

Research Article

Double Periodic Wave Solutions of the $(2 + 1)$ -Dimensional Sawada-Kotera Equation

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Based on a general Riemann theta function and Hirota's bilinear forms, we devise a straightforward way to explicitly construct double periodic wave solution of $(2 + 1)$ -dimensional nonlinear partial differential equation. The resulting theory is applied to the $(2 + 1)$ -dimensional Sawada-Kotera equation, thereby yielding its double periodic wave solutions. The relations between the periodic wave solutions and soliton solutions are rigorously established by a limiting procedure.

1. Introduction

It is always important to investigate the exact solutions for nonlinear evolution equations, which play an important role in the study of nonlinear models of natural and social phenomena. Nonlinear wave phenomena appears in various scientific and engineering areas, such as fluid mechanics, theory of solitons, hydrodynamics, and theory of turbulence, optical fibers, chaos theory, biology, and chemical physics. In the last three decades, various powerful methods have been presented, such as extended tanh method [1], homogeneous balance method [2], Lie group method [3], Wronskian technique [4, 5], Darboux transformation method [6], Hirota's bilinear method [7–10], and algebro-geometrical approach [11].

The Hirota's bilinear method provides a powerful way to derive soliton solutions to nonlinear integrable equations and its basis is the Hirota bilinear formulation. Once the corresponding bilinear forms are obtained, multisoliton solutions and rational solutions to nonlinear differential equations can be computed in quite a systematic way. In 1980s, based on Hirota bilinear forms, Nakamura proposed a comprehensive method to construct a kind of multiperiodic solutions of nonlinear equations in his papers [12, 13], such a method of solution does not need any Lax pairs and their induced Riemann surfaces for the considered equations. The

advantage of the method is that it only relies on the existed Hirota bilinear forms. Moreover, all parameters appearing in Riemann matrices are completely arbitrary, whereas algebro-geometric solutions involve specific Riemann constants, which are usually difficult to compute.

In recent years, Hon et al. have extended this method to investigate the discrete Toda lattice [14], $(2 + 1)$ -dimensional modified Bogoyavlenskii-Schiff equation [15], and the Super-symmetric KdV-Sawada-Kotera-Ramani equation [16]. Ma et al. constructed one-periodic and two-periodic wave solutions to a class of $(2 + 1)$ -dimensional Hirota bilinear equations [17]. Tian and Zhang gave the exact periodic solutions for some evolution equations with the aid of the Hirota bilinear method and theta functions identities [18, 19].

Our aim in the present work is to improve the main steps of the existing methods of Fan and Chow in [15] into the case of three dimensions. We propose a theorem, which actually provides us a direct and unifying way for applying in a class of $(2 + 1)$ -dimensional nonlinear partial differential equations. Once such an equation is written in a bilinear form, its double periodic wave solutions can be directly obtained by using this theorem.

The organization of this paper is as follows. In Section 2, we briefly introduce the Hirota bilinear operator and the Riemann theta function. In particular, we present a theorem for

constructing periodic wave solutions for $(2 + 1)$ -dimensional nonlinear partial differential equations. As applications of our method, in Section 3, we construct double periodic wave solutions to the $(2 + 1)$ -dimensional Sawada-Kotera equation. In addition, it is rigorously shown that the double periodic wave solutions tend to the soliton solutions under small amplitude limits. Finally, some conclusions and discussions are presented in Section 4.

2. Hirota Bilinear Operator and Riemann Theta Function

In this section we briefly present the notation that will be used in this paper. Here the bilinear operators $D_{x_1}, D_{x_2}, \dots, D_{x_N}, D_t$ are defined by

$$\begin{aligned} & D_{x_1}^m D_{x_2}^n, \dots, D_{x_N}^p D_t^r f(X, t) \cdot g(X, t) \\ &= (\partial_{x_1} - \partial_{x'_1})^m (\partial_{x_2} - \partial_{x'_2})^n, \dots, (\partial_{x_N} - \partial_{x'_N})^p \\ &\quad \times (\partial_t - \partial_{t'})^r f(X, t) \cdot g(X', t')|_{X=X', t=t'} \end{aligned} \quad (1)$$

with $X' = (x'_1, x'_2, \dots, x'_N)$.

