\ldots \times 10^{-4} < 0. \end{aligned} \tag{38}$$

Inequality (38) implies that $H(p)$ is strictly decreasing on $[0.1, \infty)$. Therefore, $p_0 = 0.1941\ldots$ is the unique solution of (34) on the interval $(0.1, \infty)$ which follows from (37) and the monotonicity of $H(p)$.

If $p > 0.1$ and $F_p(t) > 0$ for all $t \in (0, \pi/2)$, then (31) leads to

$$0 \leq F_p\left(\frac{\pi^+}{2}\right) = \frac{1}{p} H(p). \tag{39}$$

Therefore, $p \leq p_0$ follows from (39) and $H(p_0) = 0$ together with the monotonicity of $H(p)$ on the interval $(0.1, \infty)$. □

Lemma 6. *Let $p \in \mathbb{R}$ and $x, c, \omega \in (0, 1)$, and let $M_p(x, \omega)$ be defined by (9). Then the function $p \mapsto M_p(x, \omega)/M_p(c, \omega)$ is strictly decreasing with respect to $p \in \mathbb{R}$ if $x \in (c, 1)$.*

Proof. Let $H(p, x) = \log M_p(x, \omega) - \log M_p(c, \omega)$. Then from (9) we get

$$\frac{\partial H(p, x)}{\partial x} = \frac{\omega x^{p-1}}{\omega x^p + 1 - \omega}, \tag{40}$$

$$\frac{\partial^2 H(p, x)}{\partial p \partial x} = \frac{\omega(1-\omega) x^{p-1}}{(\omega x^p + 1 - \omega)^2} \log x < 0. \tag{41}$$

Inequality (41) and $\partial^2 H(p, x)/\partial x \partial p = \partial^2 H(p, x)/\partial p \partial x$ lead to the conclusion that $\partial H(p, x)/\partial p$ is strictly decreasing with respect to $x \in (c, 1)$. Therefore, $\partial H(p, x)/\partial p < \partial H(p, x)/\partial p|_{x=c} = 0$ for $x \in (c, 1)$, and $M_p(x, \omega)/M_p(c, \omega)$ is strictly decreasing with respect to $p \in \mathbb{R}$ if $x \in (c, 1)$. □

3. Main Results

Theorem 7. *Let $M_p(x, \omega)$ be defined by (9). Then the double inequality*

$$\lambda_p M_p\left(\cos^2 \frac{t}{2}, \frac{2}{3}\right) < \frac{\sin t}{t} < M_p\left(\cos^2 \frac{t}{2}, \frac{2}{3}\right) \tag{42}$$

holds for all $t \in (0, \pi/2)$ if and only if $p \geq 1/5$, and the double inequality

$$M_p\left(\cos^2 \frac{t}{2}, \frac{2}{3}\right) < \frac{\sin t}{t} < \lambda_p M_p\left(\cos^2 \frac{t}{2}, \frac{2}{3}\right) \tag{43}$$

holds for all $t \in (0, \pi/2)$ if and only if $p \leq p_1$, where

$$\lambda_p = \frac{2}{\pi}\left(\frac{1 + 2^{1-p}}{3}\right)^{-1/p} \quad (p \neq 0), \qquad \lambda_0 = \frac{2^{5/3}}{\pi}, \tag{44}$$

$p_1 = \log(\pi - 2)/\log 2 = 0.1910\ldots$, and $\lambda_p M_p(\cos^2(t/2), 2/3)$ is strictly decreasing with respect to $p \in \mathbb{R}$.

Proof. Let $p \in \mathbb{R}$ and $F_p(t)$ be defined on $(0, \pi/2)$ by (31) and (32). Then

$$F_p(0^+) = 0, \qquad F_p\left(\frac{\pi^-}{2}\right) = \log \lambda_p. \tag{45}$$

If $p \geq 1/5$, then inequality (42) follows from Lemma 4 (i) and (45).

If inequality (42) holds for all $t \in (0, \pi/2)$, then $F_p(t) < 0$ for all $t \in (0, \pi/2)$. It follows from Lemma 5 (i) that $p \geq 1/5$.

If $p \le p_1$, then inequality (43) follows from Lemma 4 (ii) and (45).

