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Research Article

On Solutions of Variational Inequality Problems via Iterative Methods

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We investigate an algorithm for a common point of fixed points of a finite family of Lipschitz pseudocontractive mappings and solutions of a finite family of γ-inverse strongly accretive mappings. Our theorems improve and unify most of the results that have been proved in this direction for this important class of nonlinear mappings.

1. Introduction

Let C be a subset of a real Hilbert space H. Let $A : C \to H$ be a nonlinear mapping. The *variational inequality problem* for A and C is to

$$\text{find } x^* \in C \text{ such that } \langle Ax^*, v - x^* \rangle \geq 0, \quad \forall v \in C. \tag{1}$$

The set of solutions of variational inequality problem is denoted by $\mathrm{VI}(C, A)$; that is,

$$\mathrm{VI}(C, A) = \{x^* \in C : \langle Ax^*, x - x^* \rangle \geq 0, \ \forall x \in C\}. \tag{2}$$

It is well known that variational inequality theory has emerged as an important tool in studying a wide class of numerous problems in variational inequalities, minimax problems, optimization, physics, and the Nash equilibrium problems in noncooperative games. Several numerical methods have been developed for solving variational inequalities and related optimization problems; see, for instance, [1–5] and the references therein.

A mapping $A : C \subseteq H \to H$ is said to be *γ-inverse strongly accretive* (or γ-inverse strongly monotone) if there exists a positive real number γ such that

$$\langle x - y, Ax - Ay \rangle \geq \gamma \|Ax - Ay\|^2, \quad \forall x, y \in C. \tag{3}$$

If A is γ-inverse strongly accretive, then inequality (3) implies that A is Lipschitzian with constant $L := 1/\gamma$; that is, $\|Ax - Ay\| \leq (1/\gamma)\|x - y\|$, for all $x, y \in C$. If in (3) we have that $\gamma = 0$, then A is called *accretive* (or monotone).

Let C be a closed and convex subset of a real Hilbert space H. A mapping $T : C \to H$ is called *a contraction mapping* if there exists $L \in [0, 1)$ such that $\|Tx - Ty\| \leq L\|x - y\|$ for all $x, y \in C$. If $L = 1$, then T is called *nonexpansive*. A mapping $T : C \to E$ is called *λ-strictly pseudocontractive* of Browder-Petryshyn type [6] if and only if there exists $\lambda \in (0, 1)$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + \lambda \|(I - T)x - (I - T)y\|^2 \quad \forall x, y \in C. \tag{4}$$

T is called *pseudocontractive* if

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + \|(I - T)x - (I - T)y\|^2, \quad \forall x, y \in C. \tag{5}$$

We note that inequalities (4) and (5) can be equivalently written as

$$\langle Tx - Ty, x - y \rangle \leq \|x - y\|^2 - k\|(x - Tx) - (y - Ty)\|^2 \quad \forall x, y \in C, \tag{6}$$

for some $k > 0$ and

$$\langle Tx - Ty, x - y \rangle \leq \|x - y\|^2 \quad \forall x, y \in C, \tag{7}$$

respectively. We remark that T is pseudocontractive if and only if $A := (I - T)$ is accretive. A point $x \in C$ is a *fixed point of* T if $Tx = x$ and we denote by $F(T)$ the set of fixed points of T; that is, $F(T) = \{x \in C : Tx = x\}$.

We observe that in a real Hilbert space H a class of pseudocontractive mappings includes the class of λ-strictly pseudocontractive mappings and hence the classes of nonexpansive and contraction mappings.

Closely related to the variational inequality problems is the problem of finding fixed points of nonexpansive mappings, λ-strict pseudocontraction mappings or pseudocontractive mappings which is the current interest in functional analysis. Several researchers considered a unified approach that approximates a common point of fixed point of nonlinear problems and solutions of variational inequality problems and solutions of variational inequality problems; see, for example, [7–18] and the references therein.

In [19], Takahashi and Toyoda studied the problem of finding a common point of fixed points of a nonexpansive mapping and solutions of a variational inequality problem (1) by considering the following iterative algorithm:

$$\begin{gathered} x_0 \in C, \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n) T P_C (x_n - \lambda_n A x_n), \quad n = 0, 1, \ldots, \end{gathered} \tag{8}$$

where $\{\alpha_n\}$ is a sequence in $(0, 1)$, $\{\lambda_n\}$ is a positive sequence, $T : C \to C$ is a nonexpansive mapping, and $A : C \to H$ is an γ-inverse strongly accretive mapping. They showed that the sequence $\{x_n\}$ generated by (8) converges *weakly* to some $z \in \mathrm{VI}(C, A) \cap F(S)$ provided that the control sequences satisfy some restrictions.

Iiduka and Takahashi [20] reconsidered the common element problem via the following iterative algorithm:

$$\begin{gathered} x_1 = x \in C, \\ x_{n+1} = \alpha_n x + (1 - \alpha_n) T P_C (x_n - \lambda_n A x_n), \quad n = 0, 1, \ldots, \end{gathered} \tag{9}$$

where $T : C \to C$ is a nonexpansive mapping, $A : C \to H$ is a γ-inverse-strongly accretive mapping, $\{\alpha_n\}$ is a sequence in $(0, 1)$, and $\{\lambda_n\}$ is a sequence in $(0, 2\alpha)$. They proved that the sequence $\{x_n\}$ strongly converges to some point $z \in F(T) \cap \mathrm{VI}(C, A)$.

Recently, Zegeye and Shahzad [21] investigated the problem of finding a common point of fixed points of a Lipschitz pseudocontractive mapping T and solutions of a variational inequality problem for γ-inverse strongly accretive mapping A by considering the following iterative algorithm:

$$\begin{gathered} y_n = (1 - \beta_n) x_n + \beta_n T x_n, \\ x_{n+1} = P_C [(1 - \alpha_n) (\delta_n T y_n + \theta_n x_n + \gamma_n P_C [I - \gamma A] x_n)], \end{gathered} \tag{10}$$

where P_C is a metric projection from H onto C and $\{\delta_n\}$, $\{\theta_n\}$, $\{\gamma_n\}$, $\{\alpha_n\}$, $\{\beta_n\}$ are in $(0, 1)$ satisfying certain conditions. Then, they proved that the sequence $\{x_n\}$ converges strongly to the minimum-norm point of $F(T) \cap \mathrm{VI}(C, A)$.

