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Research Article

Combinatorial Properties and Characterization of Glued Semigroups

J. I. García-García,[1] M. A. Moreno-Frías,[1] and A. Vigneron-Tenorio[2]

[1] *Departamento de Matemáticas, Universidad de Cádiz, Puerto Real, 11510 Cádiz, Spain*
[2] *Departamento de Matemáticas, Universidad de Cádiz, Jerez de la Frontera, 11405 Cádiz, Spain*

Correspondence should be addressed to M. A. Moreno-Frías; mariangeles.mfrias@gmail.com

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This work focuses on the combinatorial properties of glued semigroups and provides its combinatorial characterization. Some classical results for affine glued semigroups are generalized and some methods to obtain glued semigroups are developed.

1. Introduction

Let $S = \langle n_1, \dots, n_l\rangle$ be a finitely generated commutative semigroup with zero element which is reduced (i.e., $S \cap (-S) = \{0\}$) and cancellative (if $m, n, n' \in S$ and $m + n = m + n'$ then $n = n'$). Under these settings if S is torsion-free, then it is isomorphic to a subsemigroup of $\mathbb{N}^p$ which means it is an affine semigroup (see [1]). From now on assume that all the semigroups appearing in this work are finitely generated, commutative, reduced, and cancellative, but not necessarily torsion-free.

Let $\mathbb{K}$ be a field and $\mathbb{K}[X_1, \dots, X_l]$ the polynomial ring in l indeterminates. This polynomial ring is obviously an S-graded ring (by assigning the S-degree n_i to the indeterminate X_i, the S-degree of $X^\alpha = X_1^{\alpha_1} \cdots X_l^{\alpha_l}$ is $\sum_{i=1}^{l} \alpha_i n_i \in S$). It is well known that the ideal I_S generated by

$$\left\{X^\alpha - X^\beta \mid \sum_{i=1}^{l} \alpha_i n_i = \sum_{i=1}^{l} \beta_i n_i\right\} \subset \mathbb{K}[X_1, \dots, X_l] \qquad (1)$$

is an S-homogeneous binomial ideal called *semigroup ideal* (see [2] for details). If S is torsion-free, the ideal obtained defines a toric variety (see [3] and the references therein). By Nakayama's lemma, all the minimal generating sets of I_S have the same cardinality and the S-degrees of its elements can be determinated.

The main goal of this work is to study the semigroups which result from the gluing of other two. This concept was introduced by Rosales in [4] and it is closely related to complete intersection ideals (see [5] and the references therein). A semigroup S minimally generated by $A_1 \sqcup A_2$ (with $A_1 = \{n_1, \dots, n_r\}$ and $A_2 = \{n_{r+1}, \dots, n_l\}$) is the gluing of $S_1 = \langle A_1\rangle$ and $S_2 = \langle A_2\rangle$ if there exists a set of generators ρ of I_S of the form $\rho = \rho_1 \cup \rho_2 \cup \{X^\gamma - X^{\gamma'}\}$, where ρ_1, ρ_2 are generating sets of I_{S_1} and I_{S_2}, respectively, $X^\gamma - X^{\gamma'} \in I_S$, and the supports of γ and γ' verify $\operatorname{supp}(\gamma) \subset \{1, \dots, r\}$ and $\operatorname{supp}(\gamma') \subset \{r+1, \dots, l\}$. Equivalently, S is the gluing of S_1 and S_2 if $I_S = I_{S_1} + I_{S_2} + \langle X^\gamma - X^{\gamma'}\rangle$. A semigroup is a *glued semigroup* when it is the gluing of other two.

As seen, glued semigroups can be determinated by the minimal generating sets of I_S which can be studied by using combinatorial methods from certain simplicial complexes (see [6–8]). In this work the simplicial complexes used are defined as follows: for any $m \in S$, set

$$C_m = \left\{X^\alpha = X_1^{\alpha_1} \cdots X_l^{\alpha_l} \mid \sum_{i=1}^{l} \alpha_i n_i = m\right\}, \qquad (2)$$

and the simplicial complex

$$\nabla_m = \{F \subseteq C_m \mid \gcd(F) \neq 1\}, \qquad (3)$$

with $\gcd(F)$ as the *greatest common divisor* of the monomials in F.

Furthermore, some methods which require linear algebra and integer programming are given to obtain examples of glued semigroups.

The content of this work is organized as follows. Section 2 presents the tools to generalize to nontorsion-free semigroups a classical characterization of affine gluing semigroups (Proposition 2). In Section 3, the nonconnected simplicial complexes ∇_m associated with glued semigroups are studied. By using the vertices of the connected components of these complexes we give a combinatorial characterization of glued semigroups as well as their glued degrees (Theorem 6). Besides, in Corollary 7 we deduce the conditions for the ideal of a glued semigroup to have a unique minimal system of generators. Finally, Section 4 is devoted to the construction of glued semigroups (Corollary 10) and affine glued semigroups (Section 4.1).

2. Preliminaries and Generalizations about Glued Semigroups

A binomial of I_S is called *indispensable* if it is an element of all systems of generators of I_S (up to a scalar multiple). This kind of binomials was introduced in [9] and they have an important role in Algebraic Statistics. In [10] the authors characterize indispensable binomials by using simplicial complexes ∇_m. Note that if I_S is generated by its indispensable binomials then it is minimally generated, up to scalar multiples, in an unique way.

With the above notation, the semigroup S is associated with the lattice $\ker S$ formed by the elements $\alpha = (\alpha_1, \ldots, \alpha_l) \in \mathbb{Z}^l$ such that $\sum_{i=1}^{l} \alpha_i n_i = 0$. Given G a system of generators of I_S, the lattice $\ker S$ is generated by the elements $\alpha - \beta$ with $X^\alpha - X^\beta \in G$ and $\ker S$ also verifies that $\ker S \cap \mathbb{N}^l = \{0\}$ if and only if S is reduced. If $\mathscr{M}(I_S)$ is a minimal generating set of I_S, denote by $\mathscr{M}(I_S)_m \subset \mathscr{M}(I_S)$ the set of elements whose S-degree is equal to $m \in S$ and by Betti(S) the set of the S-degrees of the elements of $\mathscr{M}(I_S)$. When I_S is minimally generated by rank$(\ker S)$ elements, the semigroup S is called a *complete intersection* semigroup.

