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Research Article

Existence Results for Strong Mixed Vector Equilibrium Problem for Multivalued Mappings

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We consider a strong mixed vector equilibrium problem in topological vector spaces. Using generalized Fan-Browder fixed point theorem (Takahashi 1976) and generalized pseudomonotonicity for multivalued mappings, we provide some existence results for strong mixed vector equilibrium problem without using KKM-Fan theorem. The results in this paper generalize, improve, extend, and unify some existence results in the literature. Some special cases are discussed and an example is constructed.

1. Introduction

The minimax inequalities of Fan [1] are fundamental tools in proving many existence theorems in nonlinear analysis. Their equivalence to the equilibrium problems was introduced by Takahashi [2], Blum and Oettli [3], and Noor and Oettli [4]. The equilibrium theory provides a novel and unified treatment of a wide class of problems which arises in economics, finance, transportation, elasticity, optimization, and so forth. The generalization of equilibrium problem for vector valued mappings is known as vector equilibrium problem and has been studied vastly by many authors; see, for example, [5–8].

Recently, Kum and Wong [9] considered a multivalued version of generalized equilibrium problem which extends the strong vector variational inequality studied by Fang and Huang [10] in real Banach spaces. From Brouwer's fixed point theorem and Fan-Browder fixed point theorem, they derived existence results for generalized equilibrium problem with and without monotonicity in general Hausdorff topological vector spaces.

The main motivation of this paper is to establish some existence results for strong mixed vector equilibrium problem which is combination of vector equilibrium problem and a vector variational inequality. Proposition 3.3 of Ahmad and Akram [11] related to the core of a set is extended for multivalued mappings and used to prove an existence result for strong mixed vector equilibrium problem. We also prove our results with and without monotonicity assumptions.

2. Preliminaries

Throughout this paper, let X and Y be the topological vector spaces. Let K be a nonempty convex subset of X and $C \subseteq Y$ a pointed closed convex cone with $intC \neq \emptyset$. Let $f : K \times K \to 2^Y$ and $T : K \to 2^{L(X,Y)}$ be the multivalued mappings, where $L(X, Y)$ is space of all continuous and bounded mappings. We consider the following problem.

Find $x \in K$, $u \in T(x)$ such that, for all $y \in K$,

$$f(x, y) + \langle u, y - x\rangle \nsubseteq -C \setminus \{0\}. \tag{1}$$

We call problem (1) strong mixed vector equilibrium problem for multivalued mappings.

2.1. Special Cases

(1) If $T \equiv 0$, then problem (1) reduces to the problem of finding $x \in K$ such that

$$f(x, y) \nsubseteq -C \setminus \{0\}, \quad \forall y \in K. \tag{2}$$

Problem (2) is called multivalued generalized system and this problem was considered and studied by Kum and Wong [9].

(2) If $T \equiv 0$, $Y = \mathbb{R}$, $C = \mathbb{R}^+$, and f is single-valued, then problem (1) reduces to the classical equilibrium problem introduced and studied by Blum and Oettli [3] which is to find $x \in K$ such that

$$f(x, y) \geq 0, \quad \forall y \in K. \tag{3}$$

(3) If f and T are single-valued and $Y = \mathbb{R}$, $C = \mathbb{R}^+$, then problem (1) reduces to the generalized equilibrium problem of finding $x \in K$ such that

$$f(x, y) + \langle T(x), y - x \rangle \geq 0, \quad \forall y \in K, \tag{4}$$

which was studied by S. Takahashi and W. Takahashi [12].

(4) If $f \equiv 0$ and T is single-valued, then (1) reduces to strong vector variational inequality problem which is to find $x \in K$ such that

$$\langle T(x), y - x \rangle \notin -C \setminus \{0\}, \quad \forall y \in K. \tag{5}$$

Problem (5) was considered and studied by Fang and Huang [10].

It is clear that the problem under consideration is much more general than the other problems that exist in the literature.

Let us recall some definitions and results that are needed to prove the main results of this paper.