A natural question arises whether we can obtain an iterative scheme which converges strongly to a common point of fixed points of a finite family of pseudocontractive mappings and solutions of a finite family of variational inequality problems for γ-inverse strongly accretive mappings or not.

It is our purpose in this paper to introduce an algorithm and prove that the algorithm converges strongly to a common point of fixed points of a finite family of Lipschitz pseudocontractive mappings and solutions of a finite family of variational inequality problems for γ-inverse strongly accretive mappings. The results obtained in this paper improve and extend the results of Takahashi and Toyoda [19], Iiduka and Takahashi [20], and Zegeye and Shahzad [21], Theorem 3.2 of Yao et al. [22], and some other results in this direction.

2. Preliminaries

In what follows we will make use of the following lemmas.

Lemma 1. *Letting H be a real Hilbert space, the following identity holds:*

$$\|x + y\|^2 \leq \|x\|^2 + 2 \langle y, x + y \rangle, \quad \forall x, y \in H. \tag{11}$$

Lemma 2 (see [23]). *Let C be a nonempty closed and convex subset of a real Hilbert space H. Let $A : C \to E$ be a γ-inverse strongly accretive mapping. Then, for $0 < \mu < 2\gamma$, the mapping $A_\mu x := (x - \mu A x)$ is nonexpansive.*

Lemma 3 (see [24]). *Let C be a nonempty, closed, and convex subset of a smooth Banach space E. Let Q_C be a sunny nonexpansive retraction from E onto C and let A be an accretive operator of C into E. Then for all $\lambda > 0$,*

$$VI(C, A) = F(Q_C (I - \lambda A)). \tag{12}$$

Lemma 4 (see [25]). *Let C be a nonempty, closed, and convex subset of a real Hilbert space H. Let $T_i : C \to E$, $i = 1, \ldots, N$, be nonexpansive mappings such that $\cap_{i=1}^N F(T_i) \neq \emptyset$. Let $T := \theta_1 T_1 + \theta_2 T_2 + \cdots + \theta_N T_N$ with $\theta_1 + \theta_2 + \cdots + \theta_N = 1$. Then T is nonexpansive and $F(T) = \cap_{i=1}^N F(T_i)$.*

Lemma 5 (see [26]). *Let C be a convex subset of a real Hilbert space H. Let $x \in H$. Then $x_0 = P_C x$ if and only if*

$$\langle z - x_0, x - x_0 \rangle \leq 0, \quad \forall z \in C. \tag{13}$$

Lemma 6 (see [27]). *Let C be a closed convex subset of a real Hilbert space H and $A : C \to C$ be a continuous pseudocontractive mapping. Then, for $0 < \mu < 2\gamma$, the mapping $A_\mu x := (x - \mu A x)$ is nonexpansive*

(i) *$F(T)$ is a closed convex subset of C;*

(ii) *$(I - T)$ is demiclosed at zero; that is, if $\{x_n\}$ is a sequence in C such that $x_n \rightharpoonup x$ and $T x_n - x_n \to 0$, as $n \to \infty$, then $x = T(x)$.*

Lemma 7 (see [28]). *Let H be a real Hilbert space. Then for all $x_i \in H$ and $\alpha_i \in [0,1]$ for $i = 1,2,3$ such that $\alpha_1 + \alpha_2 + \alpha_3 = 1$ the following equality holds:*

$$\|\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3\|^2 = \sum_{i=1}^{3} \alpha_i \|x_i\|^2 - \sum_{1 \le i,j \le 3} \alpha_i \alpha_j \|x_i - x_j\|^2. \tag{14}$$

Lemma 8 (see [29]). *Let $\{a_n\}$ be sequences of real numbers such that there exists a subsequence $\{n_i\}$ of $\{n\}$ such that $a_{n_i} < a_{n_i+1}$ for all $i \in \mathbb{N}$. Then there exists an increasing sequence $\{m_k\} \subset \mathbb{N}$ such that $m_k \to \infty$ and the following properties are satisfied by all (sufficiently large) numbers $k \in \mathbb{N}$:*

$$a_{m_k} \le a_{m_k+1}, \qquad a_k \le a_{m_k+1}. \tag{15}$$

In fact, m_k is the largest number n in the set $\{1,2,\ldots,k\}$ such that the condition $a_n \le a_{n+1}$ holds.

Lemma 9 (see [30]). *Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying the following relation:*

$$a_{n+1} \le (1-\alpha_n) a_n + \alpha_n \delta_n, \quad n \ge n_0, \tag{16}$$

where $\{\alpha_n\} \subset (0,1)$ and $\{\delta_n\} \subset \mathbb{R}$ satisfying the following conditions: $\lim_{n\to\infty}\alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\limsup_{n\to\infty}\delta_n \le 0$. Then, $\lim_{n\to\infty}a_n = 0$.

3. Main Result

For the rest of this paper, let $\{a_n\}, \{b_n\}, \{c_n\}, \subset (c,1) \subset (0,1)$, for some $c \in (0,1)$, and $\{\alpha_n\} \subset (0,b) \subset (0,1)$, for some $b \in (0,1)$, satisfy (i) $a_n + b_n + c_n = 1$; (ii) $\lim_{n\to\infty}\alpha_n = 0$; and (iii) $\sum \alpha_n = \infty$.

Theorem 10. *Let C be a nonempty, closed, and convex subset of a real Hilbert space H. Let $T_j : C \to C$, $j = 1,2,\ldots,M$, be Lipschitz pseudocontractive mappings with Lipschitz constants L_i, respectively. Let $A_j : C \to H$, for $j = 1,2,\ldots,N$, be γ_j-inverse strongly accretive mappings. Let $f : C \to C$ be a contraction with constant α. Assume that $\mathcal{F} = [\cap_{j=1}^{M} F(T_j)] \bigcap [\cap_{j=1}^{N} VI(C, A_j)]$ is nonempty. Let a sequence $\{x_n\}$ be generated from an arbitrary $x_0 \in C$ by*

$$\begin{aligned} y_n &= (1-\lambda_n) x_n + \lambda_n T_n x_n; \\ x_{n+1} &= \alpha_n f(x_n) + (1-\alpha_n)(a_n x_n + b_n T_n y_n + c_n G x_n), \end{aligned} \tag{17}$$

where $T_n = T_{n(mod M)}$ and $G := e_0 I + e_1 P_C[I - \gamma A_1] + e_2 P_C[I - \gamma A_2] + \cdots + e_N P_C[I - \gamma A_r]$, for $\gamma \in (0, 2\gamma_0)$, for $\gamma_0 := \min_{1\le j\le N}\{\gamma_j\}$ with $e_0 + e_1 + \cdots + e_r = 1$ and $b_n + c_n \le \lambda_n \le \lambda < 1/(\sqrt{1+L^2}+1)$, $\forall n \ge 0$, for $L = \max\{L_j : 1 \le j \le M\}$. Then, $\{x_n\}$ converges strongly to a point $x^* \in \mathcal{F}$ which is the unique solution of the variational inequality $\langle (I-f)(x^*), x - x^* \rangle \ge 0$ for all $x \in \mathcal{F}$.