Let $\mathscr{C}(\nabla_m)$ be the number of connected components of ∇_m. The cardinality of $\mathscr{M}(I_S)_m$ is equal to $\mathscr{C}(\nabla_m) - 1$ (see Remark 2.6 in [6] and Theorem 3 and Corollary 4 in [8]) and the complexes associated with the elements in Betti(S) are nonconnected.

Construction 1 (see [7, Proposition 1]). For each $m \in$ Betti(S) the set $\mathscr{M}(I_S)_m$ is obtained by taking $\mathscr{C}(\nabla_m) - 1$ binomials with monomials in different connected components of ∇_m satisfying that two different binomials do not have their corresponding monomials in the same components and fulfilling that there is at least a monomial of every connected component of ∇_m. This let us construct a minimal generating set of I_S in a combinatorial way.

Let S be minimally (we consider a minimal generator set of S and in the other case S is trivially the gluing of the semigroup generated by one of its nonminimal generators and the semigroup generated by the others) generated by $A_1 \sqcup A_2$ with $A_1 = \{a_1, \ldots, a_r\}$ and $A_2 = \{b_1, \ldots, b_t\}$. From now on, identify the sets A_1 and A_2 with the matrices $(a_1 | \cdots | a_r)$ and $(b_1 | \cdots | b_t)$. Denote by $\mathbb{K}[A_1]$ and $\mathbb{K}[A_2]$ the polynomial rings $\mathbb{K}[X_1, \ldots, X_r]$ and $\mathbb{K}[Y_1, \ldots, Y_t]$, respectively. A monomial is a *pure monomial* if it has indeterminates only in $X_1, \ldots, X_r$ or only in $Y_1, \ldots, Y_t$; otherwise it is a *mixed monomial*. If S is the gluing of $S_1 = \langle A_1 \rangle$ and $S_2 = \langle A_2 \rangle$, then the binomial $X^{\gamma_X} - Y^{\gamma_Y} \in I_S$ is a *glued binomial* if $\mathscr{M}(I_{S_1}) \cup \mathscr{M}(I_{S_2}) \cup \{X^{\gamma_X} - Y^{\gamma_Y}\}$ is a generating set of I_S and in this case the element $d = S\text{-degree}(X^{\gamma_X}) \in S$ is called a *glued degree*.

It is clear that if S is a glued semigroup, the lattice $\ker S$ has a basis of the form

$$\{L_1, L_2, (\gamma_X, -\gamma_Y)\} \subset \mathbb{Z}^{r+t}, \tag{4}$$

where the supports of the elements in L_1 are in $\{1, \ldots, r\}$, the supports of the elements in L_2 are in $\{r+1, \ldots, r+t\}$, $\ker S_i = \langle L_i \rangle$ $(i = 1, 2)$ by considering only the coordinates in $\{1, \ldots, r\}$ or $\{r+1, \ldots, r+t\}$ of L_i, and $(\gamma_X, \gamma_Y) \in \mathbb{N}^{r+t}$. Moreover, since S is reduced, one has that $\langle L_1 \rangle \cap \mathbb{N}^{r+t} = \langle L_2 \rangle \cap \mathbb{N}^{r+t} = \{0\}$. Denote by $\{\rho_{1i}\}_i$ the elements in L_1 and by $\{\rho_{2i}\}_i$ the elements in L_2.

The following proposition generalizes [4, Theorem 1.4] to nontorsion-free semigroups.

Proposition 2. *The semigroup S is the gluing of S_1 and S_2 if and only if there exists $d \in (S_1 \cap S_2) \setminus \{0\}$ such that $G(S_1) \cap G(S_2) = d\mathbb{Z}$, where $G(S_1)$, $G(S_2)$, and $d\mathbb{Z}$ are the associated commutative groups of S_1, S_2, and $\{d\}$, respectively.*

Proof. Assume that S is the gluing of S_1 and S_2. In this case, $\ker S$ is generated by the set (4). Since $(\gamma_X, -\gamma_Y) \in \ker S$, the element d is equal to $A_1 \gamma_X = A_2 \gamma_Y \in S$ and $d \in S_1 \cap S_2 \subset G(S_1) \cap G(S_2)$. Let d' be in $G(S_1) \cap G(S_2)$; then there exists $(\delta_1, \delta_2) \in \mathbb{Z}^r \times \mathbb{Z}^t$ such that $d' = A_1 \delta_1 = A_2 \delta_2$. Therefore $(\delta_1, -\delta_2) \in \ker S$ because $(A_1 \mid A_2)(\delta_1, -\delta_2) = 0$ and so there exist $\lambda, \lambda_i^{\rho_1}, \lambda_i^{\rho_2} \in \mathbb{Z}$ satisfying

$$\begin{aligned} (\delta_1, 0) &= \sum_i \lambda_i^{\rho_1} \rho_{1i} + \lambda (\gamma_X, 0), \\ (0, \delta_2) &= -\sum_i \lambda_i^{\rho_2} \rho_{2i} + \lambda (0, \gamma_Y), \end{aligned} \tag{5}$$

and $d' = A_1 \delta_1 = \sum_i \lambda_i^{\rho_1} (A_1 \mid 0) \rho_{1i} + \lambda A_1 \gamma_X = \lambda d$. We conclude that $G(S_1) \cap G(S_2) = d\mathbb{Z}$ with $d \in S_1 \cap S_2$.