Definition 1 (see [13]). Let X and Y be the topological vector spaces, and let $g : X \to 2^Y$ be a multivalued mapping. Then one has the following:

(i) g is said to be upper semicontinuous at $x \in X$, if, for each $x \in X$ and each open set V in Y with $g(x) \subset V$, there exists an open neighborhood U of x in X such that $g(y) \subset V$, for each $y \in U$;

(ii) g is said to be lower semicontinuous at $x \in X$, if, for each $x \in X$ and each open set V in Y with $g(x) \cap V \neq \emptyset$, there exists an open neighborhood U of x in X such that $g(y) \cap V \neq \emptyset$, for each $y \in U$;

(iii) g is said to be continuous on X, if it is at the same time upper semicontinuous and lower semicontinuous on X. It is also known that $g : X \to 2^Y$ is lower semicontinuous if and only if, for each closed set V in Y, the set $\{x \in X : g(x) \subset V\}$ is closed in X.

Definition 2. A multivalued mapping $f : K \times K \to 2^Y$ is said to be generalized C-strongly pseudomonotone, if, for any $x, y \in K$,

$$f(x, y) \nsubseteq -C \setminus \{0\} \text{ implies } f(y, x) \subseteq -C. \tag{6}$$

Definition 3. A multivalued mapping $T : K \to 2^{L(X,Y)}$ is said to be

(i) generalized C-strongly pseudomonotone, if, for any $x, y \in K$, there exists $u \in T(x)$ such that

$$\langle u, y - x \rangle \nsubseteq -C \setminus \{0\} \tag{7}$$

implies that there exists $v \in T(y)$ such that

$$\langle v, x - y \rangle \subseteq -C; \tag{8}$$

(ii) generalized hemicontinuous, if, for any given $x, y, z \in K$ and for $\lambda \in [0, 1]$, the mapping $\lambda \mapsto \langle T(x + \lambda(y - z), z) \rangle$ is upper semicontinuous at 0^+.

Definition 4 (see [9]). Let $h : K \to 2^Y$ be a multivalued mapping. Then h is said to be

(i) C-convex, if, for all $x, y \in K$ and for $\lambda \in [0, 1]$,

$$h(\lambda x + (1 - \lambda) y) \subseteq \lambda h(x) + (1 - \lambda) h(y) - C; \tag{9}$$

(ii) generalized hemicontinuous, if, for all $x, y \in K$ and for $\lambda \in [0, 1]$, the mapping $\lambda \mapsto h(x + \lambda(y - x))$ is upper semicontinuous at 0^+.

Definition 5 (see [14]). Let $A : X \to 2^Y \cup \{\emptyset\}$ be a multivalued mapping. Then A is said to have local intersection property, if, for each $x \in X$ with $A(x) \neq \emptyset$, there exists an open neighborhood $N(x)$ of x such that $\bigcap_{y \in N(x)} A(y) \neq \emptyset$.

Lemma 6 (see [14]). *Let X and Y be the topological spaces and $A : X \to 2^Y$ a multivalued mapping. Then the following conditions are equivalent:*

(i) *A has the local intersection property;*

(ii) *there exists a multivalued mapping $F : X \to 2^Y$ such that $F(x) \subset A(x)$, for each $x \in X$; $F^{-1}(x)$ is open in X for each $y \in Y$ and $X \subseteq \bigcup_{y \in Y} F^{-1}(y)$.*

Theorem 7 (Fan-Browder fixed point theorem [15]). *Let K be a nonempty, compact, and convex subset of a Hausdorff topological vector space X and let $F : K \to 2^K$ be a mapping with nonempty convex values and open fibers (i.e., for $y \in K$, $F^{-1}(y)$ is called the fiber of F on y). Then F has a fixed point.*

The generalization of the Fan-Browder fixed point theorem [15] was derived by Balaj and Muresan [16] as follows.

Theorem 8. *Let K be a nonempty, compact, and convex subset of a topological vector space X and let $F : K \to 2^K$ be a mapping with nonempty convex values having the local intersection property. Then F has a fixed point.*

Definition 9 (see [3]). Let K and D be convex subsets of X with $D \subset K$. The core of D relative to K, denoted by $\mathrm{core}_K D$, is the set defined by $a \in \mathrm{core}_K D$ if and only if $a \in D$ and $D \cap (a, y) \neq \emptyset$, for all $y \in K \setminus D$.

