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Research Article

Iterative Computation for Solving the Variational Inequality and the Generalized Equilibrium Problem

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An iterative algorithm for solving the variational inequality and the generalized equilibrium problem has been introduced. Convergence result is given.

1. Introduction

Let $\mathbb{H}$ be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$, respectively. Let $\mathbb{C}$ be a nonempty closed convex subset of $\mathbb{H}$. Recall that a mapping $\mathbb{A}:\mathbb{C}\to\mathbb{H}$ is said to be

(i) nonexpansive $\Leftrightarrow \|\mathbb{A}u-\mathbb{A}v\|\le\|u-v\|$, for all $u,v\in\mathbb{C}$ (we use $\mathrm{Fix}(\mathbb{A})$ to denote the set of the fixed points of $\mathbb{A}$);

(ii) firmly nonexpansive $\Leftrightarrow \|\mathbb{A}u-\mathbb{A}v\|^2\le\langle u-v,\mathbb{A}u-\mathbb{A}v\rangle$, for all $u,v\in\mathbb{C}$;

(iii) L-Lipschitz $\Leftrightarrow$ there exists a constant $L>0$ such that $\|\mathbb{A}u-\mathbb{A}v\|\le L\|u-v\|$, for all $u,v\in\mathbb{C}$;

(iv) monotone $\Leftrightarrow \langle u-v,\mathbb{A}u-\mathbb{A}v\rangle\ge 0$, for all $u,v\in\mathbb{C}$;

(v) strongly monotone $\Leftrightarrow$ there exists a constant $\nu>0$ such that $\langle u-v,\mathbb{A}u-\mathbb{A}v\rangle\ge\nu\|u-v\|^2$, for all $u,v\in\mathbb{C}$;

(vi) inverse strongly monotone $\Leftrightarrow$ there exists $\zeta>0$ such that $\langle u-v,\mathbb{A}u-\mathbb{A}v\rangle\ge\zeta\|\mathbb{A}u-\mathbb{A}v\|^2$, for all $u,v\in\mathbb{C}$;

(vii) ζ-inverse strongly ϕ-monotone $\Leftrightarrow \langle\phi(u)-\phi(v),\mathbb{A}u-\mathbb{A}v\rangle\ge\zeta\|\mathbb{A}u-\mathbb{A}v\|^2$, for all $u,v\in\mathbb{C}$ and for some $\zeta>0$, where $\phi:\mathbb{C}\to\mathbb{C}$ is a nonlinear mapping.

If $\mathbb{A}$ is a multivalued mapping of H into $2^{\mathbb{H}}$, then $\mathbb{A}$ is said to be a monotone operator on $\mathbb{H}\Leftrightarrow\langle x-y,u-v\rangle\ge 0$, for all $x,y\in\mathrm{dom}(\mathbb{A})$, $u\in\mathbb{A}x$, and $v\in\mathbb{A}y$. A monotone operator $\mathbb{A}$ on $\mathbb{H}$ is said to be maximal if and only if its graph is not strictly contained in the graph of any other monotone operator on $\mathbb{H}$.

Let $\mathbb{A}$, $\mathbb{B}$, and ϕ be three nonlinear mappings on $\mathbb{C}$. Let $\theta:\mathbb{C}\times\mathbb{C}\to\mathbb{R}$ be a bifunction. Recall that the equilibrium problem is to find $x^\dagger\in\mathbb{C}$ such that

$$\theta\left(x^\dagger,y\right)+\left\langle\mathbb{A}x^\dagger,y-x^\dagger\right\rangle\ge 0,\quad\forall y\in\mathbb{C}.\tag{1}$$

The solution set of (1) is denoted by $EP(\theta,\mathbb{A})$. Now we know that the equilibrium theory provides us a natural, novel, and unified framework to study a wide class of problems arising in economics, finance, transportation, network and structural analysis, elasticity, and optimization. For related works, please refer to [1–3] and the references therein.

Recall also that the variational inequality problem is to find $u\in\mathbb{C}$, $\phi(u)\in\mathbb{C}$ such that

$$\left\langle\mathbb{B}u,\phi(v)-\phi(u)\right\rangle\ge 0,\quad\forall\phi(v)\in\mathbb{C}.\tag{2}$$

The solution set of (2) is denoted by $VI(\mathbb{B},\phi,\mathbb{C})$. It is well known that variational inequality theory has emerged as an important tool in studying a wide class of obstacle, unilateral, free, moving, and equilibrium problems arising in several branches of pure and applied sciences in a unified and general framework. Several numerical methods have been developed for solving variational inequalities and related optimization problems. For related works, please see [4–14]. Noor [15] introduced an iterative scheme and studied the approximate solutions of variational inclusion in Hilbert spaces. Glowinski and Le Tallec [16] used the iterative schemes to find the approximate solutions of the elastoviscoplasticity problem, liquid crystal theory, and eigenvalue computation. In 1998, Haubruge et al. [17] studied the convergence analysis of the iterative schemes of Glowinski and Le Tallec and applied these schemes to obtain new splitting-type algorithms for solving variational inequalities, separable convex programming, and minimization of a sum of convex functions.

Our main purpose in the present paper is to solve the following equilibrium problem and variational inequality problem: finding a point $x^{\dagger}$ such that

$$x^{\dagger} \in VI(\mathbb{B}, \phi, \mathbb{C}), \qquad \phi(x^{\dagger}) \in EP(\theta, \mathbb{A}). \tag{3}$$

Our main motivations are inspired by the following two reasons.

Firstly, it is still an interesting topic for solving the variational inequality problem and the equilibrium problem based on their applications in science and engineering. Secondly, the split problem of finding a point $\tilde{x}$ such that

$$x^{\dagger} \in \mathbb{C}, \qquad \tilde{\phi}(x^{\dagger}) \in \mathbb{D} \tag{4}$$

has received much attention. For related works, please refer to [18–20]. However, we observe that the involved operator $\tilde{\phi}$ in (4) is a bounded liner operator. In this paper, we devote to study the problem (3), where the transformation ϕ is a nonlinear mapping. For this purpose, we introduce a new iterative algorithm. Consequently, strong convergence analysis is demonstrated.

2. Preliminaries

In this section, we recall some useful lemmas.

Recall that the metric projection $\text{proj}_{\mathbb{C}} : \mathbb{H} \to \mathbb{C}$ satisfies $\|u - \text{proj}_{\mathbb{C}} u\| = \inf\{\|u - v\| : v \in \mathbb{C}\}$. The metric projection $\text{proj}_{\mathbb{C}}$ is a typical firmly nonexpansive mapping. The characteristic inequality of the projection is $\langle u - \text{proj}_{\mathbb{C}} u, v - \text{proj}_{\mathbb{C}} u\rangle \le 0$, for all $u \in \mathbb{H}$, $v \in \mathbb{C}$.