If inequality (43) holds for all $t \in (0, \pi/2)$, then $F_p(\pi/2^-) > F_p(t) > F_p(0^+) = 0$ for all $t \in (0, \pi/2)$. It follows from Lemma 5 (ii) that $p \le p_0$, where $p_0 = 0.1941\ldots$ is the unique solution of (34) on the interval $(0.1, \infty)$. We claim that $p \le p_1$; otherwise $p_1 < p \le p_0 < 1/5$, and Lemma 4 (iii) leads to the conclusion that there exists $t_0 \in (0, \pi/2)$ such that $F_p(t) > F_p(\pi/2^-)$ for $t \in [t_0, \pi/2)$.

Note that

$$\lambda_p M_p\left(\cos^2\frac{t}{2}, \frac{2}{3}\right) = \frac{2}{\pi}\frac{M_p\left(\cos^2(t/2), 2/3\right)}{M_p(1/2, 2/3)}. \tag{46}$$

It follows from Lemma 6 and (46) that $\lambda_p M_p(\cos^2(t/2), 2/3)$ is strictly decreasing with respect to $p \in \mathbb{R}$. □

From Theorem 7 we get Corollaries 8 and 9 as follows.

Corollary 8. *For all $t \in (0, \pi/2)$ one has*

$$\begin{aligned}
\frac{2}{\pi} &< \frac{2+\cos t}{\pi} = \lambda_1 M_1\left(\cos^2\frac{t}{2}, \frac{2}{3}\right) \\
&< \lambda_{1/2}\left(\frac{2}{3}\cos\frac{t}{2} + \frac{1}{3}\right)^2 \\
&< \lambda_{1/4}\left(\frac{2}{3}\cos^{1/2}\frac{t}{2} + \frac{1}{3}\right)^4 \\
&< \lambda_{1/5}\left(\frac{2}{3}\cos^{2/5}\frac{t}{2} + \frac{1}{3}\right)^5 < \frac{\sin t}{t} \\
&< \left(\frac{2}{3}\cos^{2/5}\frac{t}{2} + \frac{1}{3}\right)^5 < \left(\frac{2}{3}\cos^{1/2}\frac{t}{2} + \frac{1}{3}\right)^4 \\
&< \left(\frac{2}{3}\cos\frac{t}{2} + \frac{1}{3}\right)^2 \\
&< M_1\left(\cos^2\frac{t}{2}, \frac{2}{3}\right) = \frac{2+\cos t}{3} < 1.
\end{aligned} \tag{47}$$

Corollary 9. *For all $t \in (0, \pi/2)$ one has*

$$\begin{aligned}
\cos^2\frac{t}{2} &= M_{-\infty}\left(\cos^2\frac{t}{2}, \frac{2}{3}\right) < \frac{3(1+\cos t)}{5+\cos t} \\
&= M_{-1}\left(\cos^2\frac{t}{2}, \frac{2}{3}\right) \\
&< \frac{9\cos^2(t/2)}{(2+\cos(t/2))^2} = M_{-1/2}\left(\cos^2\frac{t}{2}, \frac{2}{3}\right) \\
&< \cos^{4/3}\frac{t}{2} = M_0\left(\cos^2\frac{t}{2}, \frac{2}{3}\right) \\
&< \left(\frac{2}{3}\cos^{1/4}\frac{t}{2} + \frac{1}{3}\right)^8 < \left(\frac{2}{3}\cos^{1/3}\frac{t}{2} + \frac{1}{3}\right)^6 \\
&< \frac{\sin t}{t} < \lambda_{1/6}\left(\frac{2}{3}\cos^{1/3}\frac{t}{2} + \frac{1}{3}\right)^6 \\
&< \lambda_{1/8}\left(\frac{2}{3}\cos^{1/4}\frac{t}{2} + \frac{1}{3}\right)^8 < \lambda_0\cos^{4/3}\frac{t}{2} \\
&< \lambda_{-1/2}\frac{9\cos^2(t/2)}{(2+\cos(t/2))^2} < \lambda_{-1}\frac{3(1+\cos t)}{5+\cos t} \\
&< \lambda_{-\infty}\cos^2\frac{t}{2} = \frac{4}{\pi}\cos^2\frac{t}{2}.
\end{aligned} \tag{48}$$