3. Existence Results

In this section, we prove some existence results for strong mixed vector equilibrium problem for multivalued mappings.

Theorem 10. *Let K be a nonempty compact convex subset of X and C a closed convex pointed cone in Y. Let $f : K \times K \to 2^Y$ and $T : K \to 2^{L(X,Y)}$ be the multivalued mappings. Suppose that the following conditions hold:*

(i) *for all $x \in K$, $0 \in f(x, x)$;*

(ii) *f is generalized C-strongly pseudomonotone and C-convex in the second argument;*

(iii) *f is generalized hemicontinuous in the first argument and lower semicontinuous;*

(iv) *T is generalized C-strongly pseudomonotone and generalized hemicontinuous.*

Then, there exists $x \in K$, $u \in T(x)$ such that

$$f(x, y) + \langle u, y - x\rangle \nsubseteq -C \setminus \{0\}, \quad \forall y \in K. \tag{10}$$

First, we prove the following lemma which is required to prove Theorem 10 for which the assumptions remain the same as in Theorem 10.

Lemma 11. *The following two problems are equivalent.*

(I) *Find $x \in K$, $u \in T(x)$ such that $f(x, y) + \langle u, y - x\rangle \nsubseteq -C \setminus \{0\}$, for all $y \in K$.*

(II) *Find $x \in K$, $v \in T(y)$ such that $f(y, x) + \langle v, x - y\rangle \subseteq -C$, for all $y \in K$.*

Proof. Suppose (I) holds. Then by using generalized C-strongly pseudomonotone of f and T, (II) follows.

Conversely, assume that (II) holds; that is,

$$f(y, x) + \langle v, x - y\rangle \subseteq -C, \quad \forall y \in K. \tag{11}$$

For any $y \in K$, set $y_\lambda = \lambda y + (1-\lambda)x$, for $\lambda \in [0, 1]$. Obviously $y_\lambda \in K$, and there exists $v' \in T(y_\lambda)$ such that

$$f(y_\lambda, x) + \left\langle v', x - y_\lambda\right\rangle \subseteq -C. \tag{12}$$

Since f is C-convex in the second argument and $0 \in f(x, x)$, using (12) we have

$$\begin{aligned}
0 &\in f(y_\lambda, y_\lambda) + (1-\lambda)\left\langle v', y_\lambda - y_\lambda\right\rangle \\
&\subseteq \lambda f(y_\lambda, y) + (1-\lambda) f(y_\lambda, x) + (1-\lambda)\left\langle v', x - y_\lambda\right\rangle \\
&\quad + (1-\lambda)\left\langle v', y_\lambda - x\right\rangle - C \\
&= \lambda f(y_\lambda, y) + \lambda(1-\lambda)\left\langle v', y - x\right\rangle \\
&\quad + (1-\lambda)\left\{f(y_\lambda, x) + \left\langle v', x - y_\lambda\right\rangle\right\} - C \\
&\subseteq \lambda f(y_\lambda, y) + \lambda(1-\lambda)\left\langle v', y - x\right\rangle + (1-\lambda)(-C) - C \\
&\subseteq \lambda f(y_\lambda, y) + \lambda(1-\lambda)\left\langle v', y - x\right\rangle - C,
\end{aligned} \tag{13}$$

which implies that

$$\lambda f(y_\lambda, y) + \lambda(1-\lambda)\left\langle v', y - x\right\rangle \subseteq C. \tag{14}$$

As C is a convex cone, we get from (14)

$$f(y_\lambda, y) + (1-\lambda)\left\langle v', y - x\right\rangle \subseteq C. \tag{15}$$