Proof. From Lemmas 2, 4, and 3 we get that G is nonexpansive mapping with $F(G) = \cap_{j=1}^{N} \mathrm{VI}(C, A_j)$. Let $p \in \mathcal{F}$. Then from (17), (5), and Lemma 7 we have that

$$\begin{aligned} \|y_n - p\|^2 &= \|(1-\lambda_n)(x_n - p) + \lambda_n (T_n x_n - p)\|^2 \\ &= (1-\lambda_n)\|x_n - p\|^2 + \lambda_n \|T_n x_n - p\|^2 \\ &\quad - \lambda_n (1-\lambda_n)\|x_n - T_n x_n\|^2 \\ &\le (1-\lambda_n)\|x_n - p\|^2 \\ &\quad + \lambda_n \left[\|x_n - p\|^2 + \|x_n - T_n x_n\|^2\right] \\ &\quad - \lambda_n (1-\lambda_n)\|x_n - T_n x_n\|^2 \\ &= \|x_n - p\|^2 + \lambda_n^2 \|x_n - T_n x_n\|^2, \end{aligned} \tag{18}$$

$$\begin{aligned} \|x_{n+1} - p\|^2 &= \|\alpha_n f(x_n) + (1-\alpha_n)(a_n x_n + b_n T_n y_n + c_n G x_n) - p\|^2 \\ &= \|\alpha_n (f(x_n) - p) + (1-\alpha_n) \\ &\qquad \times (a_n (x_n - p) + b_n (T_n y_n - p) + c_n (G x_n - p))\|^2 \\ &\le \alpha_n \|f(x_n) - p\|^2 + (1-\alpha_n) \\ &\quad \times \|a_n (x_n - p) + b_n (T_n y_n - p) + c_n (G x_n - p)\|^2 \\ &\le \alpha_n \|f(x_n) - p\|^2 + (1-\alpha_n) \\ &\quad \times \left[a_n \|x_n - p\|^2 + b_n \|T_n y_n - p\|^2\right. \\ &\qquad \left. + c_n \|G x_n - p\|^2\right] - (1-\alpha_n) b_n a_n \|T_n y_n - x_n\|^2 \\ &\le \alpha_n \|f(x_n) - p\|^2 + (1-\alpha_n)\left[(a_n + c_n)\|x_n - p\|^2\right. \\ &\qquad \left. + b_n \|T_n y_n - p\|^2\right] \\ &\quad - (1-\alpha_n) b_n a_n \|T_n y_n - x_n\|^2 \\ &\le \alpha_n \|f(x_n) - p\|^2 \\ &\quad + (1-\alpha_n)\left[(a_n + c_n)\|x_n - p\|^2\right. \\ &\qquad \left. + b_n \left(\|y_n - p\|^2 + \|y_n - T_n y_n\|^2\right)\right] \\ &\quad - (1-\alpha_n) b_n a_n \|T_n y_n - x_n\|^2. \end{aligned} \tag{19}$$

Now, substituting (18) in (19) we get that

$$\begin{aligned} \|x_{n+1} - p\|^2 &\le \alpha_n \|f(x_n) - p\|^2 \\ &\quad + (1-\alpha_n)\left[(a_n + c_n)\|x_n - p\|^2\right. \\ &\qquad + b_n \left(\|x_n - p\|^2 + \lambda_n^2 \|x_n - T_n x_n\|^2\right) \\ &\qquad \left. + b_n \|y_n - T_n y_n\|^2\right] \\ &\quad - (1-\alpha_n) b_n a_n \|T_n y_n - x_n\|^2 \end{aligned}$$

$$= \alpha_n \|f(x_n) - p\|^2 + (1-\alpha_n)\|x_n - p\|^2 + (1-\alpha_n)\lambda_n^2 b_n \|x_n - T_n x_n\|^2 + (1-\alpha_n) b_n \|y_n - T_n y_n\|^2 - (1-\alpha_n) b_n a_n \|T_n y_n - x_n\|^2. \tag{20}$$

Moreover, from (17), Lemma 7, and Lipschitz property of T_n we get that

$$\begin{aligned}
&\|y_n - T_n y_n\|^2 \\
&= \|(1-\lambda_n)(x_n - T_n y_n) + \lambda_n (T_n x_n - T_n y_n)\|^2 \\
&= (1-\lambda_n)\|x_n - T_n y_n\|^2 + \lambda_n \|T_n x_n - T_n y_n\|^2 \\
&\quad - \lambda_n (1-\lambda_n)\|x_n - T_n x_n\|^2 \\
&\le (1-\lambda_n)\|x_n - T_n y_n\|^2 + \lambda_n L^2 \|x_n - y_n\|^2 \\
&\quad - \lambda_n (1-\lambda_n)\|x_n - T_n x_n\|^2 \\
&= (1-\lambda_n)\|x_n - T_n y_n\|^2 + \lambda_n^3 L^2 \|x_n - T_n x_n\|^2 \\
&\quad - \lambda_n (1-\lambda_n)\|x_n - T_n x_n\|^2 \\
&= (1-\lambda_n)\|x_n - T_n y_n\|^2 \\
&\quad - \lambda_n \left(1 - L^2\lambda_n^2 - \lambda_n\right)\|x_n - T_n x_n\|^2.
\end{aligned} \tag{21}$$

