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Research Article

On the Hyers-Ulam Stability of Differential Equations of Second Order

Qusuay H. Alqifiary[1,2] and Soon-Mo Jung[3]

[1] *Computer Science and Mathematics College, University of Al-Qadisiyah, Al-Diwaniyah, Iraq*
[2] *College of Mathematics, University of Belgrade, Belgrade, Serbia*
[3] *Mathematics Section, College of Science and Technology, Hongik University, Sejong 339–701, Republic of Korea*

Correspondence should be addressed to Soon-Mo Jung; smjung@hongik.ac.kr

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By using of the Gronwall inequality, we prove the Hyers-Ulam stability of differential equations of second order with initial conditions.

1. Introduction

In 1940, Ulam [1] posed a problem concerning the stability of functional equations: "Give conditions in order for a linear function near an approximately linear function to exist."

A year later, Hyers [2] gave an answer to the problem of Ulam for additive functions defined on Banach spaces: let X and Y be real Banach spaces and $\varepsilon > 0$. Then for every function $f : X \to Y$ satisfying

$$\left\| f\left(x+y\right) - f\left(x\right) - f\left(y\right) \right\| \le \varepsilon \tag{1}$$

for all $x, y \in X$, there exists a unique additive function $A : X \to Y$ with the property

$$\left\| f\left(x\right) - A\left(x\right) \right\| \le \varepsilon \tag{2}$$

for all $x \in X$.

After Hyers's result, many mathematicians have extended Ulam's problem to other functional equations and generalized Hyers's result in various directions (see [3–6]). A generalization of Ulam's problem was recently proposed by replacing functional equations with differential equations: the differential equation

$$\varphi\left(f\left(t\right), y\left(t\right), y'\left(t\right), \ldots, y^{(n)}\left(t\right)\right) = 0 \tag{3}$$

has the Hyers-Ulam stability if for a given $\varepsilon > 0$ and a function y such that

$$\left|\varphi\left(f\left(t\right), y\left(t\right), y'\left(t\right), \ldots, y^{(n)}\left(t\right)\right)\right| \le \varepsilon, \tag{4}$$

there exists a solution y_0 of the differential equation such that $|y(t) - y_0(t)| \le K(\varepsilon)$ and $\lim_{\varepsilon \to 0} K(\varepsilon) = 0$.

Obloza seems to be the first author who has investigated the Hyers-Ulam stability of linear differential equations (see [7, 8]). Thereafter, Alsina and Ger published their paper [9], which handles the Hyers-Ulam stability of the linear differential equation $y'(t) = y(t)$: if a differentiable function $y(t)$ is a solution of the inequality $|y'(t) - y(t)| \le \varepsilon$ for any $t \in (a, \infty)$, then there exists a constant c such that $|y(t) - ce^t| \le 3\varepsilon$ for all $t \in (a, \infty)$.

Those previous results were extended to the Hyers-Ulam stability of linear differential equations of first order and higher order with constant coefficients in [10–13], respectively. Furthermore, Jung has also proved the Hyers-Ulam stability of linear differential equations (see [14–17]). Rus investigated the Hyers-Ulam stability of differential and integral equations using the Gronwall lemma and the technique of weakly Picard operators (see [18, 19]). Recently, the Hyers-Ulam stability problems of linear differential equations of first order and second order with constant coefficients were studied by using the method of integral factors (see [20, 21]).

The results given in [10, 15, 20] have been generalized by Cimpean and Popa [22] and by Popa and Raşa [23, 24] for the linear differential equations of nth order with constant coefficients. Furthermore, the Laplace transform method was recently applied to the proof of the Hyers-Ulam stability of linear differential equations (see [25]).

This paper consists of two main sections. In Section 2, we introduce some sufficient conditions under which each solution of the linear differential equation (11) is bounded. In Section 3, we prove the Hyers-Ulam stability of the linear differential equations of the form (11) as well as the nonlinear differential equations of the form (55) by using the Gronwall lemma that was recently introduced by Rus [18, 19] in studying the Hyers-Ulam stability of differential equations.

One of the advantages of this paper is that the authors have applied the Gronwall lemma, which is now recognized as a powerful method, for proving the Hyers-Ulam stability of various differential equations of second order.

2. Preliminaries

In this section, we first introduce and prove a lemma which is a kind of the Gronwall inequality.

Lemma 1. *Let $u, v : [0,\infty) \to [0,\infty)$ be integrable functions, let $c > 0$ be a constant, and let $t_0 \geq 0$ be given. If u satisfies the inequality*

$$u(t) \leq c + \int_{t_0}^{t} u(\tau)\, v(\tau)\, d\tau \tag{5}$$

for all $t \geq t_0$, then

$$u(t) \leq c \exp\left(\int_{t_0}^{t} v(\tau)\, d\tau\right) \tag{6}$$

for all $t \geq t_0$.