Theorem 10. *Let $M_p(x, \omega)$ be defined by (9). Then the double inequality*

$$M_p\left(\cos^2\frac{t}{2}, \frac{2}{3}\right) < \frac{\sin t}{t} < M_q\left(\cos^2\frac{t}{2}, \frac{2}{3}\right) \tag{49}$$

holds for all $t \in (0, \pi/2)$ if and only if $p \le p_0$ and $q \ge 1/5$, where $p_0 = 0.1941\ldots$ is the unique solution of (34) on the interval $(0.1, \infty)$. Moreover, the inequality

$$\frac{\sin t}{t} \le \alpha M_{p_0}\left(\cos^2\frac{t}{2}, \frac{2}{3}\right), \tag{50}$$

if and only if

$$\alpha \ge \frac{\sin t_0}{t_0 M_{p_0}\left(\cos^2(t_0/2), 2/3\right)} = 1.00004919\ldots, \tag{51}$$

where $t_0 \in (0, \pi/2)$ is defined as in Lemma 3 (iii).

Proof. Let $p \in \mathbb{R}$ and $F_p(t)$ be defined on $(0, \pi/2)$ by (31) and (32). Then Lemma 4 (iii) leads to the conclusion that $F_{p_0}(t)$ is strictly increasing on $(0, t_0]$ and strictly decreasing on $[t_0, \pi/2)$. Note that

$$F_{p_0}\left(0^+\right) = F_{p_0}\left(\frac{\pi}{2}^-\right) = 0. \tag{52}$$

It follows from the piecewise monotonicity of $F_{p_0}(t)$ and (52) that

$$0 < F_{p_0}(t) \le F_{p_0}(t_0), \tag{53}$$

for all $t \in (0, \pi/2)$. Therefore, $\sin t/t > M_{p_0}(\cos^2(t/2), 2/3)$ for all $t \in (0, \pi/2)$ follows from the first inequality of (53), while $\sin t/t < M_{1/5}(\cos^2(t/2), 2/3)$ for all $t \in (0, \pi/2)$ follows from the second inequality of (42).

Conversely, if the double inequality (49) holds for all $t \in (0, \pi/2)$, then we clearly see that the inequalities

$$F_p(t) > 0, \qquad F_q(t) < 0 \tag{54}$$

hold for all $t \in (0, \pi/2)$. Therefore, $p \le p_0$ and $q \ge 1/5$ follows from Lemma 5 and (54). Moreover, numerical computations show that $t_0 = 1.312\ldots$ and

$$e^{F_{p_0}(t_0)} = 1.00004919\ldots. \tag{55}$$

Therefore, the second conclusion of Theorem 10 follows from (55) and the second inequality of (53). □

It follows from Lemma 3 that we get Theorem 11 immediately.

Theorem 11. *The double inequalities*

$$\frac{2\cos^{2p}(t/2)+\cos t}{2\cos^{2p}(t/2)+1}<\frac{\sin t}{t}<\frac{2\cos^{2q}(t/2)+\cos t}{2\cos^{2q}(t/2)+1},$$
$$2\cos^{2p}\frac{t}{2}<\frac{\sin t-t\cos t}{t-\sin t}<2\cos^{2q}\frac{t}{2} \tag{56}$$

hold for all $t\in(0,\pi/2)$ *if and only if* $p\geq 1/5$ *and* $q\leq p_1=\log(\pi-2)/\log 2=0.1910\ldots$.

We clearly see that the function $(2\cos^{2p}(t/2)+\cos t)/(2\cos^{2p}(t/2)+1)$ is strictly decreasing with respect to $p\in\mathbb{R}$ for fixed $x\in(0,\pi/2)$. Let $p=1/2,1,2,\infty$ and $q=1/6,\ 0,\ -1/2,\ -1,\ -2,\ -\infty$; then Theorem 11 leads to the following.

Corollary 12. *The inequalities*

$$\begin{aligned}\cos t&<\frac{8\cos t+\cos 2t+3}{4\cos t+\cos 2t+7}<\frac{2\cos t+1}{\cos t+2}\\&<\frac{2\cos(t/2)+\cos t}{2\cos(t/2)+1}<\frac{\sin t}{t}\\&<\frac{\cos t+2\cos^{1/3}(t/2)}{2\cos^{1/3}(t/2)+1}<\frac{\cos t+2}{3}\\&<\frac{\cos(t/2)+\cos(3t/2)+4}{2\cos(t/2)+4}<\frac{\cos t\cos^2(t/2)+2}{\cos^2(t/2)+2}\\&<\frac{\cos t\cos^4(t/2)+2}{\cos^4(t/2)+2}<1\end{aligned} \tag{57}$$

hold for all $t\in(0,\pi/2)$.