Proposition 1. The Hirota bilinear operators $D_{x_1}, D_{x_2}, \dots, D_{x_N}, D_t$ have properties

$$\begin{aligned} & D_{x_1}^m D_{x_2}^n, \dots, D_{x_N}^p D_t^r e^{\xi_1} \cdot e^{\xi_2} \\ &= (k_1 - k_2)^m (l_1 - l_2)^n, \dots, (\sigma_1 - \sigma_2)^p (\omega_1 - \omega_2)^r e^{\xi_1 + \xi_2}, \end{aligned} \quad (2)$$

where $\xi_i = k_i x_1 + l_i x_2 + \dots + \sigma_i x_N + \omega_i t + \varepsilon_i$, $i = 1, 2$, with $k_i, l_i, \dots, \sigma_i, \omega_i, \varepsilon_i$ being constants. More generally, one has

$$\begin{aligned} & F(D_{x_1}, D_{x_2}, \dots, D_{x_N}, D_t) e^{\xi_1} \cdot e^{\xi_2} \\ &= F(k_1 - k_2, l_1 - l_2, \dots, \sigma_1 - \sigma_2, \omega_1 - \omega_2) e^{\xi_1 + \xi_2}, \end{aligned} \quad (3)$$

where $F(D_{x_1}, D_{x_2}, \dots, D_{x_N}, D_t)$ is a polynomial about operators $D_{x_1}, D_{x_2}, \dots, D_{x_N}, D_t$. These properties are useful in deriving Hirota's bilinear form and constructing periodic wave solutions of nonlinear equations.

Then, one would like to consider a general Riemann theta function and discuss its periodicity; the Riemann theta function reads

$$\vartheta \left[\begin{smallmatrix} \varepsilon \\ s \end{smallmatrix} \right] (\xi, \tau) = \sum_{m \in \mathbb{Z}} \exp \{ 2\pi i (\xi + \varepsilon) (m + s) - \pi \tau (m + s)^2 \}, \quad (4)$$

where $m \in \mathbb{Z}$, complex parameter $s, \varepsilon \in \mathbb{C}$, and complex phase variables $\xi \in \mathbb{C}$, $\tau > 0$ which is called the period matrix of the Riemann theta function.

In the definition of the theta function (4), for the case $s = \varepsilon = 0$, hereafter, one uses the notation of $\vartheta(\xi, \tau) = \vartheta \left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix} \right] (\xi, \tau)$, for simplicity. Moreover, one has $\vartheta \left[\begin{smallmatrix} \varepsilon \\ 0 \end{smallmatrix} \right] (\xi, \tau) = \vartheta(\xi + \varepsilon, \tau)$.

Definition 2. A function $g(t)$ on \mathbb{C} is said to be quasiperiodic in t with fundamental periods $T_1, T_2, \dots, T_k \in \mathbb{C}$ if

T_1, T_2, \dots, T_k being linearly dependent over \mathbb{Z} and there exists a function $G(y_1, y_2, \dots, y_k)$ in \mathbb{C}^k , such that

$$\begin{aligned} & G(y_1, y_2, \dots, y_j + T_j, \dots, y_k) \\ &= G(y_1, y_2, \dots, y_j, \dots, y_k), \quad \forall y_j \in \mathbb{C}, \quad (5) \\ & G(t, t, \dots, t, \dots, t) = g(t). \end{aligned}$$

In particular, $g(t)$ is called double periodic as $k = 2$, and it becomes periodic with the period T if and only if $T_j = m_j T$.

Proposition 3. The theta function $\vartheta(\xi, \tau)$ has the periodic properties:

$$\vartheta(\xi + 1 + i\tau, \tau) = \exp(-2\pi i \xi + \pi \tau) \vartheta(\xi, \tau). \quad (6)$$

One regards the vectors 1 and $i\tau$ as periods of the theta function $\vartheta(\xi, \tau)$ with multipliers 1 and $\exp(-2\pi i \xi + \pi \tau)$, respectively.

Proposition 4. The meromorphic functions $f(\xi)$ on \mathbb{C} is as follows:

$$f(\xi) = \partial_\xi^2 \ln \vartheta(\xi, \tau), \quad \xi \in \mathbb{C}; \quad (7)$$

then it holds that

$$f(\xi + 1 + i\tau) = f(\xi), \quad \xi \in \mathbb{C}; \quad (8)$$

that is, $f(\xi)$ is a double periodic function with 1 and $i\tau$.