Substituting (21) into (20) we obtain that

$$\begin{aligned}
&\|x_{n+1} - p\|^2 \\
&\le \alpha_n \|f(x_n) - p\|^2 + (1-\alpha_n)\|x_n - p\|^2 \\
&\quad + (1-\alpha_n) b_n \lambda_n^2 \|x_n - T_n x_n\|^2 \\
&\quad + (1-\alpha_n) b_n \left[(1-\lambda_n)\|x_n - T_n y_n\|^2 \right. \\
&\qquad \left. -\lambda_n \left(1 - L^2\lambda_n^2 - \lambda_n\right)\|x_n - T_n x_n\|^2\right] \\
&\quad - (1-\alpha_n) b_n a_n \|T_n y_n - x_n\|^2, \\
&= \alpha_n \|f(x_n) - p\|^2 + (1-\alpha_n)\|x_n - p\|^2 \\
&\quad - (1-\alpha_n)\lambda_n b_n \left[1 - L^2\lambda_n^2 - 2\lambda_n\right]\|x_n - T_n x_n\|^2 \\
&\quad + (1-\alpha_n) b_n [(1-a_n) - \lambda_n]\|T_n y_n - x_n\|^2 \\
&= \alpha_n \|f(x_n) - p\|^2 + (1-\alpha_n)\|x_n - p\|^2 \\
&\quad - (1-\alpha_n)\lambda_n b_n \left[1 - L^2\lambda_n^2 - 2\lambda_n\right]\|x_n - T_n x_n\|^2 \\
&\quad + (1-\alpha_n) b_n [b_n + c_n - \lambda_n]\|T_n y_n - x_n\|^2.
\end{aligned} \tag{22}$$

But, from the hypothesis we have that

$$\begin{gathered}
1 - 2\lambda_n - L^2\lambda_n^2 \ge 1 - 2\lambda - L^2\lambda^2 > 0, \\
b_n + c_n \le \lambda_n, \quad \forall n \ge 0,
\end{gathered} \tag{23}$$

and hence inequality (22) gives that

$$\|x_{n+1} - p\|^2 \le \alpha_n \|f(x_n) - p\|^2 + (1-\alpha_n)\|x_n - p\|^2. \tag{24}$$

But we have that

$$\begin{aligned}
&\|f(x_n) - p\|^2 \\
&= [\|f(x_n) - f(p)\| + \|f(p) - p\|]^2 \\
&\le [\alpha \|x_n - p\| + \|f(p) - p\|]^2 \\
&\le \alpha^2 \|x_n - p\|^2 + \|f(p) - p\|^2 + 2\alpha \|x_n - p\| \|f(p) - p\| \\
&\le \alpha(1+\alpha)\|x_n - p\|^2 + (1+\alpha)\|f(p) - p\|^2.
\end{aligned} \tag{25}$$

Substituting (25) into (24) we get that

$$\begin{aligned}
&\|x_{n+1} - p\|^2 \\
&\le (1 - \alpha_n(1 - \alpha(1+\alpha)))\|x_n - p\|^2 \\
&\quad + \alpha_n (1+\alpha)\|f(p) - p\|^2.
\end{aligned} \tag{26}$$

Therefore, by induction we get that

$$\begin{aligned}
&\|x_{n+1} - p\|^2 \\
&\le \max\left\{\|x_0 - p\|^2, \frac{1+\alpha}{1 - \alpha(1+\alpha)}\|f(p) - p\|^2\right\}, \\
&\qquad \forall n \ge 0,
\end{aligned} \tag{27}$$

which implies that $\{x_n\}$ and hence $\{y_n\}$ are bounded.

Let $x^* = P_{\mathcal{F}} f(x^*)$. Then, from (17), Lemmas 1 and 7, and the methods used to get (22) we obtain that

$$\begin{aligned}
&\|x_{n+1} - x^*\|^2 \\
&\quad = \|\alpha_n (f(x_n) - x^*) \\
&\qquad + (1-\alpha_n)[a_n x_n + b_n T_n y_n + c_n G x_n - x^*]\|^2 \\
&\quad \le (1-\alpha_n)\|a_n (x_n - x^*) + b_n (T_n y_n - x^*) \\
&\qquad + c_n (G x_n - x^*)\|^2 \\
&\qquad + 2\alpha_n \langle f(x_n) - x^*, x_{n+1} - x^* \rangle \\
&\quad \le (1-\alpha_n) b_n \|T_n y_n - x^*\|^2 + (1-\alpha_n) a_n \|x_n - x^*\|^2 \\
&\qquad \times (1-\alpha_n) c_n \|G x_n - p\|^2 \\
&\qquad - (1-\alpha_n) a_n b_n \|T_n y_n - x_n\|^2 \\
&\qquad - (1-\alpha_n) a_n c_n \|G x_n - x_n\|^2 \\
&\qquad + 2\alpha_n \langle f(x_n) - x^*, x_{n+1} - x^* \rangle, \\
&\|x_{n+1} - x^*\|^2 \\
&\quad \le (1-\alpha_n) b_n [\|y_n - x^*\|^2 + \|y_n - T_n y_n\|^2] \\
&\qquad + (1-\alpha_n)(a_n + c_n)\|x_n - x^*\|^2 \\
&\qquad - (1-\alpha_n) a_n b_n \|T_n y_n - x_n\|^2 \\
&\qquad - (1-\alpha_n) a_n c_n \|G x_n - x_n\|^2 \\
&\qquad + 2\alpha_n \langle f(x_n) - x^*, x_{n+1} - x^* \rangle, \\
&\quad \le (1-\alpha_n) b_n [\|x_n - x^*\|^2 + \lambda_n^2 \|x_n - T_n x_n\|^2] \\
&\qquad + (1-\alpha_n) b_n [(1-\lambda_n)\|x_n - T_n y_n\|^2 \\
&\qquad\qquad - \lambda_n (1 - L^2\lambda_n^2 - \lambda_n) \\
&\qquad\qquad \times \|x_n - T_n x_n\|^2] \\
&\qquad + (1-\alpha_n)(a_n + c_n)\|x_n - x^*\|^2 \\
&\qquad - (1-\alpha_n) a_n b_n \|T_n y_n - x_n\|^2 \\
&\qquad - (1-\alpha_n) a_n c_n \|G x_n - x_n\|^2 \\
&\qquad + 2\alpha_n \langle f(x_n) - x^*, x_{n+1} - x^* \rangle
\end{aligned} \tag{28}$$

which implies that

$$\begin{aligned}
&\|x_{n+1} - x^*\|^2 \\
&\quad \le (1-\alpha_n)\|x_n - x^*\|^2 - (1-\alpha_n) b_n \lambda_n [1 - L^2\lambda_n^2 - 2\lambda_n] \\
&\qquad \times \|x_n - T_n x_n\|^2 + (1-\alpha_n) b_n (b_n + c_n - \lambda_n)\|x_n - T_n y_n\|^2 \\
&\qquad - (1-\alpha_n) a_n d_n \|G x_n - x_n\|^2 \\
&\qquad + 2\alpha_n \langle f(x_n) - x^*, x_{n+1} - x^* \rangle
\end{aligned} \tag{29}$$

$$\le (1-\alpha_n)\|x_n - x^*\|^2 + 2\alpha_n \langle f(x_n) - x^*, x_{n+1} - x^* \rangle. \tag{30}$$