Conversely, suppose that there exists $d \in (S_1 \cap S_2) \setminus \{0\}$ such that $G(S_1) \cap G(S_2) = d\mathbb{Z}$. We see that $I_S = I_{S_1} + I_{S_2} + \langle X^{\gamma_X} - Y^{\gamma_Y} \rangle$. Trivially, $I_{S_1} + I_{S_2} + \langle X^{\gamma_X} - Y^{\gamma_Y} \rangle \subset I_S$. Let $X^\alpha Y^\beta - X^\gamma Y^\delta$ be a binomial in I_S. Its S-degree is $A_1 \alpha + A_2 \beta = A_1 \gamma + A_2 \delta$. Using $A_1(\alpha - \gamma) = A_2(\beta - \delta) \in G(S_1) \cap G(S_2) = d\mathbb{Z}$, there exists $\lambda \in \mathbb{Z}$ such that $A_1 \alpha = A_1 \gamma + \lambda d$ and $A_2 \delta = A_2 \beta + \lambda d$. We have the following cases.

(i) If $\lambda = 0$,

$$\begin{aligned} X^\alpha Y^\beta - X^\gamma Y^\delta &= X^\alpha Y^\beta - X^\gamma Y^\beta + X^\gamma Y^\beta - X^\gamma Y^\delta \\ &= Y^\beta (X^\alpha - X^\gamma) + X^\gamma \left(Y^\beta - Y^\delta\right) \in I_{S_1} + I_{S_2}. \end{aligned} \tag{6}$$

(ii) If $\lambda > 0$,

$$\begin{aligned} X^{\alpha}Y^{\beta} - X^{\gamma}Y^{\delta} = {} & X^{\alpha}Y^{\beta} - X^{\gamma}X^{\lambda\gamma_X}Y^{\beta} \\ & + X^{\gamma}X^{\lambda\gamma_X}Y^{\beta} - X^{\gamma}X^{\lambda\gamma_Y}Y^{\beta} + X^{\gamma}X^{\lambda\gamma_Y}Y^{\beta} \\ & - X^{\gamma}Y^{\delta} = Y^{\beta}\left(X^{\alpha} - X^{\gamma}X^{\lambda\gamma_X}\right) \\ & + X^{\gamma}Y^{\beta}\left(X^{\lambda\gamma_X} - Y^{\lambda\gamma_Y}\right) \\ & + X^{\gamma}\left(Y^{\lambda\gamma_Y}Y^{\beta} - Y^{\delta}\right). \end{aligned} \tag{7}$$

Using that

$$X^{\lambda\gamma_X} - Y^{\lambda\gamma_Y} = \left(X^{\gamma_X} - Y^{\gamma_Y}\right)\left(\sum_{i=0}^{\lambda-1} X^{(\lambda-1-i)\gamma_X}Y^{i\gamma_Y}\right), \tag{8}$$

the binomial $X^{\alpha}Y^{\beta} - X^{\gamma}Y^{\delta}$ belongs to $I_{S_1} + I_{S_2} + \langle X^{\gamma_X} - Y^{\gamma_Y}\rangle$.

(iii) The case $\lambda < 0$ is solved similarly.

We conclude that $I_S = I_{S_1} + I_{S_2} + \langle X^{\gamma_X} - Y^{\gamma_Y}\rangle$. □

From the above proof it is deduced that given the partition of the system of generators of S the glued degree is unique.

3. Glued Semigroups and Combinatorics

Glued semigroups by means of nonconnected simplicial complexes are characterized. For any $m \in S$, redefine C_m from (2) as

$$C_m = \left\{X^{\alpha}Y^{\beta} = X_1^{\alpha_1}\cdots X_r^{\alpha_r}Y_1^{\beta_1}\cdots Y_t^{\beta_t} \mid \sum_{i=1}^{r}\alpha_i a_i + \sum_{i=1}^{t}\beta_i b_i = m\right\}, \tag{9}$$

and consider the sets of vertices and the simplicial complexes

$$\begin{aligned} C_m^{A_1} &= \left\{X_1^{\alpha_1}\cdots X_r^{\alpha_r} \mid \sum_{i=1}^{r}\alpha_i a_i = m\right\}, \\ \nabla_m^{A_1} &= \left\{F \subseteq C_m^{A_1} \mid \gcd(F) \neq 1\right\}, \\ C_m^{A_2} &= \left\{Y_1^{\beta_1}\cdots Y_t^{\beta_t} \mid \sum_{i=1}^{t}\beta_i b_i = m\right\}, \\ \nabla_m^{A_2} &= \left\{F \subseteq C_m^{A_2} \mid \gcd(F) \neq 1\right\}, \end{aligned} \tag{10}$$

where $A_1 = \{a_1, \ldots, a_r\}$ and $A_2 = \{b_1, \ldots, b_t\}$ as in Section 2. Trivially, the relations between $\nabla_m^{A_1}$, $\nabla_m^{A_2}$, and ∇_m are

$$\nabla_m^{A_1} = \left\{F \in \nabla_m \mid F \subset C_m^{A_1}\right\}, \qquad \nabla_m^{A_2} = \left\{F \in \nabla_m \mid F \subset C_m^{A_2}\right\}. \tag{11}$$

The following result shows an important property of the simplicial complexes associated with glued semigroups.

Lemma 3. *Let S be the gluing of S_1 and S_2 and $m \in$ Betti(S). Then all the connected components of ∇_m have at least a pure monomial. In addition, all mixed monomials of ∇_m are in the same connected component.*

Proof. Suppose that there exists C, a connected component of ∇_m only with mixed monomials. By Construction 1 in all generating sets of I_S there is at least a binomial with a mixed monomial, but this does not occur in $\mathscr{M}(I_{S_1}) \cup \mathscr{M}(I_{S_2}) \cup \{X^{\gamma_X} - Y^{\gamma_Y}\}$ with $X^{\gamma_X} - Y^{\gamma_Y}$ as a glued binomial.