Proof. By using (6), we easily know that

$$\ln \vartheta(\xi + 1 + i\tau) = (-2\pi i \xi + \pi \tau) \ln \vartheta(\xi, \tau); \quad (9)$$

then differentiating it with respect to ξ , we have

$$\frac{\partial_\xi \vartheta(\xi + 1 + i\tau, \tau)}{\vartheta(\xi + 1 + i\tau, \tau)} = -2\pi i + \frac{\partial_\xi \vartheta(\xi, \tau)}{\vartheta(\xi, \tau)}, \quad (10)$$

which is equivalent to

$$\partial_\xi \ln \vartheta(\xi + 1 + i\tau, \tau) = -2\pi i + \partial_\xi \ln \vartheta(\xi, \tau). \quad (11)$$

Differentiating (11) with respect to ξ again immediately proves formula (8). \square

Theorem 5. Assuming that $\vartheta \left[\begin{smallmatrix} \varepsilon' \\ 0 \end{smallmatrix} \right] (\xi, \tau)$ and $\vartheta \left[\begin{smallmatrix} \varepsilon \\ 0 \end{smallmatrix} \right] (\xi, \tau)$ are two Riemann theta functions with $\xi = \alpha x + \beta y + \omega t + \sigma$, then Hirota bilinear operators D_x, D_y , and D_t exhibit the following perfect properties when they act on a pair of theta functions:

$$\begin{aligned} & D_x \vartheta \left[\begin{smallmatrix} \varepsilon' \\ 0 \end{smallmatrix} \right] (\xi, \tau) \cdot \vartheta \left[\begin{smallmatrix} \varepsilon \\ 0 \end{smallmatrix} \right] (\xi, \tau) \\ &= \sum_{\mu=0,1} \left[\partial_x \vartheta \left[\begin{smallmatrix} \varepsilon' - \varepsilon \\ -\frac{\mu}{2} \end{smallmatrix} \right] (2\xi, 2\tau) \Big|_{\xi=0} \right] \vartheta \left[\begin{smallmatrix} \varepsilon' + \varepsilon \\ \frac{\mu}{2} \end{smallmatrix} \right] (2\xi, 2\tau), \end{aligned} \quad (12)$$

where the notation $\sum_{\mu=0,1}$ represents two different transformations corresponding to $\mu = 0, 1$. The bilinear formulas for y, t are the same as (12) by replacing ∂_x with ∂_y and ∂_t .

In general, for a polynomial operator $F(D_x, D_y, D_t)$ with respect to D_x, D_y , and D_t , one has the following useful formula:

$$\begin{aligned} F(D_x, D_y, D_t) \vartheta \left[\begin{smallmatrix} \varepsilon' \\ 0 \end{smallmatrix} \right] (\xi, \tau) \cdot \vartheta \left[\begin{smallmatrix} \varepsilon \\ 0 \end{smallmatrix} \right] (\xi, \tau) \\ = \sum_{\mu=0,1} C(\varepsilon', \varepsilon, \mu) \vartheta \left[\begin{smallmatrix} \varepsilon' + \varepsilon \\ \frac{\mu}{2} \end{smallmatrix} \right] (2\xi, 2\tau), \end{aligned} \quad (13)$$

where

$$\begin{aligned} C(\varepsilon', \varepsilon, \mu) = \sum_{m \in \mathbb{Z}^N} F(M) \exp \left[-2\pi i \left(m - \frac{\mu}{2} \right)^2 \right. \\ \left. + 2\pi i \left(m - \frac{\mu}{2} \right) (\varepsilon' - \varepsilon) \right], \end{aligned} \quad (14)$$

and one denotes vector $M = (4\pi i(m - \mu/2)\alpha, 4\pi i(m - \mu/2)\beta, 4\pi i(m - \mu/2)\omega)$.