Since f is generalized hemicontinuous in the first argument and T is generalized hemicontinuous, therefore we have for $\lambda \to 0^+$

$$f(x, y) + \langle u, y - x\rangle \subseteq C, \quad u \in T(x). \tag{16}$$

Therefore, we get $x \in K$, $u \in T(x)$ such that

$$f(x, y) + \langle u, y - x\rangle \nsubseteq -C \setminus \{0\}, \quad \forall y \in K, \tag{17}$$

and hence (I) follows. □

Proof of Theorem 10. Consider the multivalued mappings $M, N : K \to 2^K$ for any $x \in K$ as follows:

$$\begin{aligned}
M(x) &= \left\{y \in K : f(y, x) + \left\langle v, x - y\right\rangle \nsubseteq -C\right\}, \\
&\qquad v \in T(y); \\
N(x) &= \left\{y \in K : f(x, y) + \left\langle u, y - x\right\rangle \subseteq -C \setminus \{0\}\right\} \\
&\qquad u \in T(x).
\end{aligned} \tag{18}$$

Clearly, $M(x)$ and $N(x)$ are nonempty sets as $y \in K$. By the generalized C-strong pseudomonotonicity of f and T, we have $M(x) \subseteq N(x)$.

We claim that $N(x)$ is convex. Indeed, letting $y_1, y_2 \in N(x)$, then we have

$$f(x, y_i) + \langle u, y_i - x\rangle \subseteq -C \setminus \{0\}; \quad i = 1, 2. \tag{19}$$

Since f is C-convex in the second argument, therefore, for any $\lambda \in [0, 1]$, we have

$$\begin{aligned}
&f(x, \lambda y_1 + (1-\lambda) y_2) + \langle u, \lambda y_1 + (1-\lambda) y_2 - x\rangle \\
&\quad = f(x, \lambda y_1 + (1-\lambda) y_2) \\
&\qquad + \langle u, \lambda y_1 + (1-\lambda) y_2 - \lambda x - (1-\lambda) x\rangle \\
&\quad \subseteq \lambda f(x, y_1) + (1-\lambda) f(x, y_2) + \lambda \langle u, y_1 - x\rangle \\
&\qquad + (1-\lambda)\langle u, y_2 - x\rangle - C \\
&\quad = \lambda\{f(x, y_1) + \langle u, y_1 - x\rangle\} \\
&\qquad + (1-\lambda)\{f(x, y_2) + \langle u, y_2 - x\rangle\} - C \\
&\quad \subseteq (-C \setminus \{0\}) - C \\
&\quad = -C \setminus \{0\}.
\end{aligned} \tag{20}$$

This implies that $\lambda y_1 + (1-\lambda) y_2 \in N(x)$, and hence $N(x)$ is convex.

By definition of N, we see that N has no fixed point. Indeed, suppose that there exists an $x \in K$ such that $x \in N(x)$. Thus, we have

$$f(x,x) + \langle u, x-x\rangle = f(x,x) \subseteq -C \setminus \{0\}, \tag{21}$$

which is a contradiction to hypothesis (i).

Next, we show that $M^{-1}(y)$ is open in K. For any $y \in K$, we denote the complement of $M^{-1}(y)$ by

$$\begin{aligned}\left[M^{-1}(y)\right]^c = \{x \in K : f(y,x) + \langle v, x-y\rangle \subseteq -C\},\\ v \in T(y).\end{aligned} \tag{22}$$

Let $\{x_\alpha\}$ be a net in $[M^{-1}(y)]^c$ such that $x_\alpha \to x \in K$. Then

$$f(y,x_\alpha) + \langle v, x_\alpha - y\rangle \subseteq -C. \tag{23}$$

Since f is lower semicontinuous and $\langle\cdot,\cdot\rangle$ is continuous, we get

$$f(y,x_\alpha) + \langle v, x_\alpha - y\rangle \longrightarrow f(y,x) + \langle v, x-y\rangle. \tag{24}$$

Since $(-C)^c$ is open, then there exists α_0 such that, for all $\alpha \geq \alpha_0$,

$$\{f(y,x_\alpha) + \langle v, x_\alpha - y\rangle\} \cap (-C)^c \neq \emptyset, \tag{25}$$

which contradicts (23). Hence $f(y,x) + \langle v, x-y\rangle \subseteq -C$, for $v \in T(y)$ and therefore $x \in [M^{-1}(y)]^c$. Thus $[M^{-1}(y)]^c$ is closed and accordingly $M^{-1}(y)$ is open.