4. Applications

In this section, we give some applications for our main results.

Neuman [24] proved that the Huygens type inequalities

$$\begin{aligned}2\frac{\sin t}{t}+\frac{\tan t}{t}&>\frac{\sin t}{t}+2\frac{\tan(t/2)}{t/2}\\&>2\frac{t}{\sin t}+\frac{t}{\tan t}>3,\\\left(\frac{\sin t}{t}\right)^p+2\left(\frac{\tan(t/2)}{t/2}\right)^p&\\>\left(\frac{t}{\sin t}\right)^p+2\left(\frac{t/2}{\tan(t/2)}\right)^p&\quad(p>0),\\\left(\frac{t}{\sin t}\right)^p+2\left(\frac{t/2}{\tan(t/2)}\right)^p>3&\quad(p\geq 1)\end{aligned} \tag{58}$$

hold for all $t\in(0,\pi/2)$. Note that

$$\begin{aligned}\frac{\sin t}{t}<(>)M_p\left(\cos^2\frac{t}{2},\frac{2}{3}\right)&\Longleftrightarrow\left(\frac{t}{\sin t}\right)^p\\&+2\left(\frac{t/2}{\tan(t/2)}\right)>(<)3,\\\frac{\sin t}{t}>(<)\lambda_pM_p\left(\cos^2\frac{t}{2},\frac{2}{3}\right)&\\\Longleftrightarrow\left(\frac{t}{\sin t}\right)^p+2\left(\frac{t/2}{\tan(t/2)}\right)^p&\\<(>)\left(\frac{\pi}{2}\right)^p+2\left(\frac{\pi}{4}\right)^p,&\end{aligned} \tag{59}$$

if $p>0$, and the second inequalities in (59) are reversed if $p<0$.

From Theorems 7 and 10 together with (59) we get the following.

Theorem 13. *The double inequality*

$$\left(\frac{\pi}{2}\right)^p+2\left(\frac{\pi}{4}\right)^p>\left(\frac{t}{\sin t}\right)^p+2\left(\frac{t/2}{\tan(t/2)}\right)^p>3 \tag{60}$$

holds for all $t\in(0,\pi/2)$ *if and only if* $p\geq 1/5$ *or* $p<0$, *and inequality* (60) *is reversed if and only if* $0<p\leq p_1=\log(\pi-2)/\log 2=0.1910\ldots$.

Theorem 14. *The double inequality*

$$\left(\frac{t}{\sin t}\right)^p+2\left(\frac{t/2}{\tan(t/2)}\right)^p>3>\left(\frac{t}{\sin t}\right)^q+2\left(\frac{t/2}{\tan(t/2)}\right)^q \tag{61}$$

holds for all $t\in(0,\pi/2)$ *if and only if* $0<q\leq p_0$ *and* $p\geq 1/5$ *or* $p<0$, *where* $p_0=0.1941\ldots$ *is the unique solution of* (34) *on the interval* $(0.1,\infty)$.

Neuman [24] also proved that the Wilker type inequality

$$\left(\frac{t}{\sin t}\right)^p+\left(\frac{t/2}{\tan(t/2)}\right)^{2p}>2 \tag{62}$$

holds for all $t\in(0,\pi/2)$ if $p\geq 1$.

Making use of Theorem 13 and the arithmetic-geometric means inequality

$$1+\left(\frac{t/2}{\tan(t/2)}\right)^{2p}>2\left(\frac{t/2}{\tan(t/2)}\right)^p, \tag{63}$$

we get Corollary 15 as follows.

Corollary 15. *The Wilker type inequality* (62) *holds for all* $t\in(0,\pi/2)$ *if* $p\geq 1/5$ *or* $p<0$.