Proof. Making use of the formula (2), we obtain the relation

$$\begin{aligned} D_x \vartheta \left[\begin{smallmatrix} \varepsilon' \\ 0 \end{smallmatrix} \right] (\xi, \tau) \cdot \vartheta \left[\begin{smallmatrix} \varepsilon \\ 0 \end{smallmatrix} \right] (\xi, \tau) \\ = \sum_{m', m \in \mathbb{Z}} D_x \exp \left\{ 2\pi i m' (\xi + \varepsilon') - \pi m'^2 \tau \right\} \\ \times \exp \left\{ 2\pi i m (\xi + \varepsilon) - \pi m^2 \tau \right\} \\ = \sum_{m', m \in \mathbb{Z}} 2\pi i \alpha (m' - m) \\ \times \exp \left\{ 2\pi i (m' + m) \xi + 2\pi i (m' \varepsilon' + m \varepsilon) \right. \\ \left. - \pi \tau (m'^2 + m^2) \right\}. \end{aligned} \quad (15)$$

Shifting summation index as $m = l' - m'$, then

$$\begin{aligned} &= \sum_{l', m'} 2\pi i \alpha (2m' - l') \\ &\quad \cdot \exp \left\{ 2\pi i l' \xi + 2\pi i [m' \varepsilon' + (l' - m') \varepsilon] \right. \\ &\quad \left. - \pi \tau [m'^2 + (l' - m')^2] \right\} \\ &\stackrel{l'=2l+\mu}{=} \sum_{\mu=0,1} \sum_{l, m' \in \mathbb{Z}} 2\pi i \alpha (2m' - 2l - \mu) \\ &\quad \cdot \exp \left\{ 4\pi i \xi \left(l + \frac{\mu}{2} \right) + 2\pi i [m' \varepsilon' - (m' - 2l - \mu) \varepsilon] \right. \\ &\quad \left. - \pi \tau [m'^2 + (m' - 2l - \mu)^2] \right\} \\ &\stackrel{m'=k+l}{=} \sum_{\mu=0,1} \left[\sum_{k \in \mathbb{Z}} 4\pi i \alpha \left(k - \frac{\mu}{2} \right) \right. \\ &\quad \cdot \exp \left\{ 2\pi i \left(k - \frac{\mu}{2} \right) (\varepsilon' - \varepsilon) \right. \\ &\quad \left. \left. - 2\pi \tau \left(k - \frac{\mu}{2} \right)^2 \right\} \right] \end{aligned}$$

$$\begin{aligned} &\times \left[\sum_{l \in \mathbb{Z}} \exp \left\{ 2\pi i \left(l + \frac{\mu}{2} \right) (2\xi + \varepsilon' + \varepsilon) \right. \right. \\ &\quad \left. \left. - 2\pi \tau \left(l + \frac{\mu}{2} \right)^2 \right\} \right] \\ &= \sum_{\mu=0,1} \left[\partial_x \vartheta \left[\begin{smallmatrix} \varepsilon' - \varepsilon \\ -\frac{\mu}{2} \end{smallmatrix} \right] (2\xi, 2\tau) \Big|_{\xi=0} \right] \cdot \vartheta \left[\begin{smallmatrix} \varepsilon' + \varepsilon \\ \frac{\mu}{2} \end{smallmatrix} \right] (2\xi, 2\tau). \end{aligned} \quad (16)$$

Formula (13) follows from (12). Formulas (13) and (14) imply that if the following equations are satisfied

$$C(\varepsilon, \varepsilon', 0) = 0, \quad C(\varepsilon, \varepsilon', 1) = 0, \quad (17)$$

then $\vartheta \left[\begin{smallmatrix} \varepsilon' \\ 0 \end{smallmatrix} \right] (\xi, \tau)$ and $\vartheta \left[\begin{smallmatrix} \varepsilon \\ 0 \end{smallmatrix} \right] (\xi, \tau)$ are periodic wave solutions of the bilinear equation:

$$F(D_x, D_y, D_t) \vartheta \left[\begin{smallmatrix} \varepsilon' \\ 0 \end{smallmatrix} \right] (\xi, \tau) \cdot \vartheta \left[\begin{smallmatrix} \varepsilon \\ 0 \end{smallmatrix} \right] (\xi, \tau) = 0. \quad (18)$$

□

Remark 6. Formula (17) actually provides us an unified approach to construct periodic wave solutions for nonlinear equations. Once an equation is written bilinear forms, then its periodic wave solutions can be directly obtained by solving system (17).