From the contrapositive of generalized Fan-Browder fixed point theorem and Lemma 6, we have

$$K \nsubseteq \bigcup_{y\in K} M^{-1}(y). \tag{26}$$

Hence, there exists $x_0 \in K$ such that $M(x_0) = \emptyset$ which contradicts the fact that $M(x)$ is nonempty and hence

$$f(y,x_0) + \langle v, x_0 - y\rangle \subseteq -C, \quad v \in T(y), \ \forall y \in K. \tag{27}$$

By applying Lemma 11, we get that there exists $x_0 \in K$, $u \in T(x_0)$ such that

$$f(x_0,y) + \langle u, y-x_0\rangle \nsubseteq -C \setminus \{0\}, \quad \forall y \in K. \tag{28}$$

This completes the proof. □

Example 12. Let $X = Y = \mathbb{R}$, $K = [0,1]$, and $C = \{x \in \mathbb{R} : x \geq 0\}$. Define $f : K \times K \to 2^Y$ by

$$f(x,y) = [0, x-y], \quad \forall x,y \in K. \tag{29}$$

Also $T : K \to 2^{L(X,Y)}$ is given by

$$T(x) = [-x, 0], \quad \forall x \in K. \tag{30}$$

We see that f is generalized C-strongly pseudomonotone. Indeed, suppose that $f(x,y) \nsubseteq -C \setminus \{0\}$. Therefore, $[0, x-y] \nsubseteq -C \setminus \{0\}$, which implies that $x \geq y$. It follows that $f(y,x) = [0, y-x] \subseteq -C$. Hence, f is generalized C-strongly pseudomonotone.

Similarly, to show that T is generalized C-strongly pseudomonotone, assume that $\langle T(x), y-x\rangle = [-x,0](y-x) = [x(x-y),0] \nsubseteq -C \setminus \{0\}$, which implies that $x \geq y$. It follows that $\langle T(y), x-y\rangle = [-y,0](x-y) = [y(y-x),0] \subseteq -C$. Therefore, T is generalized C-strongly pseudomonotone.

Let $x, y_1, y_2 \in K$ and $0 \leq \alpha \leq 1$. Then, we see that

$$\begin{aligned} f&(x, \alpha y_1 + (1-\alpha) y_2)\\ &= [0, x - (\alpha y_1 + (1-\alpha) y_2)]\\ &= [0, \alpha(x-y_1) + (1-\alpha)(x-y_2)] - 0\\ &\subseteq [0, \alpha(x-y_1) + (1-\alpha)(x-y_2)] - C\\ &= \alpha[0, x-y_1] + (1-\alpha)[0, x-y_2] - C\\ &= \alpha f(x,y_1) + (1-\alpha) f(x,y_2) - C.\end{aligned} \tag{31}$$

So, f is C-convex in the second argument.

It is clear that $x = 1$ is a solution of strong mixed vector equilibrium problem (1) as $f(x,y) + \langle T(x), y-x\rangle = [0, 1-y] + [0, y-1] \nsubseteq -C \setminus \{0\}$.

The following lemma is an extension of Proposition 3.3 [11] related to the core of a set for multivalued mappings.

Lemma 13. *Let K and D be the convex subsets of X with $D \subset K$. Let $\phi : K \to 2^Y$ be C-convex, $x_0 \in \mathrm{core}_K D$; $\phi(x_0) \subseteq -C$, and $\phi(y) \subseteq C$, for all $y \in D$. Then, $\phi(y) \subseteq C$, for all $y \in K$.*

Proof. On the contrary, suppose that $\phi(\overline{y}) \nsubseteq C$, for some $\overline{y} \in K \setminus D$. Then, there is $w \in \phi(\overline{y})$ such that $w \notin C$.

Since $\phi(x_0) \subseteq -C$, then there exists $u \in \phi(x_0)$ such that $u \in -C$.