But

$$\begin{aligned}
&\langle f(x_n) - x^*, x_{n+1} - x^* \rangle \\
&\quad = \langle f(x_n) - x^*, x_n - x^* \rangle + \langle f(x_n) - x^*, x_{n+1} - x_n \rangle \\
&\quad \le \langle f(x_n) - f(x^*), x_n - x^* \rangle + \langle f(x^*) - x^*, x_n - x^* \rangle \\
&\qquad + \|x_{n+1} - x_n\| \|f(x_n) - x^*\| \\
&\quad \le \alpha \|x_n - x^*\|^2 + \langle f(x^*) - x^*, x_n - x^* \rangle \\
&\qquad + \|x_{n+1} - x_n\| \|f(x_n) - x^*\|.
\end{aligned} \tag{31}$$

Thus, substituting (31) in (30) we obtain that

$$\begin{aligned}
&\|x_{n+1} - x^*\|^2 \\
&\quad \le (1 - \alpha_n (1 - 2\alpha))\|x_n - x^*\|^2 \\
&\qquad + 2\alpha_n \langle f(x^*) - x^*, x_n - x^* \rangle \\
&\qquad + 2\alpha_n \|x_{n+1} - x_n\| \cdot \|f(x_n) - x^*\|.
\end{aligned} \tag{32}$$

Next, we consider two cases.

Case 1. Suppose that there exists $n_0 \in \mathbb{N}$ such that $\{\|x_n - x^*\|\}$ is decreasing for all $n \ge n_0$. Then, we get that $\{\|x_n - x^*\|\}$ is convergent. Thus, from (29) and (23) we have that

$$x_n - T_n x_n \longrightarrow 0, \qquad G x_n - x_n \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{33}$$

Furthermore, from (17) and (33) we obtain that

$$\|y_n - x_n\| = \lambda_n \|x_n - T_n x_n\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty, \tag{34}$$

and hence Lipschitz continuity of T_n, (34), and (33) implies that

$$\begin{aligned}
&\|T_n y_n - x_n\| \\
&\quad \le \|T_n y_n - T_n x_n\| + \|T_n x_n - x_n\| \\
&\quad \le L \|y_n - x_n\| + \|T_n x_n - x_n\| \longrightarrow 0 \\
&\qquad\qquad \text{as } n \longrightarrow \infty.
\end{aligned} \tag{35}$$

Thus, from (33) and (35) we have that

$$\begin{aligned}
&\|x_{n+1} - x_n\| \\
&\quad = \|\alpha_n (f(x_n) - x_n) + (1-\alpha_n) \\
&\qquad \times (a_n x_n + b_n T_n y_n + c_n G x_n) - x_n\| \\
&\quad \le \alpha_n \|f(x_n) - x_n\| + (1-\alpha_n) b_n \|T_n y_n - x_n\| \\
&\qquad + (1-\alpha_n) c_n \|G x_n - x_n\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty.
\end{aligned} \tag{36}$$

Therefore, $\|x_{n+j} - x_n\| \to 0$, as $n \to \infty$, for all $j = 1, 2, \ldots, M$, and hence

$$\begin{aligned} &\|x_n - T_{n+j}x_n\| \\ &\quad \leq \|x_n - x_{n+j}\| + \|x_{n+j} - T_{n+j}x_{n+j}\| \\ &\qquad + L\|x_{n+j} - x_n\| \longrightarrow 0, \end{aligned} \tag{37}$$

as $n \to \infty$, for all $j \in \{1, 2, \ldots, M\}$.

Now, since $\{x_n\}$ is bounded subset of H, we can choose a subsequence $\{x_{n_m}\}$ of $\{x_n\}$ such that $x_{n_m} \rightharpoonup x$ and $\limsup_{n\to\infty}\langle f(x^*) - x^*, x_n - x^*\rangle = \lim_{m\to\infty}\langle f(x^*) - x^*, x_{n_m} - x^*\rangle$. Then, from (37) and Lemma 6 we have that $x \in F(T_j)$, for each $j = 1, 2, \ldots, M$. Hence, $x \in \cap_{j=1}^{M} F(T_j)$.

In addition, since G is nonexpansive, from Lemma 6 we get that $x \in F(G)$ and hence by Lemmas 4 and 3 we obtain that $x \in \mathrm{VI}(C, A_j)$, for each $j \in \{1, 2, \ldots, N\}$.

Therefore, by Lemma 5, we immediately obtain that

$$\begin{aligned} &\limsup_{n\to\infty} \langle f(x^*) - x^*, x_n - x^*\rangle \\ &\quad = \lim_{m\to\infty} \langle f(x^*) - x^*, x_{n_m} - x^*\rangle \\ &\quad = \langle f(x^*) - x^*, x - x^*\rangle \leq 0. \end{aligned} \tag{38}$$

Then, it follows from (32), (38), and Lemma 9 that $\|x_n - x^*\| \to 0$ as $n \to \infty$. Consequently, $x_n \to x^* = P_{\mathcal{F}}(f(x^*))$.

Case 2. Suppose that there exists a subsequence $\{n_i\}$ of $\{n\}$ such that

$$\|x_{n_i} - x^*\| < \|x_{n_i+1} - x^*\|, \tag{39}$$

for all $i \in \mathbb{N}$. Then, by Lemma 8, there exists a nondecreasing sequence $\{m_k\} \subset \mathbb{N}$ such that $m_k \to \infty$, and

$$\|x_{m_k} - x^*\| \leq \|x_{m_k+1} - x^*\|, \qquad \|x_k - x^*\| \leq \|x_{m_k+1} - x^*\|, \tag{40}$$

for all $k \in \mathbb{N}$. Now, from (29) and (23) we get that $x_{m_k} - T_{m_k}x_{m_k} \to 0$ and $Gx_{n_k} - x_{n_k} \to 0$ as $k \to \infty$. Thus, following the method in Case 1, we obtain that $x_{m_k+1} - x_{m_k} \to 0$, $x_{m_k} - T_j x_{m_k} \to 0$, and

$$\limsup_{k\to\infty} \langle f(x^*) - x^*, x_{m_k} - x^*\rangle \leq 0. \tag{41}$$