3. The (2 + 1)-Dimensional Sawada-Kotera Equation

In this section, we will focus on the following (2 + 1)-dimensional Sawada-Kotera ((2 + 1)DSK) model [20–22]:

$$\begin{aligned} u_t - \left(u_{xxxx} + 5uu_{xx} + \frac{5}{3}u^3 + 5u_{xy} \right)_x - 5uu_y \\ + 5 \int u_{yy} dx - 5u_x \int u_y dx = 0, \end{aligned} \quad (19)$$

where u is a function of the variables x, y and t , $u_t = \partial u / \partial t$ and the other quantities are similarly defined. It was widely used in many branches of physics, such as conformal field theory, two-dimensional quantum gravity gauge field, theory, and nonlinear science Liuvill flow conservation equations. When $u(x, y, t) = u(x, t)$, (19) reduces to the Sawada-Kotera equation [23]:

$$u_t - \left(u_{xxxx} + 5uu_{xx} + \frac{5}{3}u^3 \right)_x = 0. \quad (20)$$

Equation (19), a B-type Kadomtsev-Petviashvili (KP) model, has also been referred to BKP equation because it is associated with a B-type group [24]. Through the truncated Painlevé expansion and Hirota bilinear method, multisoliton solutions of (19) have been derived and graphically discussed in [25]. In the framework of Bell-polynomial manipulations, the Bell-polynomial expression and Bell-polynomial-typed BT for (19) have been given in [26]. Here we construct its double periodic wave solution and show that the one-soliton solution can be obtained as limiting case of the double periodic wave solution.

3.1. *Construct Double Periodic Wave Solutions of the (2 + 1)DSK Equation.* We consider a variable transformation

$$u = 6(\ln f(x, y, t))_{xx}. \quad (21)$$

Substituting (21) into (19) and integrating with respect to x , we then get the following Hirota's bilinear form:

$$(D_x^6 - D_x D_t + 5D_x^3 D_y - 5D_y^2 + c) f \cdot f = 0, \quad (22)$$

where c is an integration constant.

Remark 7. The constant c may be taken to be zero in the construction of soliton solutions. However, in our double periodic wave case, the nonzero constant c plays an important role and cannot be dropped.

When $c = 0$, (19) admits a one-soliton solution [21]

$$u_1 = 6(\ln(1 + e^\eta))_{xx}, \quad (23)$$

where phase variable $\eta = kx + \gamma y + ((k^6 + 5k^3\gamma - 5\gamma^2)/k)t + h$, and k, γ, h are constants. Next, we turn to see the periodicity of the solution (23); the function f is chosen to be a Riemann theta function; namely,

$$f(x, y, t) = \vartheta(\xi, \tau), \quad (24)$$

where phase variable $\xi = \alpha x + \beta y + \omega t + \sigma$. According to Proposition 4, we refer to

$$u = 6(\ln f(x, y, t))_{xx} = 6\alpha^2 \partial_\xi^2 \ln \vartheta(\xi, \tau), \quad (25)$$

which shows that the solution u is a double periodic function with two fundamental periods 1 and $i\tau$.

By introducing the notations as

$$\begin{aligned} \vartheta_1(\xi, \rho) &= \vartheta(2\xi, 2\tau) = \sum_{m \in \mathbb{Z}} \rho^{4m^2} \exp(4\pi i m \xi), \\ \vartheta_2(\xi, \rho) &= \vartheta\left[\begin{array}{c} 0 \\ -\frac{1}{2} \end{array}\right](2\xi, 2\tau) \\ &= \sum_{m \in \mathbb{Z}} \rho^{(2m-1)^2} \exp[2\pi i (2m-1)\xi], \quad \rho = e^{-\pi\tau/2}. \end{aligned} \quad (26)$$

Substituting (24) into (22), using formulas (17) and (26), leads to the following linear system:

$$\begin{aligned} & -\vartheta_1''(0, \rho) \alpha \omega + \vartheta_1^{(6)}(0, \rho) \alpha^6 + 5\vartheta_1^{(4)}(0, \rho) \alpha^3 \beta \\ & - 5\vartheta_1''(0, \rho) \beta^2 + \vartheta_1(0, \rho) c = 0, \\ & -\vartheta_2''(0, \rho) \alpha \omega + \vartheta_2^{(6)}(0, \rho) \alpha^6 + 5\vartheta_2^{(4)}(0, \rho) \alpha^3 \beta \\ & - 5\vartheta_2''(0, \rho) \beta^2 + \vartheta_2(0, \rho) c = 0, \end{aligned} \quad (27)$$

where we have denoted by the notations

$$\vartheta_j^{(k)}(0, \rho) = \left. \frac{d^k \vartheta_j(\xi, \rho)}{d\xi^k} \right|_{\xi=0}, \quad j = 1, 2; k = 1, 2, 3, 4, 5, 6. \quad (28)$$