Proof. It follows from (5) that

$$\frac{u(t)\, v(t)}{c + \int_{t_0}^{t} u(\tau)\, v(\tau)\, d\tau} \leq v(t) \tag{7}$$

for all $t \geq t_0$. Integrating both sides of the last inequality from t_0 to t, we obtain

$$\ln\left(c + \int_{t_0}^{t} u(\tau)\, v(\tau)\, d\tau\right) - \ln c \leq \int_{t_0}^{t} v(\tau)\, d\tau \tag{8}$$

or

$$c + \int_{t_0}^{t} u(\tau)\, v(\tau)\, d\tau \leq c \exp\left(\int_{t_0}^{t} v(\tau)\, d\tau\right) \tag{9}$$

for each $t \geq t_0$, which together with (5) implies that

$$u(t) \leq c \exp\left(\int_{t_0}^{t} v(\tau)\, d\tau\right) \tag{10}$$

for all $t \geq t_0$. □

In the following theorem, using Lemma 1, we investigate sufficient conditions under which every solution of the differential equation

$$u''(t) + (1 + \psi(t))\, u(t) = 0 \tag{11}$$

is bounded.

Theorem 2. *Let $\psi : [0,\infty) \to \mathbb{R}$ be a differentiable function. Every solution $u : [0,\infty) \to \mathbb{R}$ of the linear differential equation (11) is bounded provided that $\int_0^{\infty} |\psi'(t)| dt < \infty$ and $\psi(t) \to 0$ as $t \to \infty$.*

Proof. First, we choose t_0 to be large enough so that $1 + \psi(t) \geq 1/2$ for all $t \geq t_0$. Multiplying (11) by $u'(t)$ and integrating it from t_0 to t, we obtain

$$\frac{1}{2}u'(t)^2 + \frac{1}{2}u(t)^2 + \int_{t_0}^{t} \psi(\tau)\, u(\tau)\, u'(\tau)\, d\tau = c_1 \tag{12}$$

for all $t \geq t_0$. Integrating by parts, this yields

$$\begin{aligned} &\frac{1}{2}u'(t)^2 + \frac{1}{2}u(t)^2 + \frac{1}{2}\psi(t)\, u(t)^2 \\ &\quad - \frac{1}{2}\int_{t_0}^{t} \psi'(\tau)\, u(\tau)^2 d\tau = c_2 \end{aligned} \tag{13}$$

for any $t \geq t_0$. Then it follows from (13) that

$$\begin{aligned} \frac{1}{4}u(t)^2 &\leq \frac{1}{2}u'(t)^2 + \frac{1}{2}\cdot\frac{1}{2}u(t)^2 \\ &\leq \frac{1}{2}u'(t)^2 + \frac{1}{2}(1 + \psi(t))\, u(t)^2 \\ &= c_2 + \frac{1}{2}\int_{t_0}^{t} \psi'(\tau)\, u(\tau)^2 d\tau \end{aligned} \tag{14}$$

for all $t \geq t_0$. Thus, it holds that

$$\begin{aligned} u(t)^2 &\leq 4c_2 + 2\int_{t_0}^{t} \psi'(\tau)\, u(\tau)^2 d\tau \\ &\leq 4|c_2| + 2\int_{t_0}^{t} |\psi'(\tau)|\, u(\tau)^2 d\tau \end{aligned} \tag{15}$$

for any $t \geq t_0$.

In view of Lemma 1, (15), and our hypothesis, there exists a constant $M_1 > 0$ such that

$$u(t)^2 \leq 4|c_2| \exp\left(\int_{t_0}^{t} 2|\psi'(\tau)|\, d\tau\right) < M_1^2 \tag{16}$$

for all $t \geq t_0$. On the other hand, since u is continuous, there exists a constant $M_2 > 0$ such that $|u(t)| \leq M_2$ for all $0 \leq t \leq t_0$, which completes the proof. □

Corollary 3. *Let $\phi : [0,\infty) \to \mathbb{R}$ be a differentiable function satisfying $\phi(t) \to 1$ as $t \to \infty$. Every solution $u : [0,\infty) \to \mathbb{R}$ of the linear differential equation*

$$u''(t) + \phi(t)\, u(t) = 0 \tag{17}$$

is bounded provided that $\int_0^{\infty} |\phi'(t)| dt < \infty$.

3. Main Results on Hyers-Ulam Stability

Given constants $L > 0$ and $t_0 \geq 0$, let $U(L; t_0)$ denote the set of all functions $u : [t_0, \infty) \rightarrow \mathbb{R}$ with the following properties:

(i) u is twice continuously differentiable;

(ii) $u(t_0) = u'(t_0) = 0$;

(iii) $\int_{t_0}^{\infty} |u'(\tau)| d\tau \leq L$.

We now prove the Hyers-Ulam stability of the linear differential equation (11) by using the Gronwall inequality.