Assume that $\theta : \mathbb{C}\times\mathbb{C} \to \mathbb{R}$ is a bifunction which satisfies the following conditions:

(C1) $\theta(u, u) = 0$, for all $u \in \mathbb{C}$;

(C2) θ is monotone; that is, $\theta(u, v) + \theta(v, u) \le 0$, for all $u, v \in \mathbb{C}$;

(C3) for each $u, v, w \in \mathbb{C}$, $\lim_{t\downarrow 0}\theta(tw + (1-t)u, v) \le \theta(u, v)$;

(C4) for each $u \in \mathbb{C}$, $v \mapsto \theta(u, v)$ is convex and lower semicontinuous.

Lemma 1 (see [2]). *Let $\mathbb{C}$ be a nonempty closed convex subset of a real Hilbert space $\mathbb{H}$. Let $\theta : \mathbb{C}\times\mathbb{C} \to \mathbb{R}$ be a bifunction which satisfies conditions (C1)–(C4). Let $\tau > 0$ and $u \in \mathbb{C}$. Then, there exists $w \in \mathbb{C}$ such that*

$$\theta(w, v) + \frac{1}{\tau}\langle v - w, w - u\rangle \ge 0, \quad \forall v \in \mathbb{C}. \tag{5}$$

Further, if $\mathbb{S}_{\tau}(u) = \{w \in \mathbb{C} : \theta(w, v) + (1/\tau)\langle v - w, w - u\rangle \ge 0$, for all $v \in \mathbb{C}\}$, then the following hold:

(a) *$\mathbb{S}_{\tau}$ is single-valued and $\mathbb{S}_{\tau}$ is firmly nonexpansive;*

(b) *$EP(\theta)$ is closed and convex and $EP(\theta) = \text{Fix}(\mathbb{S}_{\tau})$.*

Lemma 2 (see [21]). *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space $\mathbb{X}$ and let $\{\eta_n\}$ be a sequence in $[0, 1]$ with $0 < \liminf_{n\to\infty}\eta_n \le \limsup_{n\to\infty}\eta_n < 1$. Suppose that $x_{n+1} = (1-\eta_n)y_n + \eta_n x_n$, for all $n \ge 0$, and $\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \le 0$. Then, $\lim_{n\to\infty}\|y_n - x_n\| = 0$.*

Lemma 3 (see [22]). *Assume that the sequence $\{a_n\}$ satisfies $a_n \ge 0$ and $a_{n+1} \le (1 - \nu_n)a_n + \varsigma_n\nu_n$, where $\{\nu_n\}$ is a sequence in $(0, 1)$ and $\{\varsigma_n\}$ is a sequence such that $\sum_{n=1}^{\infty}\nu_n = \infty$ and $\limsup_{n\to\infty}\varsigma_n \le 0$ (or $\sum_{n=1}^{\infty}|\varsigma_n\nu_n| < \infty$). Then, $\lim_{n\to\infty}a_n = 0$.*

3. Main Results

Let $\mathbb{C}$ be a nonempty closed convex subset of a real Hilbert space $\mathbb{H}$. Let $\mathbb{A} : \mathbb{C} \to \mathbb{H}$ be an η-inverse strongly monotone mapping. Let $\phi : \mathbb{C} \to \mathbb{C}$ be a weakly continuous and ν-strongly monotone mapping such that $R(\phi) = \mathbb{C}$. Let $\mathbb{B} : \mathbb{C} \to \mathbb{H}$ be a ζ-inverse strongly ϕ-monotone mapping. Let $\varrho : \mathbb{C} \to \mathbb{H}$ be an L-Lipschitz continuous mapping. Let $\theta : \mathbb{C}\times\mathbb{C} \to \mathbb{R}$ be a bifunction which satisfies (C1)–(C4) in the above section. Let $\{\zeta_n\} \subset [0, 1]$, $\{\eta_n\} \subset [0, 1]$, $\{\varpi_n\} \subset (0, \infty)$, and $\{\tau_n\} \subset (0, \infty)$ be four real number sequences and let $\varsigma > 0$ be a constant.

We use Δ to denote the solution set of (3). In order to solve (3), we introduce the following three-step algorithm.

Algorithm 4. Let $x_0 \in \mathbb{C}$ be an initial guess. Define the sequence $\{x_n\}$ as follows:

$$\begin{gathered} u_n = \text{proj}_{\mathbb{C}}[\phi(x_n) - \varpi_n\mathbb{B}x_n], \quad n \ge 0, \\ \theta(z_n, y) + \langle\mathbb{A}u_n, y - z_n\rangle + \frac{1}{\tau_n}\langle y - z_n, z_n - u_n\rangle \ge 0, \quad \forall y \in \mathbb{C}, \\ \phi(x_{n+1}) = \eta_n\phi(x_n) + (1-\eta_n)\,\text{proj}_{\mathbb{C}}[\zeta_n\varsigma\varrho(x_n) + (1-\zeta_n)z_n], \quad n \ge 0. \end{gathered} \tag{6}$$

Theorem 5. *Suppose that $\Delta \ne \emptyset$. Assume that the following conditions are satisfied:*

(r1) *$\varpi_n \in (a_1, a_2) \subset (0, 2\zeta)$ and $\lim_{n\to\infty}(\varpi_{n+1} - \varpi_n) = 0$;*

(r2) *$\tau_n \in (a_3, a_4) \subset (0, 2\eta)$ and $\lim_{n\to\infty}(\tau_{n+1} - \tau_n) = 0$;*

(r3) *$\eta_n \in [a_5, a_6] \subset (0, 1)$;*

(r4) *$\lim_{n\to\infty}\zeta_n = 0$ and $\sum_n\zeta_n = \infty$;*

(r5) *$\nu \in (L\varsigma, 2\zeta)$.*

Then, the sequence $\{x_n\}$ generated by (6) converges strongly to $x^* \in \Delta$ which solves the following variational inequality:

$$\langle\varsigma\varrho(x^*) - \phi(x^*), \phi(x) - \phi(x^*)\rangle \le 0, \quad \forall x \in \Delta. \tag{7}$$

Proof. First of all, we prove that the solution of the variational inequality (7) is unique. In fact, if $\tilde{x} \in \Delta$ also solves (7), then we get

$$\langle \varsigma\varrho(x^*) - \phi(x^*), \phi(\tilde{x}) - \phi(x^*) \rangle \leq 0,$$
$$\langle \varsigma\varrho(\tilde{x}) - \phi(\tilde{x}), \phi(x^*) - \phi(\tilde{x}) \rangle \leq 0. \quad (8)$$

It follows that

$$\langle \varsigma\varrho(\tilde{x}) - \phi(\tilde{x}) - \varsigma\varrho(x^*) + \phi(x^*), \phi(x^*) - \phi(\tilde{x}) \rangle \leq 0. \quad (9)$$

So,

$$\begin{aligned} \|\phi(x^*) - \phi(\tilde{x})\|^2 &\leq \varsigma \langle \varrho(x^*) - \varrho(\tilde{x}), \phi(x^*) - \phi(\tilde{x}) \rangle \\ &\leq \varsigma \|\varrho(x^*) - \varrho(\tilde{x})\| \|\phi(x^*) - \phi(\tilde{x})\|, \end{aligned} \quad (10)$$

which implies that

$$\|\phi(x^*) - \phi(\tilde{x})\| \leq \varsigma \|\varrho(x^*) - \varrho(\tilde{x})\|. \quad (11)$$