Suppose $\eta = \lambda x_0 + (1-\lambda)\overline{y}$, for $\lambda \in (0,1)$. Then $\eta \in (x_0, \overline{y})$. By using C-convexity of ϕ, we have

$$\begin{aligned}\lambda\phi(x_0) + (1-\lambda)\phi(\overline{y}) - \phi(\lambda x_0 + (1-\lambda)\overline{y}) \subseteq C\\ \Longrightarrow \lambda\phi(x_0) + (1-\lambda)\phi(\overline{y}) - \phi(\eta) \subseteq C.\end{aligned} \tag{32}$$

Then, there exists $v \in \phi(\eta)$ such that, for some $c \in C$, we have

$$\begin{aligned} v &= \lambda u + (1-\lambda) w - c\\ &\in -C + (-C) - C = -C.\end{aligned} \tag{33}$$

Therefore,

$$\phi(\eta) \subseteq -C. \tag{34}$$

Since $x_0 \in \mathrm{core}_K D$, so we have a point $z \in D \cap (x_0, \overline{y})$. By (34), we have $\phi(z) \subseteq -C$, a contradiction to the hypothesis. Thus, $\phi(y) \subseteq C$, for all $y \in K$. □

Theorem 14. *Let K be a nonempty convex subset of X and C a closed convex pointed cone in Y. Let $f : K \times K \to 2^Y$ and $T : K \to 2^{L(X,Y)}$ be the mappings satisfying the conditions which are the same as in Theorem 10. In addition, suppose that the following condition holds: there exists a nonempty convex compact subset D of K such that for $x \in D \setminus \mathrm{core}_K D$ and $z \in \mathrm{core}_K D$,*

$$f(x,z) + \langle u, z-x\rangle \subseteq -C; \quad u \in T(x). \tag{35}$$

Then, there exists a point $x \in D$ such that, for all $y \in K$,

$$f(x, y) + \langle u, y - x\rangle \nsubseteq -C \setminus \{0\}; \quad u \in T(x). \tag{36}$$

Proof. By Theorem 10, it follows that there exists $x \in D$, $u \in T(x)$ such that

$$f(x, y) + \langle u, y - x\rangle \nsubseteq -C \setminus \{0\}, \quad \forall y \in D. \tag{37}$$

Set $\phi(y) = f(x, y) + \langle u, y - x\rangle$. Then $\phi(y)$ is C-convex and $\phi(y) \subseteq C$, for all $y \in D$.

If $x \in \operatorname{core}_K D$, then choose $x_0 = x$. If $x \in D \setminus \operatorname{core}_K D$, then choose $x_0 = z$, where z is the same as in the hypothesis. In both cases, $x_0 \in \operatorname{core}_K D$ and $\phi(x_0) \subseteq -C$. Hence by Lemma 13, it follows that $\phi(y) \subseteq C$, for all $y \in K$, which implies that

$$f(x, y) + \langle u, y - x\rangle \subseteq C, \quad y \in K. \tag{38}$$

Thus, there exists at least one $x \in D$ and $u \in T(x)$ such that

$$f(x, y) + \langle u, y - x\rangle \nsubseteq -C \setminus \{0\}, \quad \forall y \in K. \tag{39}$$

This completes the proof. □

Now, we prove the existence results for strong mixed vector equilibrium problem for multivalued mappings without monotonicity.

Theorem 15. *Let K be a nonempty compact convex subset of X. Let $f : K \times K \to 2^Y$ and $T : K \to 2^{L(X,Y)}$ be the mappings such that f is C-convex in the second argument and $0 \in f(x, x)$. Assume that, for each $y \in K$, the set $\{x \in K : f(x, y) + \langle u, y - x\rangle \subseteq -C \setminus \{0\}, u \in T(x)\}$ is open. Then the strong mixed vector equilibrium problem* (1) *has a solution.*

Proof. Using the same assertions in Kum and Wong [9], one can easily prove this theorem. □

Theorem 16. *Let K be a nonempty convex subset of X. Let $f : K \times K \to 2^Y$ and $T : K \to 2^{L(X,Y)}$ be the mappings such that f is C-convex in the second argument and $0 \in f(x, x)$. Also, assume that*

(i) *K is locally compact and there is an $\gamma > 0$ and $x_0 \in K$, $\|x_0\| < \gamma$ such that, for all $y \in K$, $\|y\| = \gamma$, and $f(y, x_0) + \langle v, x_0 - y\rangle \subseteq -C$, for $v \in T(y)$;*

(ii) *for each $y \in K$, the set $\{x \in K : f(x, y) + \langle u, y - x\rangle \subseteq -C \setminus \{0\}, u \in T(x)\}$ is open.*