Theorem 4. *Given constants $L > 0$ and $t_0 \geq 0$, assume that $\psi : [t_0, \infty) \rightarrow \mathbb{R}$ is a differentiable function with $C := \int_{t_0}^{\infty} |\psi'(\tau)| d\tau < \infty$ and $\lambda := \inf_{t \geq t_0} \psi(t) > -1$. If a function $u \in U(L; t_0)$ satisfies the inequality*

$$\left| u''(t) + (1 + \psi(t)) u(t) \right| \leq \varepsilon \tag{18}$$

for all $t \geq t_0$ and for some $\varepsilon \geq 0$, then there exist a solution $u_0 \in U(L; t_0)$ of the differential equation (11) and a constant $K > 0$ such that

$$\left| u(t) - u_0(t) \right| \leq K \sqrt{\varepsilon} \tag{19}$$

for any $t \geq t_0$, where

$$K := \sqrt{\frac{2L}{1 + \lambda}} \exp\left(\frac{C}{2(1 + \lambda)} \right). \tag{20}$$

Proof. We multiply (18) with $|u'(t)|$ to get

$$\begin{aligned} -\varepsilon \left| u'(t) \right| &\leq u'(t) u''(t) + u(t) u'(t) + \psi(t) u(t) u'(t) \\ &\leq \varepsilon \left| u'(t) \right| \end{aligned} \tag{21}$$

for all $t \geq t_0$. If we integrate each term of the last inequalities from t_0 to t, then it follows from (ii) that

$$\begin{aligned} -\varepsilon \int_{t_0}^{t} \left| u'(\tau) \right| d\tau &\leq \frac{1}{2} u'(t)^2 + \frac{1}{2} u(t)^2 + \int_{t_0}^{t} \psi(\tau) u(\tau) u'(\tau) d\tau \\ &\leq \varepsilon \int_{t_0}^{t} \left| u'(\tau) \right| d\tau \end{aligned} \tag{22}$$

for any $t \geq t_0$.

Integrating by parts and using (iii), we have

$$\begin{aligned} -\varepsilon L &\leq \frac{1}{2} u'(t)^2 + \frac{1}{2} u(t)^2 + \frac{1}{2} \psi(t) u(t)^2 \\ &\quad - \frac{1}{2} \int_{t_0}^{t} \psi'(\tau) u(\tau)^2 d\tau \leq \varepsilon L \end{aligned} \tag{23}$$

for all $t \geq t_0$.

Since $1 + \lambda > 0$ holds for all $t \geq t_0$, it follows from (23) that

$$\begin{aligned} \frac{1 + \lambda}{2} u(t)^2 &\leq \frac{1}{2} u'(t)^2 + \frac{1 + \lambda}{2} u(t)^2 \\ &\leq \frac{1}{2} u'(t)^2 + \frac{1}{2} (1 + \psi(t)) u(t)^2 \\ &\leq \varepsilon L + \frac{1}{2} \int_{t_0}^{t} \psi'(\tau) u(\tau)^2 d\tau \\ &\leq \varepsilon L + \frac{1}{2} \int_{t_0}^{t} \left| \psi'(\tau) \right| u(\tau)^2 d\tau \end{aligned} \tag{24}$$

or

$$u(t)^2 \leq \frac{2L\varepsilon}{1 + \lambda} + \frac{1}{1 + \lambda} \int_{t_0}^{t} \left| \psi'(\tau) \right| u(\tau)^2 d\tau \tag{25}$$

for any $t \geq t_0$.

Applying Lemma 1, we obtain

$$\begin{aligned} u(t)^2 &\leq \frac{2L\varepsilon}{1 + \lambda} \exp\left(\frac{1}{1 + \lambda} \int_{t_0}^{t} \left| \psi'(\tau) \right| d\tau \right) \\ &\leq \frac{2L\varepsilon}{1 + \lambda} \exp\left(\frac{C}{1 + \lambda} \right) \end{aligned} \tag{26}$$

for all $t \geq t_0$. Hence, it holds that

$$|u(t)| \leq \exp\left(\frac{C}{2(1 + \lambda)} \right) \sqrt{\frac{2L\varepsilon}{1 + \lambda}} \tag{27}$$

for any $t \geq t_0$. Obviously, $u_0(t) \equiv 0$ satisfies (11) and the conditions (i), (ii), and (iii) such that

$$\left| u(t) - u_0(t) \right| \leq K \sqrt{\varepsilon} \tag{28}$$

for all $t \geq t_0$, where $K = \sqrt{2L/(1 + \lambda)} \exp(C/2(1 + \lambda))$. □

If we set $\phi(t) := 1 + \psi(t)$, then the following corollary is an immediate consequence of Theorem 4.

Corollary 5. *Given constants $L > 0$ and $t_0 \geq 0$, assume that $\phi : [t_0, \infty) \rightarrow \mathbb{R}$ is a differentiable function with $C := \int_{t_0}^{\infty} |\phi'(\tau)| d\tau < \infty$ and $\lambda := \inf_{t \geq t_0} \phi(t) > 0$. If a function $u \in U(L; t_0)$ satisfies the inequality*

$$\left| u''(t) + \phi(t) u(t) \right| \leq \varepsilon \tag{29}$$

for all $t \geq t_0$ and for some $\varepsilon \geq 0$, then there exist a solution $u_0 \in U(L; t_0)$ of the differential equation (17) and a constant $K > 0$ such that

$$\left| u(t) - u_0(t) \right| \leq K \sqrt{\varepsilon} \tag{30}$$

for any $t \geq t_0$, where $K := \exp(C/2\lambda) \sqrt{2L/\lambda}$.