Since ϕ is ν-strongly monotone, we have

$$\begin{aligned} \nu\|x^* - \tilde{x}\|^2 &\leq \langle \phi(x^*) - \phi(\tilde{x}), x^* - \tilde{x} \rangle \\ &\leq \|\phi(x^*) - \phi(\tilde{x})\| \|x^* - \tilde{x}\|. \end{aligned} \quad (12)$$

Thus,

$$\begin{aligned} \nu \|x^* - \tilde{x}\| &\leq \|\phi(x^*) - \phi(\tilde{x})\| \\ &\leq \varsigma \|\varrho(x^*) - \varrho(\tilde{x})\| \leq \varsigma L \|x^* - \tilde{x}\|. \end{aligned} \quad (13)$$

Since $\varsigma L < \nu$, we deduce the contradiction. Therefore, $x^* = \tilde{x}$. So, the solution of variational inequality (7) is unique.

Let $x^\# \in \Delta$. Hence, $x^\# \in VI(\mathbb{B}, \phi, \mathbb{C})$ and $\phi(x^\#) \in EP(\theta, \mathbb{A})$. Note that

$$x^\# \in VI(\mathbb{B}, \phi, \mathbb{C}) \Longleftrightarrow \phi(x^\#) = \text{proj}_{\mathbb{C}}(\phi(x^\#) - \nu\mathbb{B}x^\#), \quad \forall \nu > 0. \quad (14)$$

Since $\varpi_n > 0$, we have $\phi(x^\#) = \text{proj}_{\mathbb{C}}[\phi(x^\#) - \varpi_n\mathbb{B}x^\#]$, for all $n \geq 0$. For $u, v \in \mathbb{C}$, we have

$$\begin{aligned} &\|(\phi(u) - \varpi_n\mathbb{B}u) - (\phi(v) - \varpi_n\mathbb{B}v)\|^2 \\ &\quad = \|\phi(u) - \phi(v)\|^2 - 2\varpi_n \langle \mathbb{B}u - \mathbb{B}v, \phi(u) - \phi(v) \rangle \\ &\qquad + \varpi_n^2\|\mathbb{B}u - \mathbb{B}v\|^2 \\ &\quad \leq \|\phi(u) - \phi(v)\|^2 - 2\varpi_n\zeta\|\mathbb{B}u - \mathbb{B}v\|^2 + \varpi_n^2\|\mathbb{B}u - \mathbb{B}v\|^2 \\ &\quad \leq \|\phi(u) - \phi(v)\|^2 + \varpi_n(\varpi_n - 2\zeta)\|\mathbb{B}u - \mathbb{B}v\|^2. \end{aligned} \quad (15)$$

Hence,

$$\|(\phi(u) - \varpi_n\mathbb{B}u) - (\phi(v) - \varpi_n\mathbb{B}v)\| \leq \|\phi(u) - \phi(v)\|, \quad (16)$$

for all $u, v \in \mathbb{C}$. Thus,

$$\begin{aligned} &\|u_n - \phi(x^\#)\| \\ &\quad = \|\text{proj}_{\mathbb{C}}[\phi(x_n) - \varpi_n\mathbb{B}x_n] - \text{proj}_{\mathbb{C}}[\phi(x^\#) - \varpi_n\mathbb{B}x^\#]\| \\ &\quad \leq \|(\phi(x_n) - \varpi_n\mathbb{B}x_n) - (\phi(x^\#) - \varpi_n\mathbb{B}x^\#)\| \\ &\quad \leq \|\phi(x_n) - \phi(x^\#)\|. \end{aligned} \quad (17)$$

From (6), we have $z_n = \mathbb{S}_{\tau_n}(I - \tau_n\mathbb{A})u_n$, for all $n \geq 0$. Noting that $\phi(x^\#) \in EP(\theta, \mathbb{A})$, we deduce $\phi(x^\#) = \mathbb{S}_{\tau_n}(I - \tau_n\mathbb{A})\phi(x^\#)$, for all $n \geq 0$. It follows from (17) that

$$\begin{aligned} &\|\phi(x_{n+1}) - \phi(x^\#)\| \\ &\leq \eta_n \|\phi(x_n) - \phi(x^\#)\| \\ &\quad + (1 - \eta_n) \|\text{proj}_{\mathbb{C}}[\zeta_n\varsigma\varrho(x_n) + (1 - \zeta_n) z_n] - \phi(x^\#)\| \\ &\leq \eta_n \|\phi(x_n) - \phi(x^\#)\| \\ &\quad + (1 - \eta_n) \|\zeta_n(\varsigma\varrho(x_n) - \phi(x^\#)) + (1 - \zeta_n)(z_n - \phi(x^\#))\| \\ &\leq \eta_n \|\phi(x_n) - \phi(x^\#)\| \\ &\quad + (1 - \eta_n) [\zeta_n \|\varsigma\varrho(x_n) - \varsigma\varrho(x^\#)\| + \zeta_n \|\varsigma\varrho(x^\#) - \phi(x^\#)\| \\ &\qquad + (1 - \zeta_n) \\ &\qquad \times \|\mathbb{S}_{\tau_n}(I - \tau_n\mathbb{A}) u_n - \mathbb{S}_{\tau_n}(I - \tau_n\mathbb{A}) \phi(x^\#)\|] \\ &\leq \eta_n \|\phi(x_n) - \phi(x^\#)\| \\ &\quad + (1 - \eta_n) [\zeta_n\varsigma L \|x_n - x^\#\| + \zeta_n \|\varsigma\varrho(x^\#) - \phi(x^\#)\| \\ &\qquad + (1 - \zeta_n) \|u_n - \phi(x^\#)\|] \\ &\leq \eta_n \|\phi(x_n) - \phi(x^\#)\| \\ &\quad + (1 - \eta_n) \left[\frac{\zeta_n\varsigma L}{\nu} \|\phi(x_n) - \phi(x^\#)\| + \zeta_n \|\varsigma\varrho(x^\#) - \phi(x^\#)\| \right. \\ &\qquad \left. + (1 - \zeta_n) \|\phi(x_n) - \phi(x^\#)\| \right] \\ &= \left[1 - \left(1 - \frac{\varsigma L}{\nu}\right)(1 - \eta_n)\zeta_n\right] \|\phi(x_n) - \phi(x^\#)\| \\ &\quad + \left(1 - \frac{\varsigma L}{\nu}\right)(1 - \eta_n)\zeta_n \frac{\|\varsigma\varrho(x^\#) - \phi(x^\#)\|}{1 - \varsigma L/\nu}. \end{aligned} \quad (18)$$

An induction implies that

$$\left\|\phi(x_n)-\phi(x^\#)\right\| \le \max\left\{\left\|\phi(x_0)-\phi(x^\#)\right\|, \frac{\left\|\varsigma\varrho(x^\#)-\phi(x^\#)\right\|}{1-\varsigma L/\nu}\right\}. \tag{19}$$

Hence, $\{\phi(x_n)\}$ is bounded. Since ϕ is ν-strongly monotone, we can deduce $\nu\|x_n-x^\#\| \le \|\phi(x_n)-\phi(x^\#)\|$. So,

$$\begin{aligned}\left\|x_n-x^\#\right\| &\le \frac{1}{\nu}\left\|\phi(x_n)-\phi(x^\#)\right\| \\ &\le \frac{1}{\nu}\max\left\{\left\|\phi(x_0)-\phi(x^\#)\right\|, \frac{\left\|\varsigma\varrho(x^\#)-\phi(x^\#)\right\|}{1-\varsigma L/\nu}\right\}.\end{aligned} \tag{20}$$

This implies that $\{x_n\}$ is bounded.