Since S is a glued semigroup, $\ker S$ has a system of generators as (4). Let $X^{\alpha}Y^{\beta}, X^{\gamma}Y^{\delta} \in C_m$ be two monomials such that $\gcd(X^{\alpha}Y^{\beta}, X^{\gamma}Y^{\delta}) = 1$. In this case, $(\alpha,\beta) - (\gamma,\delta) \in \ker S$ and there exist $\lambda, \lambda_i^{\rho_1}, \lambda_i^{\rho_2} \in \mathbb{Z}$ satisfying

$$\begin{aligned} (\alpha - \gamma, 0) &= \sum_i \lambda_i^{\rho_1}\rho_{1i} + \lambda(\gamma_X, 0), \\ (0, \beta - \delta) &= \sum_i \lambda_i^{\rho_2}\rho_{2i} - \lambda(0, \gamma_Y). \end{aligned} \tag{12}$$

(i) If $\lambda = 0$, $\alpha - \gamma \in \ker S_1$, and $\beta - \delta \in \ker S_2$, then $A_1\alpha = A_1\gamma$, $A_2\beta = A_2\delta$, and $X^{\alpha}Y^{\delta} \in C_m$.

(ii) If $\lambda > 0$, $(\alpha, 0) = \sum_i \lambda_i^{\rho_1}\rho_{1i} + \lambda(\gamma_X, 0) + (\gamma, 0)$, and

$$A_1\alpha = \sum_i \lambda_i^{\rho_1}(A_1 \mid 0)\rho_{1i} + \lambda A_1\gamma_X + A_1\gamma = \lambda d + A_1\gamma, \tag{13}$$

then $X^{\lambda\gamma_X}X^{\gamma}Y^{\beta} \in C_m$.

(iii) The case $\lambda < 0$ is solved likewise.

In any case, $X^{\alpha}Y^{\beta}$ and $X^{\gamma}Y^{\delta}$ are in the same connected component of ∇_m. □

We now describe the simplicial complexes that correspond to the S-degrees which are multiples of the glued degree.

Lemma 4. *Let S be the gluing of S_1 and S_2, $d \in S$ the glued degree, and $d' \in S \setminus \{d\}$. Then $C_{d'}^{A_1} \neq \emptyset \neq C_{d'}^{A_2}$ if and only if $d' \in (d\mathbb{N}) \setminus \{0\}$. Furthermore, the simplicial complex $\nabla_{d'}$ has at least one connected component with elements in $C_{d'}^{A_1}$ and $C_{d'}^{A_2}$.*

Proof. If there exist $X^{\alpha}, Y^{\beta} \in C_{d'}$, then $d' = \sum_{i=1}^{r}\alpha_i a_i = \sum_{i=1}^{t}\beta_i b_i \in S_1 \cap S_2 \subset G(S_1) \cap G(S_2) = d\mathbb{Z}$. Hence, $d' \in d\mathbb{N}$.

Conversely, let $d' = jd$ with $j \in \mathbb{N}$ and $j > 1$ and let $X^{\gamma_X} - Y^{\gamma_Y} \in I_S$ be a glued binomial. It is easy to see that $X^{j\gamma_X}, Y^{j\gamma_Y} \in C_{d'}$ and thus $\{X^{j\gamma_X}, X^{(j-1)\gamma_X}Y^{\gamma_Y}\}$ and $\{X^{(j-1)\gamma_X}Y^{\gamma_Y}, Y^{j\gamma_Y}\}$ belong to $\nabla_{d'}$. □

The following lemma is a combinatorial version of [11, Lemma 9] and it is a necessary condition of Theorem 6.

Lemma 5. *Let S be the gluing of S_1 and S_2 and $d \in S$ the glued degree. Then the elements of C_d are pure monomials and $d \in$ Betti(S).*

Proof. The order $\preceq_S$ defined by $m'\preceq_S m$ if $m-m' \in S$ is a partial order on S.

Assume that there exists a mixed monomial $T \in C_d$. By Lemma 3, there exists a pure monomial Y^b in C_d such that $\{T, Y^b\} \in \nabla_d$ (the proof is analogous if we consider X^a with $\{T, X^a\} \in \nabla_d$). Now take $T_1 = \gcd(T, Y^b)^{-1}T$ and $Y^{b_1} = \gcd(T, Y^b)^{-1}Y^b$. Both monomials are in $C_{d'}$, where d' is equal to d minus the S-degree of $\gcd(T, Y^b)$. By Lemma 4, if $C_{d'}^{A_1} \neq \emptyset$, then $d' \in d\mathbb{N}$, but since $d' \prec_S d$ this is not possible. So, if T_1 is a mixed monomial and $C_{d'}^{A_1} = \emptyset$, then $C_{d'}^{A_2} \neq \emptyset$. If there exists a pure monomial in $C_{d'}^{A_2}$ connected to a mixed monomial in $C_{d'}$, we perform the same process obtaining $T_2, Y^{b_2} \in C_{d''}$ with T_2 as a mixed monomial and $d'' \prec_S d'$. This process can be repeated if there existed a pure monomial and a mixed monomial in the same connected component. By degree reasons this cannot be performed indefinitely and an element $d^{(i)} \in \text{Betti}(S)$ verifying that $\nabla_{d^{(i)}}$ is not connected having a connected component with only mixed monomials is found. This contradicts Lemma 3. □

After examining the structure of the simplicial complexes associated with glued semigroups, we enunciate a combinatorial characterization by means of the nonconnected simplicial complexes ∇_m.

Theorem 6. *The semigroup S is the gluing of S_1 and S_2 if and only if the following conditions are fulfilled.*

(1) *For all $d' \in \text{Betti}(S)$, any connected component of $\nabla_{d'}$ has at least a pure monomial.*

(2) *There exists a unique $d \in \text{Betti}(S)$ such that $C_d^{A_1} \neq \emptyset \neq C_d^{A_2}$ and the elements in C_d are pure monomials.*

(3) *For all $d' \in \text{Betti}(S) \setminus \{d\}$ with $C_{d'}^{A_1} \neq \emptyset \neq C_{d'}^{A_2}$, $d' \in d\mathbb{N}$.*

Besides, the above $d \in \text{Betti}(S)$ is the glued degree.

Proof. If S is the gluing of S_1 and S_2, the result is obtained from Lemmas 3, 4, and 5.