Example 6. Let $\phi : [0, \infty) \rightarrow \mathbb{R}$ be a constant function defined by $\phi(t) := a$ for all $t \geq 0$ and for a constant $a > 0$. Then, we have $C = \int_{0}^{\infty} |\phi'(\tau)| d\tau = 0$ and $\lambda = \inf_{t \geq 0} \phi(t) = a$.

Assume that a twice continuously differentiable function $u : [0,\infty) \to \mathbb{R}$ satisfies $u(0) = u'(0) = 0$, $\int_0^\infty |u'(\tau)|d\tau \le L$, and

$$\left|u''(t) + \phi(t)u(t)\right| = \left|u''(t) + au(t)\right| \le \varepsilon \tag{31}$$

for all $t \ge 0$ and for some $\varepsilon \ge 0$ and $L > 0$. According to Corollary 5, there exists a solution $u_0 : [0,\infty) \to \mathbb{R}$ of the differential equation, $y''(t) + ay(t) = 0$, such that

$$\left|u(t) - u_0(t)\right| \le \sqrt{\frac{2L}{a}}\varepsilon \tag{32}$$

for any $t \ge 0$.

Indeed, if we define a function $u : [0,\infty) \to \mathbb{R}$ by

$$u(t) := \frac{\alpha}{(t+1)^2}\cos\sqrt{a}t + \frac{2\alpha}{\sqrt{a}(t+1)^2}\sin\sqrt{a}t - \alpha, \tag{33}$$

where we set $\alpha = (\sqrt{a}/(a+\sqrt{a}+2))L$, then u satisfies the conditions stated in the first part of this example, as we see in the following. It follows from the definition of u that

$$\begin{aligned} u'(t) &= \left(\frac{2\alpha}{(t+1)^2} - \frac{2\alpha}{(t+1)^3}\right)\cos\sqrt{a}t \\ &\quad - \left(\frac{\sqrt{a}\alpha}{(t+1)^2} + \frac{4\alpha}{\sqrt{a}(t+1)^3}\right)\sin\sqrt{a}t \end{aligned} \tag{34}$$

and, hence, we get $u(0) = u'(0) = 0$. Moreover, we obtain

$$\begin{aligned} \left|u'(t)\right| &\le \frac{2+\sqrt{a}}{(t+1)^2}\alpha + \left(\frac{4}{\sqrt{a}} - 2\right)\frac{\alpha}{(t+1)^3}, \\ \int_0^\infty &\left|u'(\tau)\right|d\tau \\ &= \int_0^\infty \frac{2+\sqrt{a}}{(\tau+1)^2}\alpha\, d\tau + \int_0^\infty \left(\frac{4}{\sqrt{a}} - 2\right)\frac{\alpha}{(\tau+1)^3}d\tau \\ &= (2+\sqrt{a})\alpha + \left(\frac{2}{\sqrt{a}} - 1\right)\alpha = L. \end{aligned} \tag{35}$$

For any given $\varepsilon > 0$, if we choose the constant α such that $0 < \alpha \le \sqrt{a}\varepsilon/(a\sqrt{a} + 4a + 2\sqrt{a} + 12)$, then we can easily see that

$$\begin{aligned} &\left|u''(t) + au(t)\right| \\ &\quad\le \left|\left(-\frac{8}{(t+1)^3} + \frac{6}{(t+1)^4}\right)\alpha\cos\sqrt{a}t \right. \\ &\qquad \left. + \left(\frac{4\sqrt{a}}{(t+1)^3} + \frac{1}{\sqrt{a}}\frac{12}{(t+1)^4}\right)\alpha\sin\sqrt{a}t - a\alpha\right| \\ &\quad\le \left(\frac{8}{(t+1)^3} - \frac{6}{(t+1)^4}\right)\alpha \\ &\qquad + \left(\frac{4\sqrt{a}}{(t+1)^3} + \frac{1}{\sqrt{a}}\frac{12}{(t+1)^4}\right)\alpha + a\alpha \\ &\quad= \frac{a\sqrt{a} + 4a + 2\sqrt{a} + 12}{\sqrt{a}}\alpha \le \varepsilon \end{aligned} \tag{36}$$

for any $t \ge 0$.

Theorem 7. *Given constants $L > 0$ and $t_0 \ge 0$, assume that $\psi : [t_0,\infty) \to (0,\infty)$ is a monotone increasing and differentiable function. If a function $u \in U(L;t_0)$ satisfies the inequality* (18) *for all $t \ge t_0$ and for some $\varepsilon > 0$, then there exists a solution $u_0 \in U(L;t_0)$ of the differential equation* (11) *such that*

$$\left|u(t) - u_0(t)\right| \le \sqrt{\frac{2L\varepsilon}{\psi(t_0)}} \tag{37}$$

for any $t \ge t_0$.