From (6), we have

$$\theta(z_n,y)+\frac{1}{\tau_n}\langle y-z_n, z_n-(u_n-\tau_n\mathbb{A}u_n)\rangle \ge 0, \quad \forall y\in\mathbb{C}. \tag{21}$$

So,

$$\theta(z_n,z_{n+1})+\frac{1}{\tau_n}\langle z_{n+1}-z_n, z_n-(u_n-\tau_n\mathbb{A}u_n)\rangle \ge 0. \tag{22}$$

Similarly,

$$\theta(z_{n+1},z_n)+\frac{1}{\tau_{n+1}}\langle z_n-z_{n+1}, z_{n+1}-(u_{n+1}-\tau_{n+1}\mathbb{A}u_{n+1})\rangle \ge 0. \tag{23}$$

Hence,

$$\theta(z_n,z_{n+1})+\theta(z_{n+1},z_n)+\langle\mathbb{A}u_n-\mathbb{A}u_{n+1}, z_{n+1}-z_n\rangle + \left\langle z_{n+1}-z_n, \frac{z_n-u_n}{\tau_n}-\frac{z_{n+1}-u_{n+1}}{\tau_{n+1}}\right\rangle \ge 0. \tag{24}$$

Since θ is monotone, we have

$$\theta(z_n,z_{n+1})+\theta(z_{n+1},z_n)\le 0. \tag{25}$$

So,

$$\langle\mathbb{A}u_n-\mathbb{A}u_{n+1}, z_{n+1}-z_n\rangle + \left\langle z_{n+1}-z_n, \frac{z_n-u_n}{\tau_n}-\frac{z_{n+1}-u_{n+1}}{\tau_{n+1}}\right\rangle \ge 0. \tag{26}$$

Thus,

$$\tau_n\langle\mathbb{A}u_n-\mathbb{A}u_{n+1}, z_{n+1}-z_n\rangle + \left\langle z_{n+1}-z_n, z_n-z_{n+1}+z_{n+1}-u_n-\frac{\tau_n}{\tau_{n+1}}(z_{n+1}-u_{n+1})\right\rangle \ge 0. \tag{27}$$

It follows that

$$\begin{aligned}\|z_{n+1}-z_n\|^2 &\le \tau_n\langle\mathbb{A}u_n-\mathbb{A}u_{n+1}, z_{n+1}-z_n\rangle \\ &\quad + \left\langle z_{n+1}-z_n, u_{n+1}-u_n+\left(1-\frac{\tau_n}{\tau_{n+1}}\right)(z_{n+1}-u_{n+1})\right\rangle \\ &= \langle (I-\tau_n\mathbb{A})u_{n+1}-(I-\tau_n\mathbb{A})u_n, z_{n+1}-z_n\rangle \\ &\quad + \left\langle z_{n+1}-z_n, \left(1-\frac{\tau_n}{\tau_{n+1}}\right)(z_{n+1}-u_{n+1})\right\rangle \\ &\le \|(I-\tau_n\mathbb{A})u_{n+1}-(I-\tau_n\mathbb{A})u_n\|\,\|z_{n+1}-z_n\| \\ &\quad + \left|1-\frac{\tau_n}{\tau_{n+1}}\right|\|z_{n+1}-z_n\|\,\|z_{n+1}-u_{n+1}\| \\ &\le \|z_{n+1}-z_n\|\left(\|u_{n+1}-u_n\|+\left|1-\frac{\tau_n}{\tau_{n+1}}\right|\|z_{n+1}-u_{n+1}\|\right)\end{aligned} \tag{28}$$

and hence

$$\begin{aligned}\|z_{n+1}-z_n\| &\le \|u_{n+1}-u_n\|+\left|1-\frac{\tau_n}{\tau_{n+1}}\right|\|z_{n+1}-u_{n+1}\| \\ &\le \|u_{n+1}-u_n\|+\frac{1}{a_3}|\tau_{n+1}-\tau_n|\,\|z_{n+1}-u_{n+1}\|.\end{aligned} \tag{29}$$

By (6) and (16), we have

$$\begin{aligned}\|u_{n+1}-u_n\| &= \|\mathrm{proj}_{\mathbb{C}}[\phi(x_{n+1})-\varpi_{n+1}\mathbb{B}x_{n+1}]-\mathrm{proj}_{\mathbb{C}}[\phi(x_n)-\varpi_n\mathbb{B}x_n]\| \\ &\le \|[\phi(x_{n+1})-\varpi_{n+1}\mathbb{B}x_{n+1}]-[\phi(x_n)-\varpi_n\mathbb{B}x_n]\| \\ &\le \|\phi(x_{n+1})-\varpi_{n+1}\mathbb{B}x_{n+1}-(\phi(x_n)-\varpi_{n+1}\mathbb{B}x_n)\| \\ &\quad + |\varpi_{n+1}-\varpi_n|\,\|\mathbb{B}(x_n)\| \\ &\le \|\phi(x_{n+1})-\phi(x_n)\|+|\varpi_{n+1}-\varpi_n|\,\|\mathbb{B}(x_n)\|.\end{aligned} \tag{30}$$

Therefore,

$$\begin{aligned}\|z_{n+1}-z_n\| &\le \|\phi(x_{n+1})-\phi(x_n)\| \\ &\quad + |\varpi_{n+1}-\varpi_n|\,\|\mathbb{B}(x_n)\|+\frac{1}{a_3}|\tau_{n+1}-\tau_n|\,\|z_{n+1}-u_{n+1}\|.\end{aligned} \tag{31}$$

It follows that

$$\begin{aligned}\|z_{n+1}-z_n\|-\|\phi(x_{n+1})-\phi(x_n)\| \\ \le |\varpi_{n+1}-\varpi_n|\,\|\mathbb{B}(x_n)\|+\frac{1}{a_3}|\tau_{n+1}-\tau_n|\,\|z_{n+1}-u_{n+1}\|.\end{aligned} \tag{32}$$