In addition, power series expansions show that

$$\left(\frac{t}{\sin t}\right)^p+\left(\frac{t/2}{\tan(t/2)}\right)^{2p}-2=\frac{p(20p-3)}{720}t^4+o\left(t^4\right). \tag{64}$$

Therefore, we conjecture that inequality (62) *holds for all* $t \in (0, \pi/2)$ *if and only if* $p \geq 3/20$ *or* $p < 0$. *We leave it to the readers for further discussion.*

The Schwab-Borchardt mean $SB(a,b)$ *[25–27] of two distinct positive real numbers* a *and* b *is defined by*

$$SB(a,b) = \begin{cases} \dfrac{\sqrt{b^2-a^2}}{\cos^{-1}(a/b)}, & a < b, \\ \dfrac{\sqrt{a^2-b^2}}{\cosh^{-1}(a/b)}, & a > b, \end{cases} \tag{65}$$

where $\cos^{-1}(x)$ *and* $\cosh^{-1}(x) = \log(x + \sqrt{x^2-1})$ *are the inverse cosine and inverse hyperbolic cosine functions, respectively.*

Let $b > a > 0$, $A(a,b) = (a+b)/2$ *be the arithmetic mean of* a *and* b, *and* $t = \cos^{-1}(a/b) \in (0, \pi/2)$. *Then simple computations lead to*

$$\frac{\sin t}{t} = \frac{SB(a,b)}{b},$$

$$M_p\left(\cos^2\frac{t}{2}, \frac{2}{3}\right) = \frac{1}{b}\left(\frac{2}{3}A^p(a,b) + \frac{b^p}{3}\right)^{1/p}, \tag{66}$$

$$\frac{2\cos^{2p}(t/2) + \cos t}{2\cos^{2p}(t/2) + 1} = \frac{2A^p(a,b) + ab^{p-1}}{2A^p(a,b) + b^p}.$$

It follows from Theorems 7, 10, and 11 together with (66) *that we have the following.*

Theorem 16. *Let* $p_1 = \log(\pi-2)/\log 2 = 0.1910\ldots$, λ_p *and* $p_0 = 0.1941\ldots$ *be defined as in Theorems 7 and 10, respectively. Then for all* $b > a > 0$, *the following statements are true.*

(i) *The double inequality*

$$\lambda_p\left(\frac{2}{3}A^p(a,b) + \frac{1}{3}b^p\right)^{1/p} < SB(a,b) < \left(\frac{2}{3}A^p(a,b) + \frac{1}{3}b^p\right)^{1/p} \tag{67}$$

holds if and only if $p \geq 1/5$, *and inequality* (67) *is reversed if and only if* $p \leq p_1$.

(ii) *The double inequality*

$$\left(\frac{2}{3}A^p(a,b) + \frac{1}{3}b^p\right)^{1/p} < SB(a,b) < \left(\frac{2}{3}A^q(a,b) + \frac{1}{3}b^q\right)^{1/q} \tag{68}$$

holds if and only if $p \leq p_0$ *and* $q \geq 1/5$.

(iii) *The double inequality*

$$\frac{2A^p(a,b) + ab^{p-1}}{2A^p(a,b) + b^p}b < SB(a,b) < \frac{2A^q(a,b) + ab^{q-1}}{2A^q(a,b) + b^q}b \tag{69}$$

holds if and only if $p \geq 1/5$ *and* $q \leq p_1$.

Let $b > a > 0$, $G(a,b) = \sqrt{ab}$, $Q(a,b) = \sqrt{(a^2+b^2)/2}$, $P(a,b) = (b-a)/[2\sin^{-1}((b-a)/(b+a))]$, $T(a,b) = (b-a)/[2\tan^{-1}((b-a)/(b+a))]$, and $Y(a,b) = (b-a)/[\sqrt{2}\tan^{-1}((b-a)/\sqrt{2ab})]$ be the geometric, quadratic, first Seiffert [28], second Seiffert [29], and Yang [15] means of a and b, respectively. Then it is easy to check that $P(a,b) = \mathrm{SB}(G(a,b), A(a,b))$, $T(a,b) = \mathrm{SB}(A(a,b), Q(a,b))$, and $Y(a,b) = \mathrm{SB}(G(a,b), Q(a,b))$. Therefore, Theorem 16 leads to Corollary 17.