Proof. We multiply (18) with $|u'(t)|$ to get

$$\begin{aligned} &-\varepsilon\left|u'(t)\right| \\ &\quad\le u'(t)u''(t) + u(t)u'(t) + \psi(t)u(t)u'(t) \\ &\quad\le \varepsilon\left|u'(t)\right| \end{aligned} \tag{38}$$

for all $t \ge t_0$. If we integrate each term of the last inequalities from t_0 to t, then it follows from (ii) that

$$\begin{aligned} &-\varepsilon\int_{t_0}^t \left|u'(\tau)\right|d\tau \\ &\quad\le \frac{1}{2}u'(t)^2 + \frac{1}{2}u(t)^2 + \int_{t_0}^t \psi(\tau)u(\tau)u'(\tau)d\tau \\ &\quad\le \varepsilon\int_{t_0}^t \left|u'(\tau)\right|d\tau \end{aligned} \tag{39}$$

for any $t \ge t_0$.

Integrating by parts, the last inequalities together with (iii) yield

$$\begin{aligned} -\varepsilon L &\le \frac{1}{2}u'(t)^2 + \frac{1}{2}u(t)^2 + \frac{1}{2}\psi(t)u(t)^2 \\ &\quad - \frac{1}{2}\int_{t_0}^t \psi'(\tau)u(\tau)^2 d\tau \le \varepsilon L \end{aligned} \tag{40}$$

for all $t \ge t_0$. Then we have

$$\begin{aligned} \frac{1}{2}\psi(t)u(t)^2 &\le \frac{1}{2}\int_{t_0}^t \psi'(\tau)u(\tau)^2 d\tau + \varepsilon L \\ &\le \varepsilon L + \int_{t_0}^t \frac{\psi'(\tau)}{\psi(\tau)}u(\tau)^2\frac{\psi(\tau)}{2}d\tau \end{aligned} \tag{41}$$

for any $t \ge t_0$.

Applying Lemma 1, we obtain

$$\frac{1}{2}\psi(t)u(t)^2 \le \varepsilon L\exp\left(\int_{t_0}^t \frac{\psi'(\tau)}{\psi(\tau)}d\tau\right) = \varepsilon L\frac{\psi(t)}{\psi(t_0)} \tag{42}$$

for all $t \ge t_0$, since $\psi : [t_0,\infty) \to (0,\infty)$ is a monotone increasing function. Hence, it holds that

$$|u(t)| \le \sqrt{\frac{2L\varepsilon}{\psi(t_0)}} \tag{43}$$

for any $t \ge t_0$. Obviously, $u_0(t) \equiv 0$ satisfies (11), $u_0 \in U(L;t_0)$, and the inequality (37) for all $t \ge t_0$. □

Corollary 8. *Given constants $L > 0$ and $t_0 \geq 0$, assume that $\phi : [t_0, \infty) \to (1, \infty)$ is a monotone increasing and differentiable function with $\phi(t_0) = 2$. If a function $u \in U(L; t_0)$ satisfies the inequality*

$$\left|u''(t) + \phi(t) u(t)\right| \leq \varepsilon \tag{44}$$

for all $t \geq t_0$ and for some $\varepsilon > 0$, then there exists a solution $u_0 \in U(L; t_0)$ of the differential equation (17) such that

$$\left|u(t) - u_0(t)\right| \leq \sqrt{2L\varepsilon} \tag{45}$$

for any $t \geq t_0$.

If we set $\phi(t) := -\psi(t)$, then the following corollary is an immediate consequence of Theorem 7.

Corollary 9. *Given constants $L > 0$ and $t_0 \geq 0$, assume that $\phi : [t_0, \infty) \to (-\infty, 0)$ is a monotone decreasing and differentiable function with $\phi(t_0) = -1$. If a function $u \in U(L; t_0)$ satisfies the inequality*

$$\left|u''(t) + (1 - \phi(t)) u(t)\right| \leq \varepsilon \tag{46}$$

for all $t \geq t_0$ and for some $\varepsilon > 0$, then there exists a solution $u_0 \in U(L; t_0)$ of the differential equation

$$u''(t) + (1 - \phi(t)) u(t) = 0 \tag{47}$$

such that

$$\left|u(t) - u_0(t)\right| \leq \sqrt{2L\varepsilon} \tag{48}$$

for any $t \geq t_0$.

Example 10. Let $\phi : [0, \infty) \to (-\infty, 0)$ be a monotone decreasing function defined by $\phi(t) := e^{-t} - 2$ for all $t \geq 0$. Then, we have $\phi(0) = -1$. Assume that a twice continuously differentiable function $u : [0, \infty) \to \mathbb{R}$ satisfies $u(0) = u'(0) = 0$, $\int_0^\infty |u'(\tau)| d\tau \leq L$, and

$$\left|u''(t) + (1 - \phi(t)) u(t)\right| = \left|u''(t) + \left(3 - e^{-t}\right) u(t)\right| \leq \varepsilon \tag{49}$$

for all $t \geq 0$ and for some $\varepsilon > 0$ and $L > 0$. According to Corollary 9, there exists a solution $u_0 : [0, \infty) \to \mathbb{R}$ of the differential equation, $y''(t) + (3 - e^{-t}) y(t) = 0$, such that

$$\left|u(t) - u_0(t)\right| \leq \sqrt{2L\varepsilon} \tag{50}$$

for any $t \geq 0$.