This system admits an explicit solution

$$\begin{aligned} \omega &= \left(-(\alpha^6 \vartheta_1^{(6)} + 5\alpha^3 \beta \vartheta_1^{(4)} - 5\beta^2 \vartheta_1'') \vartheta_2 \right. \\ & \quad \left. + (\alpha^6 \vartheta_2^{(6)} + 5\alpha^3 \beta \vartheta_2^{(4)} - 5\beta^2 \vartheta_2'') \vartheta_1 \right) \\ & \quad \times (-\alpha \vartheta_1'' \vartheta_2 + \alpha \vartheta_2'' \vartheta_1)^{-1}, \\ c &= \left(\vartheta_1'' (\vartheta_2^{(6)} \alpha^6 + 5\alpha^3 \beta \vartheta_2^{(4)} - 5\vartheta_2'' \beta^2) \right. \\ & \quad \left. - \vartheta_2'' (\alpha^6 \vartheta_1^{(6)} + 5\alpha^3 \beta \vartheta_1^{(4)} - 5\beta^2 \vartheta_1'') \right) \\ & \quad \times (-\vartheta_1'' \vartheta_2 + \vartheta_2'' \vartheta_1)^{-1}, \end{aligned} \quad (29)$$

where we have omitted the notation $(0, \rho)$ after ϑ_1, ϑ_2 for simplicity of formula (29). Therefore, we get a double periodic wave solution of (19) which reads

$$u = 6(\ln \vartheta(\xi, \tau))_{xx} \quad (30)$$

with the theta function $\vartheta(\xi, \tau)$ given by (4) for the case $s = \varepsilon = 0$, and parameters ω, c by (29), while other parameters α, β, σ are free.

3.2. *Feature and Asymptotic Property of Double Periodic Waves.* The double periodic wave solution (30) possesses a simple characterization as follows.

- (i) It has a single phase variable ξ ; that is, it is one-dimensional.
- (ii) It has two fundamental periods 1 and $i\tau$ in the phase variable ξ .
- (iii) The speed parameter of ξ is given by

$$\begin{aligned} \omega &= \left(-(\alpha^6 \vartheta_1^{(6)} + 5\alpha^3 \beta \vartheta_1^{(4)} - 5\beta^2 \vartheta_1'') \vartheta_2 \right. \\ & \quad \left. + (\alpha^6 \vartheta_2^{(6)} + 5\alpha^3 \beta \vartheta_2^{(4)} - 5\beta^2 \vartheta_2'') \vartheta_1 \right) \\ & \quad \times (-\alpha \vartheta_1'' \vartheta_2 + \alpha \vartheta_2'' \vartheta_1)^{-1}. \end{aligned} \quad (31)$$

- (iv) It has only one wave pattern for all time and it can be viewed as a parallel superposition of overlapping one-soliton waves, placed one period apart.

Now, we further consider the asymptotic properties of the double periodic wave solution. The relation between the periodic wave solution (30) and the one-soliton solution (23) can be established as follows.

Theorem 8. *If the vector $(\omega, c)^T$ is a solution of the system (27) and for the double periodic wave solution (30), we let*

$$\alpha = \frac{k}{2\pi i}, \quad \beta = \frac{\gamma}{2\pi i}, \quad \sigma = \frac{h + \pi\tau}{2\pi i}, \quad (32)$$

where k, γ , and h are given in (23). Then one has the following asymptotic properties

$$\begin{aligned} c &\longrightarrow 0, \quad \xi \longrightarrow \frac{\eta + \pi\tau}{2\pi i}, \quad \vartheta(\xi, \tau) \longrightarrow 1 + e^\eta, \\ &\text{when } \rho \longrightarrow 0. \end{aligned} \quad (33)$$

It implies that the double periodic solution (30) tends to the one-soliton solution (23) under a small amplitude limit. In other words, the periodic solution (30) tends to a solution under a small amplitude limit; namely

$$u \longrightarrow u_1 = 6\partial_x^2 \ln(1 + e^\eta), \quad \text{as } \rho \longrightarrow 0. \quad (34)$$