Conversely, by hypotheses 1 and 3, given that $d' \in \text{Betti}(S) \setminus \{d\}$, the set $\mathcal{M}(I_{S_1})_{d'}$ is constructed from $C_{d'}^{A_1}$ and $\mathcal{M}(I_{S_2})_{d'}$ from $C_{d'}^{A_2}$ as in Construction 1. Analogously, if $d \in \text{Betti}(S)$, the set $\mathcal{M}(I_S)_d$ is obtained from the union of $\mathcal{M}(I_{S_1})_d$, $\mathcal{M}(I_{S_2})_d$ and the binomial $X^{\gamma_X} - Y^{\gamma_Y}$ with $X^{\gamma_X} \in C_d^{A_1}$ and $Y^{\gamma_Y} \in C_d^{A_2}$. Finally

$$\coprod_{m\in\text{Betti}(S)} \left(\mathcal{M}\left(I_{S_1}\right)_m \sqcup \mathcal{M}\left(I_{S_2}\right)_m\right) \sqcup \{X^{\gamma_X} - Y^{\gamma_Y}\} \tag{14}$$

is a generating set of I_S and S is the gluing of S_1 and S_2. □

From Theorem 6 we obtain an equivalent property to Theorem 12 in [11] by using the *language* of monomials and binomials.

Corollary 7. *Let S be the gluing of S_1 and S_2 and $X^{\gamma_X} - Y^{\gamma_Y} \in I_S$ a glued binomial with S-degree d. The ideal I_S is minimally generated by its indispensable binomials if and only if the following conditions are fulfilled.*

(i) *The ideals I_{S_1} and I_{S_2} are minimally generated by their indispensable binomials.*

(ii) *The element $X^{\gamma_X} - Y^{\gamma_Y}$ is an indispensable binomial of I_S.*

(iii) *For all $d' \in \text{Betti}(S)$, the elements of $C_{d'}$ are pure monomials.*

Proof. Suppose that I_S is generated by its indispensable binomials. By [10, Corollary 6], for all $m \in \text{Betti}(S)$ the simplicial complex ∇_m has only two vertices. By Construction 1 $\nabla_d = \{\{X^{\gamma_X}\}, \{Y^{\gamma_Y}\}\}$ and by Theorem 6 for all $d' \in \text{Betti}(S) \setminus \{d\}$ the simplicial $\nabla_{d'}$ is equal to $\nabla_{d'}^{A_1}$ or $\nabla_{d'}^{A_2}$. In any case, $X^{\gamma_X} - Y^{\gamma_Y} \in I_S$ is an indispensable binomial, and I_{S_1}, I_{S_2} are generated by their indispensable binomials.

Conversely, suppose that I_S is not generated by its indispensable binomials. Then, there exists $d' \in \text{Betti}(S) \setminus \{d\}$ such that $\nabla_{d'}$ has more than two vertices in at least two different connected components. By hypothesis, there are not mixed monomials in $\nabla_{d'}$ and thus

(i) if $\nabla_{d'}$ is equal to $\nabla_{d'}^{A_1}$ (or $\nabla_{d'}^{A_2}$), then I_{S_1} (or I_{S_2}) is not generated by its indispensable binomials;

(ii) otherwise, $C_{d'}^{A_1} \neq \emptyset \neq C_{d'}^{A_2}$ and by Lemma 4, $d' = jd$ with $j \in \mathbb{N}$, therefore $X^{(j-1)\gamma_X}Y^{\gamma_Y} \in C_{d'}$ which contradicts the hypothesis.

We conclude that I_S is generated by its indispensable binomials. □

The following example taken from [5] illustrates the above results.

Example 8. Let $S \subset \mathbb{N}^2$ be the semigroup generated by the set

$$\{(13,0), (5,8), (2,11), (0,13), (4,4), (6,6), (7,7), (9,9)\}. \tag{15}$$

In this case, Betti(S) is

$$\begin{gathered}\{(15,15), (14,14), (12,12), (18,18), \\ (10,55), (15,24), (13,52), (13,13)\}.\end{gathered} \tag{16}$$

Using the appropriated notation for the indeterminates in the polynomial ring $\mathbb{K}[x_1, \ldots, x_4, y_1, \ldots, y_4]$ (x_1, x_2, x_3, and x_4 for the first four generators of S and y_1, y_2, y_3, y_4 for the others), the simplicial complexes associated with the elements in Betti(S) are those that appear in Figure 1. From Figure 1 and by using Theorem 6, the semigroup S is the gluing of $\langle(13,0), (5,8), (2,11), (0,13)\rangle$ and $\langle(4,4), (6,6), (7,7), (9,9)\rangle$ and the glued degree is $(13,13)$. From Corollary 7, the ideal I_S is not generated by its indispensable binomials (I_S has only four indispensable binomials).

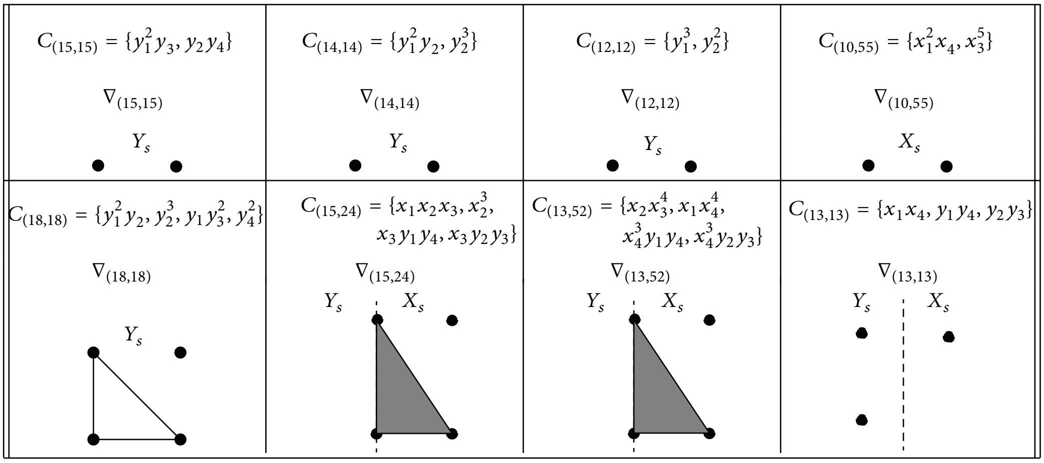

Figure 1: Nonconnected simplicial complexes associated with Betti(S).