Indeed, if we define a function $u : [0, \infty) \to \mathbb{R}$ by

$$u(t) := \frac{\alpha}{(t+1)^3} \sin t + \frac{1}{2} \frac{\alpha}{(t+1)^2} \cos t - \frac{\alpha}{2}, \tag{51}$$

where α is a real number with $|\alpha| \leq (2/43)\varepsilon$, then u satisfies the conditions stated in the first part of this example, as we see in the following. It follows from the definition of u that

$$u'(t) = -\frac{3\alpha}{(t+1)^4} \sin t - \frac{1}{2} \frac{\alpha}{(t+1)^2} \sin t \tag{52}$$

and, hence, we get $u(0) = u'(0) = 0$. Moreover, we obtain

$$\begin{aligned} \left|u'(t)\right| &\leq \frac{3|\alpha|}{(t+1)^4} + \frac{1}{2} \frac{|\alpha|}{(t+1)^2}, \\ \int_0^\infty \left|u'(\tau)\right| d\tau &\leq \int_0^\infty \frac{3|\alpha|}{(\tau+1)^4} d\tau + \int_0^\infty \frac{1}{2} \frac{|\alpha|}{(\tau+1)^2} d\tau \\ &=: L < \infty. \end{aligned} \tag{53}$$

We can see that

$$\begin{aligned} &\left|u''(t) + \left(3 - e^{-t}\right) u(t)\right| \\ &\leq \left| \frac{12\alpha}{(t+1)^5} \sin t - \frac{3\alpha}{(t+1)^4} \cos t \right. \\ &\qquad + \left(4 - e^{-t}\right) \frac{\alpha}{(t+1)^3} \sin t \\ &\qquad \left. + \frac{2 - e^{-t}}{2} \frac{\alpha}{(t+1)^2} \cos t - \frac{3 - e^{-t}}{2} \alpha \right| \\ &\leq \frac{12|\alpha|}{(t+1)^5} + \frac{3|\alpha|}{(t+1)^4} + \frac{4|\alpha|}{(t+1)^3} + \frac{|\alpha|}{(t+1)^2} + \frac{3}{2}|\alpha| \\ &\leq \frac{43}{2}|\alpha| \leq \varepsilon \end{aligned} \tag{54}$$

for any $t \geq 0$.

Now, we investigate the Hyers-Ulam stability of the nonlinear differential equation

$$u''(t) + F(t, u(t)) = 0. \tag{55}$$

Theorem 11. *Given constants $L > 0$ and $t_0 \geq 0$, assume that $F : [t_0, \infty) \times \mathbb{R} \to (0, \infty)$ is a function satisfying $F'(t, u(t))/F(t, u(t)) > 0$ and $F(t, 0) = 1$ for all $t \geq t_0$ and $u \in U(L; t_0)$. If a function $u : [t_0, \infty) \to [0, \infty)$ satisfies $u \in U(L; t_0)$ and the inequality*

$$\left|u''(t) + F(t, u(t))\right| \leq \varepsilon \tag{56}$$

for all $t \geq t_0$ and for some $\varepsilon > 0$, then there exists a solution $u_0 : [t_0, \infty) \to [0, \infty)$ of the differential equation (55) such that

$$\left|u(t) - u_0(t)\right| \leq L\varepsilon \tag{57}$$

for any $t \geq t_0$.

Proof. We multiply (56) with $|u'(t)|$ to get

$$-\varepsilon \left|u'(t)\right| \leq u'(t) u''(t) + F(t, u(t)) u'(t) \leq \varepsilon \left|u'(t)\right| \tag{58}$$

for all $t \geq t_0$. If we integrate each term of the last inequalities from t_0 to t, then it follows from (ii) that

$$\begin{aligned} -\varepsilon \int_{t_0}^t \left|u'(\tau)\right| d\tau &\leq \frac{1}{2} u'(t)^2 + \int_{t_0}^t F(\tau, u(\tau)) u'(\tau) d\tau \\ &\leq \varepsilon \int_{t_0}^t \left|u'(\tau)\right| d\tau \end{aligned} \tag{59}$$

for any $t \geq t_0$.

Integrating by parts and using (iii), the last inequalities yield

$$-\varepsilon L \leq \frac{1}{2}u'(t)^2 + F(t,u(t))\,u(t) - \int_{t_0}^{t} F'(\tau,u(\tau))\,u(\tau)\,d\tau \leq \varepsilon L \tag{60}$$

for all $t \geq t_0$. Then we have

$$\begin{aligned} F(t,u(t))\,u(t) &\leq \varepsilon L + \int_{t_0}^{t} F'(\tau,u(\tau))\,u(\tau)\,d\tau \\ &\leq \varepsilon L + \int_{t_0}^{t} \frac{F'(\tau,u(\tau))}{F(\tau,u(\tau))} F(\tau,u(\tau))\,u(\tau)\,d\tau \end{aligned} \tag{61}$$

for any $t \geq t_0$.