Since $\lim_{n\to\infty}(\varpi_{n+1} - \varpi_n) = 0$, $\lim_{n\to\infty}(\tau_{n+1} - \tau_n) = 0$, and the sequences $\{\varrho(x_n)\}$, $\{\phi(x_n)\}$, $\{z_n\}$, $\{u_n\}$, and $\{\mathbb{B}x_n\}$ are bounded, we have

$$\limsup_{n\to\infty}\left(\|z_{n+1} - z_n\| - \|\phi(x_{n+1}) - \phi(x_n)\|\right) \le 0. \tag{33}$$

From Lemma 2, we obtain

$$\lim_{n\to\infty}\|z_n - \phi(x_n)\| = 0. \tag{34}$$

Note that

$$\begin{aligned}\|\phi(x_{n+1}) - \phi(x_n)\| \le{}& (1-\eta_n)\zeta_n\|\varsigma\varrho(x_n) - \phi(x_n)\| \\ &+ (1-\eta_n)(1-\zeta_n)\|z_n - \phi(x_n)\|.\end{aligned} \tag{35}$$

Hence,

$$\lim_{n\to\infty}\|\phi(x_{n+1}) - \phi(x_n)\| = 0. \tag{36}$$

This together with the ν-strong monotonicity of ϕ implies that

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0. \tag{37}$$

From (18), we have

$$\begin{aligned}&\|\phi(x_{n+1}) - \phi(x^{\#})\| \\ &\le \eta_n\|\phi(x_n) - \phi(x^{\#})\| \\ &\quad + (1-\eta_n)\left[\zeta_n\varsigma L\|x_n - x^{\#}\| + \zeta_n\|\varsigma\varrho(x^{\#}) - \phi(x^{\#})\|\right. \\ &\qquad\left. + (1-\zeta_n)\|u_n - \phi(x^{\#})\|\right] \\ &\le \eta_n\|\phi(x_n) - \phi(x^{\#})\| \\ &\quad + (1-\eta_n)\left[\frac{\zeta_n\varsigma L}{\nu}\|\phi(x_n) - \phi(x^{\#})\|\right. \\ &\qquad + \zeta_n\|\varsigma\varrho(x^{\#}) - \phi(x^{\#})\| \\ &\qquad\left. + (1-\zeta_n)\|u_n - \phi(x^{\#})\|\right] \\ &= \left[1 - (1-\eta_n)\left(1 - \frac{\varsigma L\zeta_n}{\nu}\right)\right]\|\phi(x_n) - \phi(x^{\#})\| \\ &\quad + (1-\zeta_n)(1-\eta_n)\|u_n - \phi(x^{\#})\| \\ &\quad + (1-\eta_n)\zeta_n\left(1 - \frac{\varsigma L}{\nu}\right)\frac{\|\varsigma\varrho(x^{\#}) - \phi(x^{\#})\|}{1 - \varsigma L/\nu}.\end{aligned} \tag{38}$$

By the convexity of the norm and (17), we have

$$\begin{aligned}&\|\phi(x_{n+1}) - \phi(x^{\#})\|^2 \\ &\le \left[1 - (1-\eta_n)\left(1 - \frac{\varsigma L\zeta_n}{\nu}\right)\right]\|\phi(x_n) - \phi(x^{\#})\|^2 \\ &\quad + (1-\zeta_n)(1-\eta_n)\|u_n - \phi(x^{\#})\|^2 \\ &\quad + (1-\eta_n)\zeta_n\left(1 - \frac{\varsigma L}{\nu}\right)\left(\frac{\|\varsigma\varrho(x^{\#}) - \phi(x^{\#})\|}{1 - \varsigma L/\nu}\right)^2 \\ &\le \left[1 - (1-\eta_n)\left(1 - \frac{\varsigma L\zeta_n}{\nu}\right)\right]\|\phi(x_n) - \phi(x^{\#})\|^2 \\ &\quad + (1-\zeta_n)(1-\eta_n) \\ &\quad \times \|(\phi(x_n) - \varpi_n\mathbb{B}x_n) - (\phi(x^{\#}) - \varpi_n\mathbb{B}x^{\#})\|^2 \\ &\quad + (1-\eta_n)\zeta_n\left(1 - \frac{\varsigma L}{\nu}\right)\left(\frac{\|\varsigma\varrho(x^{\#}) - \phi(x^{\#})\|}{1 - \varsigma L/\nu}\right)^2 \\ &\le \left[1 - (1-\eta_n)\left(1 - \frac{\varsigma L\zeta_n}{\nu}\right)\right]\|\phi(x_n) - \phi(x^{\#})\|^2 \\ &\quad + (1-\zeta_n)(1-\eta_n) \\ &\quad \times \left(\|\phi(x_n) - \phi(x^{\#})\|^2 + \varpi_n(\varpi_n - 2\zeta)\|\mathbb{B}x_n - \mathbb{B}x^{\#}\|^2\right) \\ &\quad + (1-\eta_n)\zeta_n\left(1 - \frac{\varsigma L}{\nu}\right)\left(\frac{\|\varsigma\varrho(x^{\#}) - \phi(x^{\#})\|}{1 - \varsigma L/\nu}\right)^2 \\ &\le \|\phi(x_n) - \phi(x^{\#})\|^2 \\ &\quad + (1-\zeta_n)(1-\eta_n)\varpi_n(\varpi_n - 2\zeta)\|\mathbb{B}x_n - \mathbb{B}x^{\#}\|^2 \\ &\quad + (1-\eta_n)\zeta_n\left(1 - \frac{\varsigma L}{\nu}\right)\left(\frac{\|\varsigma\varrho(x^{\#}) - \phi(x^{\#})\|}{1 - \varsigma L/\nu}\right)^2.\end{aligned} \tag{39}$$

So,

$$\begin{aligned}&(1-\eta_n)(1-\zeta_n)\varpi_n(2\zeta - \varpi_n)\|\mathbb{B}x_n - \mathbb{B}x^{\#}\|^2 \\ &\le \|\phi(x_n) - \phi(x^{\#})\|^2 - \|\phi(x_{n+1}) - \phi(x^{\#})\|^2 \\ &\quad + (1-\eta_n)\zeta_n\left(1 - \frac{\varsigma L}{\nu}\right)\left(\frac{\|\varsigma\varrho(x^{\#}) - \phi(x^{\#})\|}{1 - \varsigma L/\nu}\right)^2 \\ &\le \left(\|\phi(x_n) - \phi(x^{\#})\| + \|\phi(x_{n+1}) - \phi(x^{\#})\|\right) \\ &\quad \times \|\phi(x_{n+1}) - \phi(x_n)\| \\ &\quad + (1-\eta_n)\zeta_n\left(1 - \frac{\varsigma L}{\nu}\right)\left(\frac{\|\varsigma\varrho(x^{\#}) - \phi(x^{\#})\|}{1 - \varsigma L/\nu}\right)^2.\end{aligned} \tag{40}$$

Since $\zeta_n \to 0$, $\|\phi(x_{n+1}) - \phi(x_n)\| \to 0$, and $\liminf_{n\to\infty}(1-\eta_n)(1-\zeta_n)\varpi_n(2\zeta - \varpi_n) > 0$, we obtain

$$\lim_{n\to\infty} \left\|\mathbb{B}x_n - \mathbb{B}x^{\#}\right\| = 0. \tag{41}$$