4. Generating Glued Semigroups

In this section, an algorithm to obtain examples of glued semigroups is given. Consider $A_1 = \{a_1, \ldots, a_r\}$ and $A_2 = \{b_1, \ldots, b_t\}$ as two minimal generator sets of the semigroups T_1 and T_2 and let $L_j = \{\rho_{ji}\}_i$ be a basis of $\ker T_j$ with $j = 1, 2$. Assume that I_{T_1} and I_{T_2} are nontrivial proper ideals of their corresponding polynomial rings. Consider γ_X and γ_Y be two nonzero elements in $\mathbb{N}^r$ and $\mathbb{N}^t$, respectively, (note that $\gamma_X \notin \ker T_1$ and $\gamma_Y \notin \ker T_2$ because these semigroups are reduced) and the integer matrix

$$A = \left(\begin{array}{c|c} L_1 & 0 \\ \hline 0 & L_2 \\ \hline \gamma_X & -\gamma_Y \end{array} \right). \tag{17}$$

Let S be a semigroup such that $\ker S$ is the lattice generated by the rows of matrix A. This semigroup can be computed by using the Smith Normal Form (see [1, Chapter 3]). Denote by B_1, B_2 two sets of cardinality r and t, respectively, satisfying $S = \langle B_1, B_2 \rangle$ and $\ker(\langle B_1, B_2 \rangle)$ is generated by the rows of A.

The following proposition shows that the semigroup S satisfies one of the necessary conditions to be a glued semigroup.

Proposition 9. *The semigroup S verifies $G(\langle B_1 \rangle) \cap G(\langle B_2 \rangle) = (B_1 \gamma_X)\mathbb{Z} = (B_2 \gamma_Y)\mathbb{Z}$ with $d = B_1 \gamma_X \in \langle B_1 \rangle \cap \langle B_2 \rangle$.*

Proof. Use that $\ker S$ has a basis as (4) and proceed as in the proof of the necessary condition of Proposition 2. □

Because $B_1 \cup B_2$ may not be a minimal generating set, this condition does not assure that S is a glued semigroup. For instance, taking the numerical semigroups $T_1 = \langle 3, 5 \rangle$, $T_2 = \langle 2, 7 \rangle$, and $(\gamma_X, \gamma_Y) = (1, 0, 2, 0)$, the matrix obtained from formula (17) is

$$\left(\begin{array}{cc|cc} 5 & -3 & 0 & 0 \\ \hline 0 & 0 & 7 & -2 \\ \hline 1 & 0 & -2 & 0 \end{array} \right), \tag{18}$$

and $B_1 \cup B_2 = \{12, 20, 6, 21\}$ is not a minimal generating set. The following result solves this issue.

Corollary 10. *The semigroup S is a glued semigroup if*

$$\sum_{i=1}^{r} \gamma_{Xi} > 1, \qquad \sum_{i=1}^{t} \gamma_{Yi} > 1. \tag{19}$$

Proof. Suppose that the set of generators $B_1 \cup B_2$ of S is nonminimal and thus one of its elements is a natural combination of the others. Assume that this element is the first of $B_1 \cup B_2$ and then there exist $\lambda_2, \ldots, \lambda_{r+t} \in \mathbb{N}$ such that $B_1(1, -\lambda_2, \ldots, -\lambda_r) = B_2(\lambda_{r+1}, \ldots, \lambda_{r+t}) \in G(\langle B_1 \rangle) \cap G(\langle B_2 \rangle)$. By Proposition 9, there exists $\lambda \in \mathbb{Z}$ satisfying $B_1(1, -\lambda_2, \ldots, -\lambda_r) = B_2(\lambda_{r+1}, \ldots, \lambda_{r+t}) = B_1(\lambda \gamma_X)$. Since $B_2(\lambda_{r+1}, \ldots, \lambda_{r+t}) \in S$, $\lambda \geq 0$ and thus

$$\nu = \left(1 - \lambda \gamma_{X1}, \underbrace{-\lambda_2 - \lambda \gamma_{X2}, \ldots, -\lambda_r - \lambda \gamma_{Xr}}_{\leq 0} \right) \tag{20}$$

$$\in \ker(\langle B_1 \rangle) = \ker T_1,$$

with the following cases.

(i) If $\lambda \gamma_{X1} = 0$, then T_1 is not minimally generated which it is not possible by hypothesis.

(ii) If $\lambda \gamma_{X1} > 1$, then $0 > \nu \in \ker T_1$, but this is not possible because T_1 is a reduced semigroup.

(iii) If $\lambda \gamma_{X1} = 1$, then $\lambda = \gamma_{X1} = 1$ and

$$\nu = \left(0, \underbrace{-\lambda_2 - \gamma_{X2}, \ldots, -\lambda_r - \gamma_{Xr}}_{\leq 0} \right) \in \ker T_1. \tag{21}$$

If $\lambda_i + \gamma_{Xi} \neq 0$ for some $i = 2, \ldots, r$, then T_1 is not a reduced semigroup. This implies that $\lambda_i = \gamma_{Xi} = 0$ for all $i = 2, \ldots, r$.

We have just proved that $\gamma_X = (1, 0, \ldots, 0)$. In the general case, if S is not minimally generated it is because either γ_X or γ_Y are elements in the canonical bases of $\mathbb{N}^r$ or $\mathbb{N}^t$, respectively. To avoid this situation, it is sufficient to take γ_X and γ_Y satisfying $\sum_{i=1}^{r} \gamma_{Xi} > 1$ and $\sum_{i=1}^{t} \gamma_{Yi} > 1$. □

From the above result we obtain a characterization of glued semigroups: S is a glued semigroup if and only if $\ker S$ has a basis as (4) satisfies Condition (19).