Corollary 17. *Let* $p_1 = \log(\pi-2)/\log 2 = 0.1910\ldots$, λ_p *and* $p_0 = 0.1941\ldots$ *be defined as in Theorems 7 and 10, respectively. Then for all* $b > a > 0$, *the following statements are true.*

(i) *The double inequalities*

$$\begin{aligned} &\lambda_p\left[\frac{2}{3}\left(\frac{G(a,b)+A(a,b)}{2}\right)^p + \frac{1}{3}A^p(a,b)\right]^{1/p} \\ &\quad < P(a,b) \\ &\quad < \left[\frac{2}{3}\left(\frac{G(a,b)+A(a,b)}{2}\right)^p + \frac{1}{3}A^p(a,b)\right]^{1/p} \\ &\lambda_p\left[\frac{2}{3}\left(\frac{A(a,b)+Q(a,b)}{2}\right)^p + \frac{1}{3}Q^p(a,b)\right]^{1/p} \\ &\quad < T(a,b) \\ &\quad < \left[\frac{2}{3}\left(\frac{A(a,b)+Q(a,b)}{2}\right)^p + \frac{1}{3}Q^p(a,b)\right]^{1/p}, \\ &\lambda_p\left[\frac{2}{3}\left(\frac{G(a,b)+Q(a,b)}{2}\right)^p + \frac{1}{3}Q^p(a,b)\right]^{1/p} \\ &\quad < Y(a,b) \\ &\quad < \left[\frac{2}{3}\left(\frac{G(a,b)+Q(a,b)}{2}\right)^p + \frac{1}{3}Q^p(a,b)\right]^{1/p}, \end{aligned} \tag{70}$$

hold if and only if $p \geq 1/5$, *and all inequalities in* (70) *are reversed if and only if* $p \leq p_1$.

(ii) *The double inequalities*

$$\begin{aligned} &\left[\frac{2}{3}\left(\frac{G(a,b)+A(a,b)}{2}\right)^p + \frac{1}{3}A^p(a,b)\right]^{1/p} \\ &\quad < P(a,b) \\ &\quad < \left[\frac{2}{3}\left(\frac{G(a,b)+A(a,b)}{2}\right)^q + \frac{1}{3}A^q(a,b)\right]^{1/q}, \\ &\left[\frac{2}{3}\left(\frac{A(a,b)+Q(a,b)}{2}\right)^p + \frac{1}{3}Q^p(a,b)\right]^{1/p} \\ &\quad < T(a,b) \end{aligned}$$

$$< \left[\frac{2}{3}\left(\frac{A(a,b)+Q(a,b)}{2}\right)^q + \frac{1}{3}Q^q(a,b)\right]^{1/q},$$

$$\left[\frac{2}{3}\left(\frac{G(a,b)+Q(a,b)}{2}\right)^p + \frac{1}{3}Q^p(a,b)\right]^{1/p} < Y(a,b) < \left[\frac{2}{3}\left(\frac{G(a,b)+Q(a,b)}{2}\right)^q + \frac{1}{3}Q^q(a,b)\right]^{1/q} \quad (71)$$

hold if and only if $p \le p_0$ *and* $q \ge 1/5$.

(iii) *The double inequalities*

$$\frac{2^{1-p}(G(a,b)+A(a,b))^p A(a,b) + G(a,b)A^p(a,b)}{2^{1-p}(G(a,b)+A(a,b))^p + A^p(a,b)} < P(a,b) < \frac{2^{1-q}(G(a,b)+A(a,b))^q A(a,b) + G(a,b)A^q(a,b)}{2^{1-q}(G(a,b)+A(a,b))^q + A^q(a,b)},$$

$$\frac{2^{1-p}(A(a,b)+Q(a,b))^p Q(a,b) + A(a,b)Q^p(a,b)}{2^{1-p}(A(a,b)+Q(a,b))^p + Q^p(a,b)} < T(a,b) < \frac{2^{1-q}(A(a,b)+Q(a,b))^q Q(a,b) + A(a,b)Q^q(a,b)}{2^{1-q}(A(a,b)+Q(a,b))^q + Q^q(a,b)},$$

$$\frac{2^{1-p}(G(a,b)+Q(a,b))^p Q(a,b) + G(a,b)Q^p(a,b)}{2^{1-p}(G(a,b)+Q(a,b))^p + Q^p(a,b)} < Y(a,b) < \frac{2^{1-q}(G(a,b)+Q(a,b))^q Q(a,b) + G(a,b)Q^q(a,b)}{2^{1-q}(G(a,b)+Q(a,b))^q + Q^q(a,b)} \quad (72)$$

hold if and only if $p \ge 1/5$ *and* $q \le p_1$.