Then, there exists $x \in K$, $u \in T(x)$ such that

$$f(x, y) + \langle u, y - x\rangle \nsubseteq -C \setminus \{0\}, \quad \forall y \in K. \tag{40}$$

Proof. Let $K_\gamma = \{x \in K : \|x\| \le \gamma\}$. Since K is locally compact, therefore K_γ is compact. By applying Theorem 10, we can see that there exists $\tilde{x} \in K_\gamma$ and $u \in T(\tilde{x})$ such that

$$f(\tilde{x}, y) + \langle u, y - \tilde{x}\rangle \nsubseteq -C \setminus \{0\}, \quad \forall y \in K. \tag{41}$$

To show that $\tilde{x}$ is the solution of problem (41), consider the following two cases.

(I) If $\|\tilde{x}\| = \gamma$, then, by assumption (i), we have, for $u \in T(\tilde{x})$,

$$f(\tilde{x}, x_0) + \langle u, x_0 - \tilde{x}\rangle \subseteq -C. \tag{42}$$

Now, for any $y \in K$, set $y_\lambda = \lambda y + (1 - \lambda)x_0$, for $\lambda \in [0, 1]$. Obviously, $y_\lambda \in K_\gamma$ and it follows that for $u \in T(\tilde{x})$

$$f(\tilde{x}, y_\lambda) + \langle u, y_\lambda - \tilde{x}\rangle \nsubseteq -C \setminus \{0\}. \tag{43}$$

Since f is C-convex in the second argument, we have

$$\begin{aligned} f(\tilde{x}, y_\lambda) &+ \langle u, y_\lambda - \tilde{x}\rangle \\ &= f(\tilde{x}, \lambda y + (1 - \lambda) x_0) \\ &\quad + \langle u, \lambda y + (1 - \lambda) x_0 - \lambda\tilde{x} - (1 - \lambda)\tilde{x}\rangle \\ &\subseteq \lambda \{f(\tilde{x}, y) + \langle u, y - \tilde{x}\rangle\} \\ &\quad + (1 - \lambda)\{f(\tilde{x}, x_0) + \langle u, x_0 - \tilde{x}\rangle\} - C. \end{aligned} \tag{44}$$

Therefore, using (42) we conclude that

$$\begin{aligned} \lambda \{f(\tilde{x}, y) &+ \langle u, y - \tilde{x}\rangle\} \\ &\subseteq [Y \setminus (-C \setminus \{0\})] + (1 - \lambda) C + C \\ &\subseteq Y \setminus (-C \setminus \{0\}), \end{aligned} \tag{45}$$

which implies that

$$f(\tilde{x}, y) + \langle u, y - \tilde{x}\rangle \nsubseteq -C \setminus \{0\}, \quad \forall y \in K. \tag{46}$$

(II) If $\|\tilde{x}\| < \gamma$, then, for any $y \in K$, set $y_\lambda = \lambda y + (1 - \lambda)\tilde{x}$, for $\lambda \in [0, 1]$. Clearly, $y_\lambda \in K_\gamma$ and it follows that for $u \in T(\tilde{x})$

$$f(\tilde{x}, y_\lambda) + \langle u, y_\lambda - \tilde{x}\rangle \nsubseteq -C \setminus \{0\}. \tag{47}$$

Using C-convexity of f in the second argument and $0 \in f(x, x)$, we have

$$\begin{aligned} f(\tilde{x}, y_\lambda) &+ \langle u, y_\lambda - \tilde{x}\rangle \\ &\subseteq \lambda f(\tilde{x}, y) + (1 - \lambda) f(\tilde{x}, \tilde{x}) + \lambda \langle u, y - \tilde{x}\rangle - C \\ &\subseteq \lambda \{f(\tilde{x}, y) + \langle u, y - \tilde{x}\rangle\} - C, \end{aligned} \tag{48}$$

which implies that, for $u \in T(\tilde{x})$, we have

$$f(\tilde{x}, y) + \langle u, y - \tilde{x}\rangle \nsubseteq -C \setminus \{0\}, \quad \forall y \in K. \tag{49}$$

This completes the proof. □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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