Example 11. Let $T_1 = \langle (-7,2), (11,1), (5,0), (0,1) \rangle \subset \mathbb{Z}^2$ and $T_2 = \langle 3,5,7 \rangle \subset \mathbb{N}$ be two reduced affine semigroups. We compute their associated lattices

$$\begin{aligned} \ker T_1 &= \langle (1,2,-3,-4), (2,-1,5,-3) \rangle, \\ \ker T_2 &= \langle (-4,1,1), (-7,0,3) \rangle. \end{aligned} \tag{22}$$

If we take $\gamma_X = (2,0,2,0)$ and $\gamma_Y = (1,2,1)$, the matrix A is

$$\begin{pmatrix} 1 & 2 & -3 & -4 & 0 & 0 & 0 \\ 2 & -1 & 5 & -3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -4 & 1 & 1 \\ 0 & 0 & 0 & 0 & -7 & 0 & 3 \\ 2 & 0 & 2 & 0 & -1 & -2 & -1 \end{pmatrix} \tag{23}$$

and the semigroup $S \subset \mathbb{Z}_4 \times \mathbb{Z}^2$ is generated by

$$\Big\{ \underbrace{(1,-5,35), (3,12,-55), (1,5,-25), (0,1,0)}_{B_1}, \underbrace{(2,0,3), (2,0,5), (2,0,7)}_{B_2} \Big\}. \tag{24}$$

The semigroup S is the gluing of the semigroups $\langle B_1 \rangle$ and $\langle B_2 \rangle$ and $\ker S$ is generated by the rows of the above matrix. The ideal $I_S \subset \mathbb{C}[x_1, \ldots, x_4, y_1, \ldots, y_3]$ is generated (see [12] to compute I_S when S has torsion) by

$$\Big\{ x_1 x_3^8 x_4 - x_2^3, x_1 x_2^2 - x_3^3 x_4^4, x_1^2 x_3^5 - x_2 x_4^3, x_1^3 x_2 x_3^2 - x_7^7, y_1 y_3 - y_2^2, y_1^3 y_2 - y_3^2, y_1^4 - y_2 y_3, \underbrace{x_1^2 x_3^2 - y_1^5 y_2}_{\text{glued binomial}} \Big\}; \tag{25}$$

then S is really a glued semigroup.

4.1. Generating Affine Glued Semigroups. From Example 11 it be can deduced that the semigroup S is not necessarily torsion-free. In general, a semigroup T is affine (or equivalently it is torsion-free) if and only if the *invariant factors* (the invariant factors of a matrix are the diagonal elements of its Smith Normal Form (see [13, Chapter 2] and [1, Chapter 2])) of the matrix whose rows are a basis of $\ker T$ are equal to one. Assume that zero-columns of the Smith Normal Form of a matrix are located on its right side. We now show conditions for S being torsion-free.

Take L_1 and L_2 as the matrices whose rows form a basis of $\ker T_1$ and $\ker T_2$, respectively, and let P_1, P_2, Q_1, and Q_2 be some matrices with determinant ± 1 (i.e., unimodular matrices) such that $D_1 = P_1 L_1 Q_1$ and $D_2 = P_2 L_2 Q_2$ are the Smith Normal Form of L_1 and L_2, respectively. If T_1 and T_2 are two affine semigroups, the invariant factors of L_1 and L_2 are equal to 1. Then

$$\left(\begin{array}{c|c} D_1 & 0 \\ \hline 0 & D_2 \\ \hline \gamma'_X & \gamma'_Y \end{array} \right) = \left(\begin{array}{c|c|c} P_1 & 0 & 0 \\ \hline 0 & P_2 & 0 \\ \hline 0 & 0 & 1 \end{array} \right) \underbrace{\left(\begin{array}{c|c} L_1 & 0 \\ \hline 0 & L_2 \\ \hline \gamma_X & -\gamma_Y \end{array} \right)}_{=:A} \left(\begin{array}{c|c} Q_1 & 0 \\ \hline 0 & Q_2 \end{array} \right), \tag{26}$$

where $\gamma'_X = \gamma_X Q_1$ and $\gamma'_Y = -\gamma_Y Q_2$. Let s_1 and s_2 be the numbers of zero-columns of D_1 and D_2 ($s_1, s_2 > 0$ because T_1 and T_2 are reduced, see [1, Theorem 3.14]).

Lemma 12. *The semigroup S is an affine semigroup if and only if*

$$\gcd \left(\{\gamma'_{Xi}\}_{i=r-s_1}^{r} \cup \{\gamma'_{Yi}\}_{i=t-s_2}^{t} \right) = 1. \tag{27}$$

Proof. With the conditions fulfilled by T_1, T_2, and (γ_X, γ_Y), the necessary and sufficient condition for the invariant factors of A to be all equal to one is $\gcd(\{\gamma'_{Xi}\}_{i=r-s_1}^{r} \cup \{\gamma'_{Yi}\}_{i=t-s_2}^{t}) = 1$. □

The following corollary gives the explicit conditions that γ_X and γ_Y must satisfy to construct an affine semigroup.

Corollary 13. *The semigroup S is an affine glued semigroup if and only if*

(1) *T_1 and T_2 are two affine semigroups;*

(2) $(\gamma_X, \gamma_Y) \in \mathbb{N}^{r+t}$;

(3) $\sum_{i=1}^{r} \gamma_{Xi}, \sum_{i=1}^{t} \gamma_{Yi} > 1$;

(4) *there exist $f_{r-s_1}, \ldots, f_r, g_{t-s_2}, \ldots, g_t \in \mathbb{Z}$ such that*

$$\begin{aligned} &\left(f_{r-s_1}, \ldots, f_r \right) \cdot \left(\gamma'_{X(r-s_1)}, \ldots, \gamma'_{Xr} \right) \\ &\quad + \left(g_{t-s_2}, \ldots, g_t \right) \cdot \left(\gamma'_{Y(t-s_2)}, \ldots, \gamma'_{Yt} \right) = 1. \end{aligned} \tag{28}$$

Proof. It is trivial by the given construction, Corollary 10 and Lemma 12. □

Therefore, to obtain an affine glued semigroup it is enough to take two affine semigroups and any solution (γ_X, γ_Y) of the equations of the above corollary.