Applying Lemma 1, we obtain

$$\begin{aligned} F(t,u(t))\,u(t) &\leq \varepsilon L \exp\left(\int_{t_0}^{t} \frac{F'(\tau,u(\tau))}{F(\tau,u(\tau))}d\tau\right) \\ &= \varepsilon L F(t,u(t)) \end{aligned} \tag{62}$$

for all $t \geq t_0$. Hence, it holds that $|u(t)| \leq L\varepsilon$ for any $t \geq t_0$. Obviously, $u_0(t) \equiv 0$ satisfies (55) and $u_0 \in U(L;t_0)$ such that

$$\left|u(t) - u_0(t)\right| \leq L\varepsilon \tag{63}$$

for all $t \geq t_0$. □

In the following theorem, we investigate the Hyers-Ulam stability of the Emden-Fowler nonlinear differential equation of second order

$$u''(t) + h(t)\,u(t)^{\alpha} = 0 \tag{64}$$

for the case where α is a positive odd integer.

Theorem 12. *Given constants $L > 0$ and $t_0 \geq 0$, assume that $h : [t_0,\infty) \to (0,\infty)$ is a differentiable function. Let α be an odd integer larger than 0. If a function $u : [t_0,\infty) \to [0,\infty)$ satisfies $u \in U(L;t_0)$ and the inequality*

$$\left|u''(t) + h(t)\,u(t)^{\alpha}\right| \leq \varepsilon \tag{65}$$

for all $t \geq t_0$ and for some $\varepsilon > 0$, then there exists a solution $u_0 : [t_0,\infty) \to [0,\infty)$ of the differential equation (64) such that

$$\left|u(t) - u_0(t)\right| \leq \left(\frac{\beta L\varepsilon}{h(t_0)}\right)^{1/\beta} \tag{66}$$

for any $t \geq t_0$, where $\beta := \alpha + 1$.

Proof. We multiply (65) with $|u'(t)|$ to get

$$-\varepsilon\left|u'(t)\right| \leq u'(t)\,u''(t) + h(t)\,u(t)^{\alpha}u'(t) \leq \varepsilon\left|u'(t)\right| \tag{67}$$

for all $t \geq t_0$. If we integrate each term of the last inequalities from t_0 to t, then it follows from (ii) that

$$\begin{aligned} -\varepsilon\int_{t_0}^{t}\left|u'(\tau)\right|d\tau &\leq \frac{1}{2}u'(t)^2 + \int_{t_0}^{t} h(\tau)\,u(\tau)^{\alpha}u'(\tau)\,d\tau \\ &\leq \varepsilon\int_{t_0}^{t}\left|u'(\tau)\right|d\tau \end{aligned} \tag{68}$$

for any $t \geq t_0$.

Integrating by parts and using (iii), the last inequalities yield

$$-\varepsilon L \leq \frac{1}{2}u'(t)^2 + h(t)\frac{u(t)^{\alpha+1}}{\alpha+1} - \int_{t_0}^{t} h'(\tau)\frac{u(\tau)^{\alpha+1}}{\alpha+1}d\tau \leq \varepsilon L \tag{69}$$

for all $t \geq t_0$. Then we have

$$\begin{aligned} h(t)\frac{u(t)^{\alpha+1}}{\alpha+1} &\leq \varepsilon L + \int_{t_0}^{t} h'(\tau)\frac{u(\tau)^{\alpha+1}}{\alpha+1}d\tau \\ &\leq \varepsilon L + \int_{t_0}^{t} \frac{h'(\tau)}{h(\tau)}h(\tau)\frac{u(\tau)^{\alpha+1}}{\alpha+1}d\tau \end{aligned} \tag{70}$$

for any $t \geq t_0$.

Applying Lemma 1, we obtain

$$h(t)\frac{u(t)^{\alpha+1}}{\alpha+1} \leq \varepsilon L \exp\left(\int_{t_0}^{t}\frac{h'(\tau)}{h(\tau)}d\tau\right) \leq \varepsilon L\frac{h(t)}{h(t_0)} \tag{71}$$

for all $t \geq t_0$, from which we have

$$u(t)^{\alpha+1} \leq \frac{(\alpha+1)\,L\varepsilon}{h(t_0)} \tag{72}$$

for all $t \geq t_0$. Hence, it holds that

$$|u(t)| \leq \left(\frac{\beta L\varepsilon}{h(t_0)}\right)^{1/\beta} \tag{73}$$

for any $t \geq t_0$, where we set $\beta = \alpha + 1$. Obviously, $u_0(t) \equiv 0$ satisfies (64) and $u_0 \in U(L;t_0)$. Moreover, we get

$$\left|u(t) - u_0(t)\right| \leq \left(\frac{\beta L\varepsilon}{h(t_0)}\right)^{1/\beta} \tag{74}$$

for all $t \geq t_0$. □

Given constants $L \geq 0$, $M > 0$, and $t_0 \geq 0$, let $U(L;M;t_0)$ denote the set of all functions $u : [t_0,\infty) \to \mathbb{R}$ with the following properties:

(i′) u is twice continuously differentiable;

(ii′) $u(t_0) = u'(t_0) = 0$;

(iii′) $|u(t)| \leq L$ for all $t \geq t_0$;

(iv′) $\int_{t_0}^{\infty}|u'(\tau)|d\tau \leq M$ for all $t \geq t_0$.