Furthermore, from (32) and (40) we obtain that

$$\begin{aligned} &\alpha_{m_k}(1 - 2\alpha)\|x_{m_k} - x^*\|^2 \\ &\quad \leq \|x_{m_k} - x^*\|^2 - \|x_{m_k+1} - x^*\|^2 \\ &\qquad + 2\alpha_{m_k}\langle f(x^*) - x^*, x_{m_k} - x^*\rangle \\ &\qquad + 2\alpha_{m_k}\|x_{m_k+1} - x_{m_k}\|\|f(x_{m_k}) - x^*\| \\ &\quad \leq 2\alpha_{m_k}\langle f(x^*) - x^*, x_{m_k} - x^*\rangle \\ &\qquad + 2\alpha_{m_k}\|x_{m_k+1} - x_{m_k}\|\|f(x_{m_k}) - x^*\|. \end{aligned} \tag{42}$$

Now, using the fact that $\alpha_{m_k} > 0$ and (41) we get that

$$\|x_{m_k} - x^*\|^2 \longrightarrow 0 \quad \text{as } k \longrightarrow \infty, \tag{43}$$

and this together with (32) implies that $\|x_{m_k+1} - x^*\| \to 0$ as $k \to \infty$. Since $\|x_k - x^*\| \leq \|x_{m_k+1} - x^*\|$ for all $k \in \mathbb{N}$, we obtain that $x_k \to x^*$. Hence, from the above two cases, we can conclude that $\{x_n\}$ converges strongly to a point $x^* = P_{\mathcal{F}}f(x^*)$, which satisfies the variational inequality $\langle (I - f)(x^*), x - x^*\rangle \geq 0$, for all $x \in \mathcal{F}$. The proof is complete. □

If, in Theorem 10, we assume that $f(x) = u \in C$, a constant mapping, then we get the following corollary.

Corollary 11. *Let C be a nonempty, closed, and convex subset of a real Hilbert space H. Let $T_j : C \to C$, $j = 1, 2, \ldots, M$, be Lipschitz pseudocontractive mappings with Lipschitz constants L_j, respectively. Let $A_j : C \to H$, for $j = 1, 2, \ldots, N$, be γ_j-inverse strongly accretive mappings. Assume that $\mathcal{F} = [\cap_{j=1}^{M} F(T_j)] \bigcap [\cap_{j=1}^{N} VI(C, A_j)]$ is nonempty. Let a sequence $\{x_n\}$ be generated from an arbitrary $x_0, u \in C$ by*

$$\begin{aligned} y_n &= (1 - \lambda_n)x_n + \lambda_n T_n x_n; \\ x_{n+1} &= \alpha_n u + (1 - \alpha_n)(a_n x_n + b_n T_n y_n + c_n G x_n), \end{aligned} \tag{44}$$

where $T_n = T_{n(modM)}$, $G := e_0 I + e_1 P_C[I - \gamma A_1] + e_2 P_C[I - \gamma A_2] + \cdots + e_N P_C[I - \gamma A_r]$, for $\gamma \in (0, 2\gamma_0)$, for $\gamma_0 := \min_{1\leq j\leq N}\{\gamma_j\}$ with $e_0 + e_1 + \cdots + e_r = 1$, and $b_n + c_n \leq \lambda_n \leq \lambda < 1/(\sqrt{1 + L^2} + 1)$, $\forall n \geq 0$, for $L = \max\{L_j : 1 \leq j \leq M\}$. Then, $\{x_n\}$ converges strongly to a unique point $x^* \in C$ satisfying $x^* = P_{\mathcal{F}}(u)$, which is the unique solution of the variational inequality $\langle x^* - u, x - x^*\rangle \geq 0$ for all $x \in \mathcal{F}$.

If, in Theorem 10, we assume that $N = 1$ and $M = 1$, then we get the following corollary which is Theorem 3.1 of [21].

Corollary 12. *Let C be a nonempty, closed, and convex subset of a real Hilbert space H. Let $T : C \to C$ be Lipschitz pseudocontractive mappings with Lipschitz constant L and $A : C \to H$ an γ-inverse strongly accretive mapping. Let $f : C \to C$ be a contraction with constant α. Assume that $\mathcal{F} = F(T) \bigcap VI(C, A)$ is nonempty. Let a sequence $\{x_n\}$ be generated from an arbitrary $x_0 \in C$ by*

$$\begin{aligned} y_n &= (1 - \lambda_n)x_n + \lambda_n T x_n; \\ x_{n+1} &= \alpha_n f(x_n) \\ &\qquad + (1 - \alpha_n)(a_n x_n + b_n T y_n + c_n P_C[I - rA]x_n), \end{aligned} \tag{45}$$

where $r \in (0, 2\gamma)$ and $b_n + c_n \leq \lambda_n \leq \lambda < 1/(\sqrt{1 + L^2} + 1)$, $\forall n \geq 0$. Then, $\{x_n\}$ converges strongly to a point $x^* \in \mathcal{F}$, which is the unique solution of the variational inequality $\langle (I - f)(x^*), x - x^*\rangle \geq 0$ for all $x \in \mathcal{F}$.

If, in Theorem 10, we assume that T_i's are strictly pseudocontractive mappins, then we get the following corollary.

Corollary 13. *Let C be a nonempty, closed, and convex subset of a real Hilbert space H. Let $T_i : C \to C$, $i = 1, 2, \ldots M$, be λ_i-strictly pseudocontractive mappings and let $A_i : C \to H$, for $i = 1, 2, \ldots, N$, be an γ_i-inverse strongly accretive mappings. Let $f : C \to C$ be a contraction with constant α. Assume that $\mathcal{F} = [\cap_{i=1}^{M} F(T_i)] \bigcap [\cap_{i=1}^{N} VI(C, A_i)]$ is nonempty. Let a sequence $\{x_n\}$ be generated from an arbitrary $x_0 \in C$ by*

$$\begin{aligned} y_n &= (1 - \lambda_n) x_n + \lambda_n T_n x_n; \\ x_{n+1} &= \alpha_n f(x_n) \\ &\quad + (1 - \alpha_n)(a_n x_n + b_n T_n y_n + c_n G x_n), \end{aligned} \tag{46}$$

where $T_n = T_{n(modM)}$, $G := c_0 I + e_1 P_C[I - \gamma A_1] + e_2 P_C[I - \gamma A_2] + \cdots + e_N P_C[I - \gamma A_r]$, for $\gamma \in (0, 2\gamma_0)$, for $\gamma_0 := \min_{1 \le i \le N}\{\gamma_i\}$ with $e_0 + e_1 + \cdots + e_r = 1$, and $b_n + c_n \le \lambda_n \le \lambda < 1/(\sqrt{1 + L^2} + 1), \forall n \ge 0$, $L = \max\{(1 + \lambda_i)/\lambda_i\}$. Then, $\{x_n\}$ converges strongly to a point $x^* \in \mathcal{F}$, which is the unique solution of the variational inequality $\langle (I - f)(x^*), x - x^* \rangle \ge 0$ for all $x \in \mathcal{F}$.