For $x \in (0,1)$, *the following Shafer-Fink type inequality can be found in the literature [1, 30]:*

$$\sin^{-1}x > \frac{6\left(\sqrt{1+x}-\sqrt{1-x}\right)}{4+\sqrt{1+x}+\sqrt{1-x}} > \frac{3x}{2+\sqrt{1-x^2}}. \quad (73)$$

Fink [31] proved that the double inequality

$$\frac{3x}{2+\sqrt{1-x^2}} \le \sin^{-1}x \le \frac{\pi x}{2+\sqrt{1-x^2}} \quad (74)$$

holds for all $x \in [0,1]$. *It was generalized and improved by Zhu [32].*

Let $t \in (0,\pi/2)$, $x = \sin t \in (0,1)$. Then Theorems 7, 10, and 11 lead to Corollary 18 as follows.

Corollary 18. *Let* $p_1 = \log(\pi-2)/\log 2 = 0.1910\ldots$, λ_p *and* $p_0 = 0.1941\ldots$ *be defined as in Theorems 7 and 10, respectively. Then for all* $x \in (0,1)$, *the following statements are true.*

(i) *The double inequality*

$$\frac{x}{M_p\left(\left(1+\sqrt{1-x^2}\right)/2, 2/3\right)} < \sin^{-1}(x) < \frac{x}{\lambda_p M_p\left(\left(1+\sqrt{1-x^2}\right)/2, 2/3\right)} \quad (75)$$

holds if and only if $p \ge 1/5$, *and inequality* (75) *is reversed if and only if* $p \le p_1$.

(ii) *The double inequality*

$$\frac{x}{M_p\left(\left(1+\sqrt{1-x^2}\right)/2, 2/3\right)} < \sin^{-1}(x) < \frac{x}{M_q\left(\left(1+\sqrt{1-x^2}\right)/2, 2/3\right)} \quad (76)$$

holds if and only if $p \ge 1/5$ *and* $q \le p_0$.

(iii) *The double inequality*

$$\frac{2^{1-2p}\left(\sqrt{1+x}+\sqrt{1-x}\right)^{2p}+1}{2^{1-2p}\left(\sqrt{1+x}+\sqrt{1-x}\right)^{2p}+\sqrt{1-x^2}} < \sin^{-1}(x) < \frac{2^{1-2q}\left(\sqrt{1+x}+\sqrt{1-x}\right)^{2q}+1}{2^{1-2q}\left(\sqrt{1+x}+\sqrt{1-x}\right)^{2q}+\sqrt{1-x^2}} \quad (77)$$

holds if and only if $p \le p_1$ *and* $q \ge 1/5$.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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References

[1] D. S. Mitrinović, *Analytic Inequalities*, Springer, New York, NY, USA, 1970.

[2] C. Huygens, *Oeuvres Completes 1888–1940*, Sociéte Hollondaise des Science, Haga, Sweden, 1940.

[3] C. Chen and W. Cheung, "Sharp Cusa and Becker-Stark inequalities," *Journal of Inequalities and Applications*, vol. 2011, article 136, 2011.

[4] C. Mortici, "The natural approach of Wilker-Cusa-Huygens inequalities," *Mathematical Inequalities & Applications*, vol. 14, no. 3, pp. 535–541, 2011.

[5] C. Mortici, "A subtly analysis of Wilker inequality," *Applied Mathematics and Computation*, vol. 231, pp. 516–520, 2014.

[6] E. Neuman and J. Sándor, "On some inequalities involving trigonometric and hyperbolic functions with emphasis on the Cusa-Huygens, Wilker, and Huygens inequalities," *Mathematical Inequalities & Applications*, vol. 13, no. 4, pp. 715–723, 2010.

[7] J. Sándor and M. Bencze, "On Huygen's trigonometric inequality," *RGMIA Research Report Collection*, vol. 8, no. 3, article 14, 2005.

[8] L. Zhu, "A source of inequalities for circular functions," *Computers & Mathematics with Applications*, vol. 58, no. 10, pp. 1998–2004, 2009.

[9] F. Qi, L. H. Cui, and S. L. Xu, "Some inequalities constructed by Tchebysheff's integral inequality," *Mathematical Inequalities & Applications*, vol. 2, no. 4, pp. 517–528, 1999.

[10] Y. Lv, G. Wang, and Y. Chu, "A note on Jordan type inequalities for hyperbolic functions," *Applied Mathematics Letters*, vol. 25, no. 3, pp. 505–508, 2012.

[11] E. Neuman, "Refinements and generalizations of certain inequalities involving trigonometric and hyperbolic functions," *Advances in Inequalities and Applications*, vol. 1, no. 1, pp. 1–11, 2012.