We now investigate the Hyers-Ulam stability of the differential equation of the form

$$u''(t) + u(t) + h(t)\, u(t)^{\beta} = 0, \tag{75}$$

where β is a positive odd integer.

Theorem 13. *Given constants $L \geq 0$, $M > 0$, and $t_0 \geq 0$, assume that $h : [t_0, \infty) \to [0, \infty)$ is a function satisfying $C := \int_{t_0}^{\infty} |h'(\tau)| d\tau < \infty$. Let β be an odd integer larger than 0. If a function $u \in U(L; M; t_0)$ satisfies the inequality*

$$\left| u''(t) + u(t) + h(t)\, u(t)^{\beta} \right| \leq \varepsilon \tag{76}$$

for all $t \geq t_0$ and for some $\varepsilon > 0$, then there exists a solution $u_0 : [t_0, \infty) \to \mathbb{R}$ of the differential equation (75) such that

$$\left| u(t) - u_0(t) \right| \leq \sqrt{2M\varepsilon} \exp\left(\frac{CL^{\beta-1}}{\beta+1} \right) \tag{77}$$

for any $t \geq t_0$.

Proof. We multiply (76) with $|u'(t)|$ to get

$$\begin{aligned} -\varepsilon \left| u'(t) \right| &\leq u'(t)\, u''(t) + u(t)\, u'(t) + h(t)\, u(t)^{\beta} u'(t) \\ &\leq \varepsilon \left| u'(t) \right| \end{aligned} \tag{78}$$

for all $t \geq t_0$. If we integrate each term of the last inequalities from t_0 to t, then it follows from (ii$'$) that

$$\begin{aligned} &-\varepsilon \int_{t_0}^{t} \left| u'(\tau) \right| d\tau \\ &\quad \leq \frac{1}{2} u'(t)^2 + \frac{1}{2} u(t)^2 + \int_{t_0}^{t} h(\tau)\, u(\tau)^{\beta} u'(\tau)\, d\tau \\ &\quad \leq \varepsilon \int_{t_0}^{t} \left| u'(\tau) \right| d\tau \end{aligned} \tag{79}$$

for any $t \geq t_0$.

Integrating by parts and using (ii$'$) and (iv$'$), the last inequalities yield

$$\begin{aligned} -\varepsilon M \leq \frac{1}{2} u'(t)^2 + \frac{1}{2} u(t)^2 + h(t) \frac{1}{\beta+1} u(t)^{\beta+1} \\ - \frac{1}{\beta+1} \int_{t_0}^{t} h'(\tau)\, u(\tau)^{\beta+1} d\tau \leq \varepsilon M \end{aligned} \tag{80}$$

for all $t \geq t_0$. Then it follows from (iii$'$) that

$$\begin{aligned} \frac{1}{2} u(t)^2 &\leq \varepsilon M + \frac{1}{\beta+1} \int_{t_0}^{t} h'(\tau)\, u(\tau)^{\beta+1} d\tau \\ &\leq \varepsilon M + \frac{2}{\beta+1} \int_{t_0}^{t} \frac{1}{2} u(\tau)^2 h'(\tau)\, u(\tau)^{\beta-1} d\tau \\ &\leq \varepsilon M + \frac{2}{\beta+1} \int_{t_0}^{t} \frac{1}{2} u(\tau)^2 \left| h'(\tau) \right| \left| u(\tau) \right|^{\beta-1} d\tau \\ &\leq \varepsilon M + \frac{2L^{\beta-1}}{\beta+1} \int_{t_0}^{t} \frac{1}{2} u(\tau)^2 \left| h'(\tau) \right| d\tau \end{aligned} \tag{81}$$

for any $t \geq t_0$.

Applying Lemma 1, we obtain

$$\begin{aligned} \frac{1}{2} u(t)^2 &\leq \varepsilon M \exp\left(\int_{t_0}^{t} \frac{2L^{\beta-1}}{\beta+1} \left| h'(\tau) \right| d\tau \right) \\ &\leq \varepsilon M \exp\left(\frac{2CL^{\beta-1}}{\beta+1} \right) \end{aligned} \tag{82}$$

for all $t \geq t_0$. Hence, it holds that

$$|u(t)| \leq \sqrt{2M\varepsilon} \exp\left(\frac{CL^{\beta-1}}{\beta+1} \right) \tag{83}$$

for any $t \geq t_0$. Obviously, $u_0(t) \equiv 0$ satisfies (75) and $u_0 \in U(L; M; t_0)$. Furthermore, we get

$$\left| u(t) - u_0(t) \right| \leq \sqrt{2M\varepsilon} \exp\left(\frac{CL^{\beta-1}}{\beta+1} \right) \tag{84}$$

for all $t \geq t_0$.

□

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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