If, in Theorem 10, we assume that T_i's are nonexpansive mapping, then we get the following corollary.

Corollary 14. *Let C be a nonempty, closed, and convex subset of a real Hilbert space H. Let $T_i : C \to C$, $i = 1, 2, \ldots, M$, be nonexpansive mappings and let $A_i : C \to H$, for $i = 1, 2, \ldots, N$, be an γ_i-inverse strongly accretive mappings. Let $f : C \to C$ be a contraction with constant α. Assume that $\mathcal{F} = [\cap_{i=1}^{M} F(T_i)] \bigcap [\cap_{i=1}^{N} VI(C, A_i)]$ is nonempty. Let a sequence $\{x_n\}$ be generated from an arbitrary $x_0 \in C$ by*

$$\begin{aligned} y_n &= (1 - \lambda_n) x_n + \lambda_n T_n x_n; \\ x_{n+1} &= \alpha_n f(x_n) \\ &\quad + (1 - \alpha_n)(a_n x_n + b_n T_n y_n + c_n G x_n), \end{aligned} \tag{47}$$

where $T_n = T_{n(modM)}$, $G = c_0 I + e_1 P_C[I - \gamma A_1] + e_2 P_C[I - \gamma A_2] + \cdots + e_N P_C[I - \gamma A_r]$, for $\gamma \in (0, 2\gamma_0)$, for $\gamma_0 := \min_{1 \le i \le N}\{\gamma_i\}$ with $e_0 + e_1 + \cdots + e_r = 1$, $a_n + b_n + c_n = 1$, and $b_n + c_n \le \lambda_n \le \lambda < 1/(\sqrt{2} + 1), \forall n \ge 0$. Then, $\{x_n\}$ converges strongly to point $x^* \in \mathcal{F}$, which is the unique solution of the variational inequality $\langle (I - f)(x^*), x - x^* \rangle \ge 0$ for all $x \in \mathcal{F}$.

We note that the method of proof of Theorem 10 provides the following theorem which is a convergence theorem for a minimum norm point of common fixed points of a finite family of Lipschitz pseudocontractive mappings and common solutions of a finite family of variational inequality problems for accretive mappings.

Theorem 15. *Let C be a nonempty, closed, and convex subset of a real Hilbert space H. Let $T_j : C \to C$, $j = 1, 2, \ldots, M$, be Lipschitz pseudocontractive mappings with Lipschitz constants L_j, respectively. Let $A_j : C \to H$, for $j = 1, 2, \ldots, N$, be γ_j-inverse strongly accretive mappings. Assume that $\mathcal{F} = [\cap_{j=1}^{M} F(T_j)] \bigcap [\cap_{j=1}^{N} VI(C, A_j)]$ is nonempty. Let a sequence $\{x_n\}$ be generated from an arbitrary $x_0 \in C$ by*

$$\begin{aligned} y_n &= (1 - \lambda_n) x_n + \lambda_n T_n x_n; \\ x_{n+1} &= P_C[(1 - \alpha_n)(a_n x_n + b_n T_n y_n + c_n G x_n)], \end{aligned} \tag{48}$$

where $T_n = T_{n(modM)}$, $G := e_0 I + e_1 P_C[I - \gamma A_1] + e_2 P_C[I - \gamma A_2] + \cdots + e_N P_C[I - \gamma A_r]$, for $\gamma \in (0, 2\gamma_0)$, for $\gamma_0 := \min_{1 \le j \le N}\{\gamma_j\}$ with $e_0 + e_1 + \cdots + e_r = 1$, and $b_n + c_n \le \lambda_n \le \lambda < 1/(\sqrt{1 + L^2} + 1), \forall n \ge 0$, for $L = \max\{L_j : 1 \le j \le M\}$. Then, $\{x_n\}$ converges strongly to a unique minimum norm point x^* of $\mathcal{F}$ (i.e., $x^* = P_{\mathcal{F}}(0)$), which is the unique solution of the variational inequality $\langle x^*, x - x^* \rangle \ge 0$ for all $x \in \mathcal{F}$.

4. Numerical Example

Now, we give an example of two Lipschitz pseudocontractive mappings and two γ-inverse strongly accretive mappings satisfying Theorem 10 and some numerical experiment result to explain the conclusion of the theorem as follows.

Example 1. Let $H = \mathbb{R}$ with absolute value norm. Let $C = [-2, 2]$ and let $T_1, T_2 : C \to C$ be defined by

$$\begin{aligned} T_1 x &:= \begin{cases} x + x^2, & x \in [-2, 0], \\ x, & x \in (0, 2], \end{cases} \\ T_2 x &:= \begin{cases} x, & x \in \left[-2, \dfrac{1}{2}\right], \\ x - \left(\dfrac{16}{9}\right)\left(x - \dfrac{1}{2}\right)^2, & x \in \left(\dfrac{1}{2}, 2\right]. \end{cases} \end{aligned} \tag{49}$$

Clearly, for $x, y \in C$ we have that

$$\begin{aligned} \langle (I - T_1) x - (I - T_1) y, x - y \rangle &\ge 0, \\ \langle (I - T_2) x - (I - T_2) y, x - y \rangle &\ge 0 \end{aligned} \tag{50}$$

which show that both mappings are pseudocontractive. Next, we show that T_1 is Lipschitz with $L = 5$. If $x, y \in [-2, 0]$, then

$$\begin{aligned} |T_1 x - T_1 y| &= |x + x^2 - y - y^2| \\ &= |(x + y) + 1| \, |x - y| \le 3 |x - y|. \end{aligned} \tag{51}$$

If $x, y \in (0, 2]$, then

$$|T_1 x - T_1 y| = |x - y|. \tag{52}$$

If $x \in [-2, 0]$ and $y \in (0, 2]$, then

$$\begin{aligned} &|T_1 x - T_1 y| \\ &\quad = |x + x^2 - y| \\ &\quad = |x - y + x^2| = |x - y + x^2 - y^2 + y^2| \\ &\quad = |x - y + x^2 - y^2| + y^2 \\ &\quad \le |x + y + 1| \cdot |x - y| + |y - x|^2 \\ &\quad = (|x + y + 1| + |x + y|) \cdot |x - y| \le 5 |x - y|. \end{aligned} \tag{53}$$

Thus, we get that T_1 is Lipschitz pseudocontractive with $L = 5$ and $F(T_1) = [0, 2]$ which is not nonexpansive, since if we take $x = -2$ and $y = -1.9$, we have that $|T_1x - T_2y| = 0.29 > 0.1 = |x-y|$. Similarly, we can show that T_2 is Lipschitz pseudocontractive with $L = 4$ and $F(T_2) = [-2, 1/2]$ which is not nonexpansive.