[12] E. Neuman, "Inequalities for the Schwab-Borchardt mean and their applications," *Journal of Mathematical Inequalities*, vol. 5, no. 4, pp. 601–609, 2011.

[13] R. Klén, M. Visuri, and M. Vuorinen, "On Jordan type inequalities for hyperbolic functions," *Journal of Inequalities and Applications*, vol. 2010, Article ID 362548, 14 pages, 2010.

[14] Z.-H. Yang, "New sharp Jordan type inequalities and their applications," *Gulf Journal of Mathematics*, vol. 2, no. 1, pp. 1–10, 2014.

[15] Z.-H. Yang, "Three families of two-parameter means constructed by trigonometric functions," *Journal of Inequalities and Applications*, vol. 2013, article 541, 27 pages, 2013.

[16] Z. Yang, "Refinements of a two-sided inequality for trigonometric functions," *Journal of Mathematical Inequalities*, vol. 7, no. 4, pp. 601–615, 2013.

[17] Zh.-H. Yang, "Sharp bounds for Seiffert mean in terms of weighted power means of arithmetic mean and geometric mean," *Mathematical Inequalities & Applications*, vol. 17, no. 2, pp. 499–511, 2014.

[18] K. S. K. Iyengar, B. S. Madhava Rao, and T. S. Nanjundiah, "Some trigonometrical inequalities," *The Half-Yearly Journal of the Mysore University B*, vol. 6, pp. 1–12, 1945.

[19] S.-H. Wu and L. Debnath, "A new generalized and sharp version of Jordan's inequality and its applications to the improvement of the Yang Le inequality," *Applied Mathematics Letters*, vol. 19, no. 12, pp. 1378–1384, 2006.

[20] S. Wu, "Sharpness and generalization of Jordan's inequality and its application," *Taiwanese Journal of Mathematics*, vol. 12, no. 2, pp. 325–336, 2008.

[21] S.-H. Wu and Á. Baricz, "Generalizations of Mitrinović, Adamović and Lazarević 's inequalities and their applications," *Publicationes Mathematicae Debrecen*, vol. 75, no. 3-4, pp. 447–458, 2009.

[22] F. Qi, D. Niu, and B. Guo, "Refinements, generalizations, and applications of Jordan's inequality and related problems," *Journal of Inequalities and Applications*, vol. 2009, Article ID 271923, 52 pages, 2009.

[23] P. S. Bullen, D. S. Mitrinović, and P. M. Vasić, *Means and Their Inequalties*, D. Reidel Publishing, Dordrecht, The Netherlands, 1988.

[24] E. Neuman, "On Wilker and Huygens type inequalities," *Mathematical Inequalities & Applications*, vol. 15, no. 2, pp. 271–279, 2012.

[25] E. Neuman and J. Sándor, "On the Schwab-Borchardt mean," *Mathematica Pannonica*, vol. 14, no. 2, pp. 253–266, 2003.

[26] E. Neuman and J. Sándor, "On the schwab-borchardt mean II," *Mathematica Pannonica*, vol. 17, no. 1, pp. 49–59, 2006.

[27] E. Neuman, "On some means derived from the Schwab-Borchardt mean," *Journal of Mathematical Inequalities*, vol. 8, no. 1, pp. 171–181, 2014.

[28] H. Seiffert, "Werte zwischen dem geometrischen und dem arithmetischen Mittel zweier Zahlen," *Elemente der Mathematik Revue de Math ematiques El ementaires Rivista de Matematica Elementare*, vol. 42, no. 4, pp. 105–107, 1987.

[29] H.-J. Seiffert, "Aufgabe β 16," *Die Wurzel*, vol. 29, pp. 221–222, 1995.

[30] R. E. Shafer, L. S. Grinstein, and D. C. B. Marsh, "Problems and solutions: solutions of elementary problems: E1867," *The American Mathematical Monthly*, vol. 74, no. 6, pp. 726–727, 1967.

[31] A. M. Fink, "Two inequalities," *Univerzitet u Beogradu. Publikacije Elektrotehni v ckog Fakulteta: Serija Matematika*, vol. 6, pp. 48–49, 1995.

[32] L. Zhu, "On Shafer-Fink inequalities," *Mathematical Inequalities and Applications*, vol. 8, no. 4, pp. 571–574, 2005.