Note that

$$\begin{aligned}
&\left\|u_n - \phi(x^{\#})\right\|^2 \\
&= \left\|\text{proj}_{\mathbb{C}}\left[\phi(x_n) - \varpi_n\mathbb{B}x_n\right] - \text{proj}_{\mathbb{C}}\left[\phi(x^{\#}) - \varpi_n\mathbb{B}x^{\#}\right]\right\|^2 \\
&\le \left\langle \phi(x_n) - \varpi_n\mathbb{B}x_n - \left(\phi(x^{\#}) - \varpi_n\mathbb{B}x^{\#}\right), u_n - \phi(x^{\#})\right\rangle \\
&= \frac{1}{2}\left\{\left\|\phi(x_n) - \varpi_n\mathbb{B}x_n - \left(\phi(x^{\#}) - \varpi_n\mathbb{B}x^{\#}\right)\right\|^2\right. \\
&\qquad + \left\|u_n - \phi(x^{\#})\right\|^2 \\
&\qquad \left. - \left\|\phi(x_n) - \varpi_n\mathbb{B}x_n - \left(\phi(x^{\#}) - \varpi_n\mathbb{B}x^{\#}\right) - u_n + \phi(x^{\#})\right\|^2\right\} \\
&\le \frac{1}{2}\left\{\left\|\phi(x_n) - \phi(x^{\#})\right\|^2 + \left\|u_n - \phi(x^{\#})\right\|^2\right. \\
&\qquad \left. - \left\|\phi(x_n) - u_n - \varpi_n\left(\mathbb{B}x_n - \mathbb{B}x^{\#}\right)\right\|^2\right\} \\
&= \frac{1}{2}\left\{\left\|\phi(x_n) - \phi(x^{\#})\right\|^2 + \left\|u_n - \phi(x^{\#})\right\|^2\right. \\
&\qquad - \left\|\phi(x_n) - u_n\right\|^2 - \varpi_n^2\left\|\mathbb{B}x_n - \mathbb{B}x^{\#}\right\|^2 \\
&\qquad \left. + 2\varpi_n\left\langle \phi(x_n) - u_n, \mathbb{B}x_n - \mathbb{B}x^{\#}\right\rangle\right\}.
\end{aligned} \tag{42}$$

It follows that

$$\begin{aligned}
&\left\|u_n - \phi(x^{\#})\right\|^2 \\
&\quad \le \left\|\phi(x_n) - \phi(x^{\#})\right\|^2 - \left\|\phi(x_n) - u_n\right\|^2 - \varpi_n^2\left\|\mathbb{B}x_n - \mathbb{B}x^{\#}\right\|^2 \\
&\qquad + 2\varpi_n\left\|\phi(x_n) - u_n\right\|\left\|\mathbb{B}x_n - \mathbb{B}x^{\#}\right\|.
\end{aligned} \tag{43}$$

From (39) and (43), we have

$$\begin{aligned}
&\left\|\phi(x_{n+1}) - \phi(x^{\#})\right\|^2 \\
&\quad \le \left[1 - (1-\eta_n)\left(1 - \frac{\varsigma L\zeta_n}{\nu}\right)\right]\left\|\phi(x_n) - \phi(x^{\#})\right\|^2 \\
&\qquad + (1-\zeta_n)(1-\eta_n)\left\|u_n - \phi(x^{\#})\right\|^2 \\
&\qquad + (1-\eta_n)\zeta_n\left(1 - \frac{\varsigma L}{\nu}\right)\left(\frac{\left\|\varsigma\varrho(x^{\#}) - \phi(x^{\#})\right\|}{1 - \varsigma L/\nu}\right)^2 \\
&\quad \le \left[1 - (1-\eta_n)\left(1 - \frac{\varsigma L\zeta_n}{\nu}\right)\right]\left\|\phi(x_n) - \phi(x^{\#})\right\|^2 \\
&\qquad + (1-\zeta_n)(1-\eta_n)\left\|\phi(x_n) - \phi(x^{\#})\right\|^2 \\
&\qquad - (1-\zeta_n)(1-\eta_n)\left\|\phi(x_n) - u_n\right\|^2 \\
&\qquad + 2(1-\zeta_n)(1-\eta_n)\varpi_n\left\|\phi(x_n) - u_n\right\|\left\|\mathbb{B}x_n - \mathbb{B}x^{\#}\right\| \\
&\qquad + (1-\eta_n)\zeta_n\left(1 - \frac{\varsigma L}{\nu}\right)\left(\frac{\left\|\varsigma\varrho(x^{\#}) - \phi(x^{\#})\right\|}{1 - \varsigma L/\nu}\right)^2 \\
&\quad \le \left\|\phi(x_n) - \phi(x^{\#})\right\|^2 - (1-\zeta_n)(1-\eta_n)\left\|\phi(x_n) - u_n\right\|^2 \\
&\qquad + 2\varpi_n\left\|\phi(x_n) - u_n\right\|\left\|\mathbb{B}x_n - \mathbb{B}x^{\#}\right\| \\
&\qquad + (1-\eta_n)\zeta_n\left(1 - \frac{\varsigma L}{\nu}\right)\left(\frac{\left\|\varsigma\varrho(x^{\#}) - \phi(x^{\#})\right\|}{1 - \varsigma L/\nu}\right)^2.
\end{aligned} \tag{44}$$

Then, we obtain

$$\begin{aligned}
&(1-\zeta_n)(1-\eta_n)\left\|\phi(x_n) - u_n\right\|^2 \\
&\quad \le \left(\left\|\phi(x_n) - \phi(x^{\#})\right\| + \left\|\phi(x_{n+1}) - \phi(x^{\#})\right\|\right) \\
&\qquad \times \left\|\phi(x_{n+1}) - \phi(x_n)\right\| \\
&\qquad + 2\varpi_n\left\|\phi(x_n) - u_n\right\|\left\|\mathbb{B}x_n - \mathbb{B}x^{\#}\right\| \\
&\qquad + (1-\eta_n)\zeta_n\left(1 - \frac{\varsigma L}{\nu}\right)\left(\frac{\left\|\varsigma\varrho(x^{\#}) - \phi(x^{\#})\right\|}{1 - \varsigma L/\nu}\right)^2.
\end{aligned} \tag{45}$$