Proof. We explicitly expand the coefficients of system (27) as follows:

$$\begin{aligned} \vartheta_1(0, \rho) &= 1 + 2\rho^4 + \cdots, \\ \vartheta_1''(0, \rho) &= -32\pi^2 \rho^4 + \cdots, \\ \vartheta_1^{(4)}(0, \rho) &= 512\pi^4 \rho^4 + \cdots, \\ \vartheta_1^{(6)}(0, \rho) &= -8192\pi^6 \rho^4 + \cdots, \\ \vartheta_2(0, \rho) &= 2\rho + 2\rho^9 + \cdots, \\ \vartheta_2''(0, \rho) &= -8\pi^2 \rho + \cdots, \\ \vartheta_2^{(4)}(0, \rho) &= 32\pi^4 \rho + \cdots, \\ \vartheta_2^{(6)}(0, \rho) &= -128\pi^6 \rho + \cdots. \end{aligned} \quad (35)$$

Let the solution of the system (27) be in the form

$$\begin{aligned} \omega &= \omega_0 + \omega_1 \rho + \omega_2 \rho^2 + \cdots = \omega_0 + o(\rho), \\ c &= c_0 + c_1 \rho + c_2 \rho^2 + \cdots = c_0 + o(\rho). \end{aligned} \quad (36)$$

Substituting the expansions (35) and (36) into the system (27), where the second equation is divided by ρ , and letting $\rho \rightarrow 0$, we immediately obtain the following relations:

$$c_0 = 0, \quad -8\pi^2 \alpha \omega_0 - 128\pi^6 \alpha^6 + 160\pi^4 \alpha^3 \beta + 40\pi^2 \beta^2 = 0, \quad (37)$$

which have a solution

$$c_0 = 0, \quad \omega_0 = \frac{16\pi^4 \alpha^6 - 20\pi^2 \alpha^3 \beta - 5\beta^2}{\alpha}. \quad (38)$$

Combining (32) and (38) leads to

$$c \longrightarrow 0, \quad 2\pi i \omega \longrightarrow \frac{k^6 + 5k^3 \gamma - 5\gamma^2}{k}, \quad \text{as } \rho \longrightarrow 0. \quad (39)$$

Hence we conclude that

$$\begin{aligned} \hat{\xi} &= 2\pi i \xi - \pi \tau = kx + \gamma y \\ &+ 2\pi i \omega t + h \longrightarrow kx \\ &+ \gamma y + \frac{k^6 + 5k^3 \gamma - 5\gamma^2}{k} t + h = \eta, \quad \text{as } \rho \longrightarrow 0. \end{aligned} \quad (40)$$

In the following, we consider asymptotic properties of the double periodic wave solution (30) under the limit $\rho \rightarrow 0$. For this purpose, we expand the Riemann theta function $\vartheta(\xi, \tau)$ and make use of the expression (40); it follows that

$$\begin{aligned} \vartheta(\xi, \tau) &= 1 + (e^{2\pi i \xi} + e^{-2\pi i \xi}) \rho^2 + (e^{4\pi i \xi} + e^{-4\pi i \xi}) \rho^8 + \cdots \\ &= 1 + e^{\hat{\xi}} + (e^{-\hat{\xi}} + e^{2\hat{\xi}}) \rho^4 + (e^{-2\hat{\xi}} + e^{3\hat{\xi}}) \rho^{12} \\ &+ \cdots \longrightarrow 1 + e^{\hat{\xi}} \longrightarrow 1 + e^\eta, \quad \text{as } \rho \longrightarrow 0. \end{aligned} \quad (41)$$

Therefore we conclude that the periodic solution (30) just goes to the soliton solution (23) as the amplitude $\rho \rightarrow 0$. \square

4. Conclusions

In this paper, based on the Hirota's bilinear method, combining the theory of a general Riemann theta function, we have derived a method of constructing double periodic wave solutions for $(2 + 1)$ -dimensional nonlinear partial differential equations. As application of our method, we construct double periodic wave solutions to the $(2 + 1)$ -dimensional Sawada-Kotera equation. The double periodic wave solutions obtained in this paper are theta function series solutions. By making a limiting procedure, we have analyzed asymptotic behavior of the double periodic waves, obtaining the relations between the periodic wave solutions and soliton solutions. We note that this method can be generalized to the case of $N \geq 2$ to construct N -periodic wave solutions. But more constraint equations need to be satisfied, so the calculation will be more complicated.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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