Example 14. Let T_1 and T_2 be the semigroups of Example 11. We compute two elements $\gamma_X = (a_1, a_2, a_3, a_4)$ and $\gamma_Y = (b_1, b_2, b_3)$ in order to obtain an affine semigroup. First of all, we perform a decomposition of the matrix as (26) by computing the integer Smith Normal Form of L_1 and L_2:

$$\left(\begin{array}{cccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ \hline a_1 & a_1 - 2a_2 - a_3 & -7a_1 + 11a_2 + 5a_3 & 2a_1 + a_2 + a_4 & -b_1 & b_1 + 2b_2 + 3b_3 & -3b_1 - 5b_2 - 7b_3 \end{array}\right)$$

$$= \left(\begin{array}{cc|cc|c} 1 & 0 & 0 & 0 & 0 \\ 2 & -1 & 0 & 0 & 0 \\ \hline 0 & 0 & -2 & 1 & 0 \\ 0 & 0 & 7 & -4 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right) \left(\begin{array}{cccc|ccc} 1 & 2 & -3 & -4 & 0 & 0 & 0 \\ 2 & -1 & 5 & -3 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & -4 & 1 & 1 \\ 0 & 0 & 0 & 0 & -7 & 0 & 3 \\ \hline a_1 & a_2 & a_3 & a_4 & -b_1 & -b_2 & -b_3 \end{array}\right) \left(\begin{array}{cccc|ccc} 1 & 1 & -7 & 2 & 0 & 0 & 0 \\ 0 & -2 & 11 & 1 & 0 & 0 & 0 \\ 0 & -1 & 5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 & -1 & 3 \\ 0 & 0 & 0 & 0 & 0 & -2 & 5 \\ 0 & 0 & 0 & 0 & 0 & -3 & 7 \end{array}\right). \tag{29}$$

Second, by Corollary 13, we must find a solution to the system:

$$\begin{aligned} a_1 + a_2 + a_3 + a_4 &> 1, \\ b_1 + b_2 + b_3 &> 1, \\ f_1, f_2, g_1 &\in \mathbb{Z}, \\ f_1 \left(-7a_1 + 11a_2 + 5a_3\right) + f_2 \left(2a_1 + a_2 + a_4\right) & \\ + g_1 \left(-3b_1 - 5b_2 - 7b_3\right) &= 1, \end{aligned} \tag{30}$$

with $a_1, a_2, a_3, a_4, b_1, b_2, b_3 \in \mathbb{N}$. Such solution is computed (in less than a second) using `FindInstance` of Wolfram Mathematica (see [14]):

$$\begin{aligned} &\texttt{FindInstance}\left[\left(-7a_1 + 11a_2 + 5a_3\right) * f_1\right. \\ &\quad + \left(2a_1 + a_2 + a_4\right) * f_2 \\ &\quad + \left(-3b_1 - 5b_2 - 7b_3\right) * g_1 == 1 \\ &\quad \&\& a1 + a2 + a3 + a4 > 1 \\ &\quad \&\& b_1 + b_2 + b_3 > 1 \&\& a_1 \geq 0 \&\& a_2 \geq 0 \\ &\quad \&\& a_3 \geq 0 \&\& a_4 \geq 0 \&\& b_1 \geq 0 \\ &\quad \&\& b_2 \geq 0 \&\& b_3 \geq 0, \\ &\quad \left\{a_1, a_2, a_3, a_4, b_1, b_2, b_3, f_1, f_2, g_1\right\}, \\ &\quad \left.\text{Integers}\right] \\ &\qquad\qquad \Downarrow \\ &\{\{a_1 \longrightarrow 0, a_2 \longrightarrow 0, a_3 \longrightarrow 3, a_4 \longrightarrow 0, b_1 \longrightarrow 1, \\ &\quad b_2 \longrightarrow 1, b_3 \longrightarrow 0, f_1 \longrightarrow 1, f_2 \longrightarrow 0, g_1 \longrightarrow 0\}\}. \end{aligned} \tag{31}$$

We now take $\gamma_X = (0, 0, 3, 0)$ and $\gamma_Y = (1, 1, 0)$, and construct the matrix

$$A = \begin{pmatrix} 1 & 2 & -3 & -4 & 0 & 0 & 0 \\ 2 & -1 & 5 & -3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -4 & 1 & 1 \\ 0 & 0 & 0 & 0 & -7 & 0 & 3 \\ 0 & 0 & 3 & 0 & -1 & -1 & 0 \end{pmatrix}. \tag{32}$$

We have the affine semigroup $S \subset \mathbb{Z}^2$ which is minimally generated by

$$\Big\{ \underbrace{(2, -56), (1, 88), (0, 40), (1, 0)}_{B_1}, \underbrace{(0, 45), (0, 75), (0, 105)}_{B_2} \Big\} \tag{33}$$

satisfying that $\ker S$ is generated by the rows of A and it is the result of gluing the semigroups $\langle B_1 \rangle$ and $\langle B_2 \rangle$. The ideal I_S is generated by

$$\begin{aligned} \Big\{ & x_1 x_3^8 x_4 - x_2^3, x_1 x_2^2 - x_3^3 x_4^4, x_1^2 x_3^5 - x_2 x_4^3, x_1^3 x_2 x_3^2 - x_4^7, \\ & y_1 y_3 - y_2^2, y_1^3 y_2 - y_3^2, y_1^4 - y_2 y_3, \underbrace{x_3^3 - y_1 y_2}_{\text{glued binomial}} \Big\}; \end{aligned} \tag{34}$$

therefore, S is a glued semigroup.

All glued semigroups have been computed by using our program `Ecuaciones` which is available in [15] (this program requires Wolfram Mathematica 7 or above to run).

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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