Since $\lim_{n\to\infty}\zeta_n = 0$, $\lim_{n\to\infty}\|\phi(x_{n+1}) - \phi(x_n)\| = 0$, and $\lim_{n\to\infty}\|\mathbb{B}x_n - \mathbb{B}x^{\#}\| = 0$, we have

$$\lim_{n\to\infty}\left\|\phi(x_n) - u_n\right\| = 0. \tag{46}$$

Next, we prove that $\limsup_{n\to\infty}\langle\varsigma\varrho(x^*) - \phi(x^*), u_n - \phi(x^*)\rangle \le 0$, where x^* is the unique solution of (7). Let $\{u_{n_i}\}$ be a subsequence of $\{u_n\}$ such that

$$\begin{aligned}
&\limsup_{n\to\infty}\left\langle\varsigma\varrho(x^*) - \phi(x^*), u_n - \phi(x^*)\right\rangle \\
&\quad = \lim_{i\to\infty}\left\langle\varsigma\varrho(x^*) - \phi(x^*), u_{n_i} - \phi(x^*)\right\rangle \\
&\quad = \lim_{i\to\infty}\left\langle\varsigma\varrho(x^*) - \phi(x^*), \phi(x_{n_i}) - \phi(x^*)\right\rangle.
\end{aligned} \tag{47}$$

By the boundedness of $\{x_{n_i}\}$, there exists a subsequence $\{x_{n_{i_j}}\}$ of $\{x_{n_i}\}$ which converges weakly to some point $z \in \mathbb{C}$. Without loss of generality, we may assume that $x_{n_i} \rightharpoonup z$. From the weak continuity of ϕ, we deduce $\phi(x_{n_i}) \rightharpoonup \phi(z)$. Next, we prove $z \in \Delta$. We firstly show $z \in EP(\theta, \mathbb{A})$. Noting that $z_n = \mathbb{S}_{\tau_n}(u_n - \tau_n\mathbb{A}u_n)$, for any $y \in \mathbb{C}$, we have

$$\theta(z_n, y) + \frac{1}{\tau_n}\left\langle y - z_n, z_n - (u_n - \tau_n\mathbb{A}u_n)\right\rangle \ge 0. \tag{48}$$

Since θ is monotone, we have

$$\frac{1}{\tau_n}\langle y-z_n, z_n-(u_n-\tau_n\mathbb{A}u_n)\rangle \geq \theta(y,z_n), \quad \forall y\in\mathbb{C}. \tag{49}$$

Hence,

$$\left\langle y-z_{n_i}, \frac{z_{n_i}-u_{n_i}}{\tau_{n_i}}+\mathbb{A}u_{n_i}\right\rangle \geq \theta(y,z_{n_i}), \quad \forall y\in\mathbb{C}. \tag{50}$$

Let $v_t=ty+(1-t)z$, for all $t\in(0,1]$ and $y\in C$. We have $v_t\in C$. So, from (50) we have

$$\begin{aligned}
&\langle v_t-z_{n_i}, \mathbb{A}v_t\rangle \\
&\quad\geq \langle v_t-z_{n_i}, \mathbb{A}v_t\rangle - \left\langle v_t-z_{n_i}, \frac{z_{n_i}-u_{n_i}}{\tau_{n_i}}+\mathbb{A}u_{n_i}\right\rangle \\
&\qquad+\theta(v_t,z_{n_i}) \\
&\quad= \langle v_t-z_{n_i}, \mathbb{A}v_t-\mathbb{A}z_{n_i}\rangle + \langle v_t-z_{n_i}, \mathbb{A}z_{n_i}-\mathbb{A}u_{n_i}\rangle \\
&\qquad-\left\langle v_t-z_{n_i}, \frac{z_{n_i}-u_{n_i}}{\tau_{n_i}}\right\rangle+\theta(v_t,z_{n_i}).
\end{aligned} \tag{51}$$

Note that $\|\mathbb{A}z_{n_i}-\mathbb{A}u_{n_i}\|\leq(1/\eta)\|z_{n_i}-u_{n_i}\|\rightarrow 0$. Further, from monotonicity of $\mathbb{A}$, we have $\langle v_t-z_{n_i}, \mathbb{A}v_t-\mathbb{A}z_{n_i}\rangle\geq 0$. Letting $i\rightarrow\infty$ in (51), we have $\langle v_t-z, \mathbb{A}v_t\rangle\geq\theta(v_t,z)$. This together with (C1) and (C4) implies that

$$\begin{aligned}
0=\theta(v_t,v_t)&\leq t\theta(v_t,y)+(1-t)\theta(v_t,z)\\
&\leq t\theta(v_t,y)+(1-t)\langle v_t-z,\mathbb{A}v_t\rangle\\
&= t\theta(v_t,y)+(1-t)t\langle y-z,\mathbb{A}v_t\rangle
\end{aligned} \tag{52}$$

and hence $0\leq\theta(v_t,y)+(1-t)\langle\mathbb{A}v_t, y-z\rangle$. Letting $t\rightarrow 0$, we have $0\leq\theta(z,y)+\langle y-z,\mathbb{A}z\rangle$. This implies that $z\in EP(\theta,\mathbb{A})$. Next, we prove $z\in VI(\mathbb{B},\phi,\mathbb{C})$. Set

$$Rv=\begin{cases}\mathbb{B}v+\mathbb{N}_{\mathbb{C}}(v), & v\in\mathbb{C},\\ \emptyset, & v\notin\mathbb{C}.\end{cases} \tag{53}$$

It is well known that R is maximal ϕ-monotone. Let $(v,w)\in G(R)$ (the graph of R). Since $w-\mathbb{B}v\in\mathbb{N}_{\mathbb{C}}(v)$ and $x_n\in\mathbb{C}$, we have $\langle\phi(v)-\phi(x_n), w-\mathbb{B}v\rangle\geq 0$. Noting that $u_n=\mathrm{proj}_{\mathbb{C}}[\phi(x_n)-\varpi_n\mathbb{B}x_n]$, we get

$$\langle\phi(v)-u_n, u_n-[\phi(x_n)-\varpi_n\mathbb{B}x_n]\rangle\geq 0. \tag{54}$$

It follows that

$$\left\langle\phi(v)-u_n, \frac{u_n-\phi(x_n)}{\varpi_n}+\mathbb{B}x_n\right\rangle\geq 0. \tag{55}$$

Then,

$$\begin{aligned}
&\langle\phi(v)-\phi(x_{n_i}), w\rangle\\
&\quad\geq\langle\phi(v)-\phi(x_{n_i}),\mathbb{B}v\rangle\\
&\quad\geq\langle\phi(v)-\phi(x_{n_i}),\mathbb{B}v\rangle-\left\langle\phi(v)-u_{n_i},\frac{u_{n_i}-\phi(x_{n_i})}{\varpi_{n_i}}\right\rangle\\
&\qquad-\langle\phi(v)-u_{n_i},\mathbb{B}x_{n_i}\rangle\\
&\quad=\langle\phi(v)-\phi(x_{n_i}),\mathbb{B}v-\mathbb{B}x_{n_i}\rangle+\langle\phi(v)-\phi(x_{n_i}),\mathbb{B}x_{n_i}\rangle\\
&\qquad-\left\langle\phi(v)-u_{n_i},\frac{u_{n_i}-\phi(x_{n_i})}{\varpi_{n_i}}\right\rangle-\langle\phi(v)-u_{n_i},\mathbb{B}x_{n_i}\rangle\\
&\quad\geq-\left\langle\phi(v)-u_{n_i},\frac{u_{n_i}-\phi(x_{n_i})}{\varpi_{n_i}}\right\rangle-\langle\phi(x_{n_i})-u_{n_i},\mathbb{B}x_{n_i}\rangle.
\end{aligned} \tag{56}$$

Since $\|\phi(x_{n_i})-u_{n_i}\|\rightarrow 0$ and $\phi(x_{n_i})\rightharpoonup\phi(z)$, we deduce that $\langle\phi(v)-\phi(z), w\rangle\geq 0$ by taking $i\rightarrow\infty$ in (56). Thus, $z\in R^{-1}0$ by the maximal ϕ-monotonicity of R. Hence, $z\in VI(\mathbb{B},\phi,\mathbb{C})$. Therefore, $z\in\Delta$. From (47), we obtain

$$\begin{aligned}
&\limsup_{n\rightarrow\infty}\langle\varsigma\varrho(x^*)-\phi(x^*), u_n-\phi(x^*)\rangle\\
&\quad=\lim_{i\rightarrow\infty}\langle\varsigma\varrho(x^*)-\phi(x^*),\phi(x_{n_i})-\phi(x^*)\rangle\\
&\quad=\langle\varsigma\varrho(x^*)-\phi(x^*),\phi(z)-\phi(x^*)\rangle\leq 0.
\end{aligned} \tag{57}$$

Set $y_n=\mathrm{proj}_{\mathbb{C}}[\zeta_n\varsigma\varrho(x_n)+(1-\zeta_n)z_n]$, for all $n\geq 0$. Then, we have

$$\begin{aligned}
&\|y_n-\phi(x^*)\|^2\\
&\quad\leq\langle\zeta_n\varsigma\varrho(x_n)+(1-\zeta_n)z_n-\phi(x^*), y_n-\phi(x^*)\rangle\\
&\quad\leq\zeta_n\varsigma\langle\varrho(x_n)-\varrho(x^*), y_n-\phi(x^*)\rangle\\
&\qquad+\zeta_n\langle\varsigma\varrho(x^*)-\phi(x^*), y_n-\phi(x^*)\rangle\\
&\qquad+(1-\zeta_n)\langle z_n-\phi(x^*), y_n-\phi(x^*)\rangle\\
&\quad\leq\frac{\zeta_n L\varsigma}{\nu}\|\phi(x_n)-\phi(x^*)\|\|y_n-\phi(x^*)\|\\
&\qquad+\zeta_n\langle\varsigma\varrho(x^*)-\phi(x^*), y_n-\phi(x^*)\rangle\\
&\qquad+(1-\zeta_n)\|z_n-\phi(x^*)\|\|y_n-\phi(x^*)\|
\end{aligned}$$

$$\leq \zeta_n \left(\frac{\varsigma L}{\nu}\right) \|\phi(x_n) - \phi(x^*)\| \|y_n - \phi(x^*)\|$$
$$+ \zeta_n \langle \varsigma\varrho(x^*) - \phi(x^*), y_n - \phi(x^*)\rangle$$
$$+ (1-\zeta_n) \|\phi(x_n) - \phi(x^*)\| \|y_n - \phi(x^*)\|$$
$$= \left[1 - \left(1 - \frac{L\varsigma}{\nu}\right)\zeta_n\right] \|\phi(x_n) - \phi(x^*)\| \|y_n - \phi(x^*)\|$$
$$+ \zeta_n \langle \varsigma\varrho(x^*) - \phi(x^*), y_n - \phi(x^*)\rangle$$
$$= \frac{1-(1-L\varsigma/\nu)\zeta_n}{2} \|\phi(x_n) - \phi(x^*)\|^2 + \frac{1}{2}\|y_n - \phi(x^*)\|^2$$
$$+ \zeta_n \left\langle \varsigma\varrho(x^*) - \phi(x^*), y_n - \phi(x^\#)\right\rangle. \quad (58)$$

It follows that

$$\|y_n - \phi(x^*)\|^2 \leq \left[1 - \left(1 - \frac{L\varsigma}{\nu}\right)\zeta_n\right] \|\phi(x_n) - \phi(x^*)\|^2 + 2\zeta_n \langle \varsigma\varrho(x^*) - \phi(x^*), y_n - \phi(x^*)\rangle. \quad (59)$$

Therefore,

$$\begin{aligned}
&\|\phi(x_{n+1}) - \phi(x^*)\|^2 \\
&\leq \eta_n \|\phi(x_n) - \phi(x^*)\|^2 + (1-\eta_n) \|y_n - \phi(x^*)\|^2 \\
&\leq \eta_n \|\phi(x_n) - \phi(x^*)\|^2 \\
&\quad + (1-\eta_n)\left[1 - \left(1 - \frac{\varsigma L}{\nu}\right)\zeta_n\right] \|\phi(x_n) - \phi(x^*)\|^2 \\
&\quad + 2(1-\eta_n)\zeta_n \langle \varsigma\varrho(x^*) - \phi(x^*), y_n - \phi(x^*)\rangle \\
&= \left[1 - \left(1 - \frac{\varsigma L}{\nu}\right)(1-\eta_n)\zeta_n\right] \|\phi(x_n) - \phi(x^*)\|^2 \\
&\quad + 2(1-\eta_n)\zeta_n \langle \varsigma\varrho(x^*) - \phi(x^*), y_n - \phi(x^*)\rangle \\
&= \left[1 - \left(1 - \frac{\varsigma L}{\nu}\right)(1-\eta_n)\zeta_n\right] \|\phi(x_n) - \phi(x^*)\|^2 \\
&\quad + \left(1 - \frac{\varsigma L}{\nu}\right)(1-\eta_n)\zeta_n \\
&\quad \times \left(\frac{2}{1-\varsigma L/\nu} \langle \varsigma\varrho(x^*) - \phi(x^*), y_n - \phi(x^*)\rangle\right) \\
&= (1-\nu_n)\|\phi(x_n) - \phi(x^*)\|^2 + \varsigma_n \nu_n,
\end{aligned} \quad (60)$$

where $\nu_n = (1-\varsigma L/\nu)(1-\eta_n)\zeta_n$ and $\varsigma_n = (2/(1-\varsigma L/\nu)) \langle \varsigma\varrho(x^*) - \phi(x^*), y_n - \phi(x^*)\rangle$. It is easily seen that $\sum_n \nu_n = \infty$. Since

$$\begin{aligned}
\|y_n - u_n\| &\leq \|y_n - z_n\| + \|z_n - u_n\| \\
&\leq \zeta_n \|\varsigma\phi(x_n) - z_n\| + \|z_n - u_n\| \\
&\longrightarrow 0
\end{aligned} \quad (61)$$

and by $\limsup_{n\to\infty}\langle \varsigma\varrho(x^*) - \phi(x^*), u_n - \phi(x^*)\rangle \leq 0$, we get $\limsup_{n\to\infty}\varsigma_n \leq 0$. We can therefore apply Lemma 3 to conclude that $\phi(x_n) \to \phi(x^*)$ and $x_n \to x^*$. This completes the proof. □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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