Furthermore, for $C = [-2, 2]$, let $A_1, A_2 : C \to \mathbb{R}$ be defined by

$$A_1x := \begin{cases} -\left(x - \frac{1}{2}\right)^2, & x \in \left[-2, \frac{1}{2}\right), \\ 0, & x \in \left[\frac{1}{2}, 2\right], \end{cases}$$
$$A_2x := \begin{cases} 0, & x \in \left[-2, \frac{2}{3}\right], \\ 3\left(x - \frac{2}{3}\right)^2, & x \in \left(\frac{2}{3}, 2\right]. \end{cases} \quad (54)$$

Then we first show that A_1 is γ-inverse strongly accretive mapping with $\gamma = 1/5$.

If $x, y \in [-2, 1/2)$, then

$$\begin{aligned}
&\langle A_1x - A_1y, x - y\rangle \\
&= \left\langle -\left(x - \frac{1}{2}\right)^2 + \left(y - \frac{1}{2}\right)^2, x - y\right\rangle \\
&= \left[\left(x - \frac{1}{2}\right)^2 - \left(y - \frac{1}{2}\right)^2\right](y - x) \\
&= \left[\left(x - \frac{1}{2}\right)^2 - \left(y - \frac{1}{2}\right)^2\right]\left[\left(y - \frac{1}{2}\right) - \left(x - \frac{1}{2}\right)\right] \\
&= \left[\left(x - \frac{1}{2}\right)^2 - \left(y - \frac{1}{2}\right)^2\right]\frac{\left[(y - 1/2)^2 - (x - 1/2)^2\right]}{(y - 1/2) + (x - 1/2)} \\
&= \left[\left(x - \frac{1}{2}\right)^2 - \left(y - \frac{1}{2}\right)^2\right]\frac{\left[(x - 1/2)^2 - (y - 1/2)^2\right]}{(1/2 - x) + (1/2 - y)} \\
&\geq \frac{1}{5}\left|\left(x - \frac{1}{2}\right)^2 - \left(y - \frac{1}{2}\right)^2\right|^2 \\
&= \frac{1}{5}\left|A_1x - A_1y\right|^2.
\end{aligned} \quad (55)$$

If $x \in [-2, 1/2)$ and $y \in [1/2, 2]$, we get that

$$\begin{aligned}
&\langle A_1x - A_1y, x - y\rangle \\
&= \left\langle -\left(x - \frac{1}{2}\right)^2, x - y\right\rangle = \left(x - \frac{1}{2}\right)^2(y - x) \\
&= \left(x - \frac{1}{2}\right)^2\left[\left(y - \frac{1}{2}\right) - \left(x - \frac{1}{2}\right)\right] \\
&\geq \left(x - \frac{1}{2}\right)^2\left(\frac{1}{2} - x\right) \\
&= \left(x - \frac{1}{2}\right)^2\frac{(1/2 - x)^2}{(1/2 - x)} \geq \frac{2}{5}\left|\left(x - \frac{1}{2}\right)^2\right|^2 \\
&\geq \frac{1}{5}\left|A_1x - A_1y\right|^2.
\end{aligned} \quad (56)$$

Table 1

$u = 0.6$	$x_0 = 1$	$u = 0.8$	$x_0 = -1$
n	x_n	n	x_n
0	1.0000	0	−1.0000
500	0.6112	5000	0.0627
10,000	0.5137	10,000	0.4282
12,000	0.5121	15,000	0.4540
14,000	0.5110	20,000	0.4686
18,000	0.5093	25,000	0.4782
20,000	0.5087	35,000	0.4905

If $x, y \in [1/2, 2]$, then we get that $|A_1x - A_1y| = 0$ and hence

$$\langle A_1x - A_1y, x - y\rangle \geq \frac{1}{5}\left|A_1x - A_1y\right|^2. \quad (57)$$

Therefore, A_1 is γ-inverse strongly accretive mapping with $\gamma = 1/5$ and $\mathrm{VI}(C, A_1) = [1/2, 2]$. Similarly, we can show that A_2 is γ-inverse strongly accretive mapping with $\gamma = 1/2$ and $\mathrm{VI}(C, A_2) = [-2, 2/3]$.

Note that we have $F(T_1) \cap F(T_2) \cap \mathrm{VI}(C, A_1) \cap \mathrm{VI}(C, A_2) = \{1/2\}$.

Thus, taking $\alpha_n = 1/(10n + 100)$, $\lambda_n = 2/(n + 100) + 0.065$, $b_n = c_n = 1/(n + 100) + 0.01$, $a_n = 1 - 2/(n + 100) - 0.02$, and $f(x) = u \in C$, we observe that conditions of Theorem 10 are satisfied and Scheme (17) provides the following Table 1 and Figures 1(a) and 1(b) for $u = 0.6$ and $u = 0.8$, respectively.

We observe that the data provides strong convergence of the sequence to the common point of fixed points of both pseudocontractive mappings and solutions of both variational inequality problems for γ-inverse strongly accretive mappings.

Remark 2. Theorem 10 provides an iteration scheme which converges strongly to a common point of fixed points of a finite family of Lipschitzian pseudocontractive mappings and solutions of a finite family of variational inequality problems in Hilbert spaces.

Remark 3. Theorem 10 improves Theorem 3.1 of Takahashi and Toyoda [19], Iiduka and Takahashi [20], and Zegeye and Shahzad [21] and Theorem 3.2 of Yao et al. [22] in the sense that our convergence is to a common point of fixed points of a finite family of Lipschitzian pseudocontractive mappings and solutions of a finite family of variational inequality problems.

Conflict of Interests

The authors declare that they have no conflict of interests regarding the publication of this paper.

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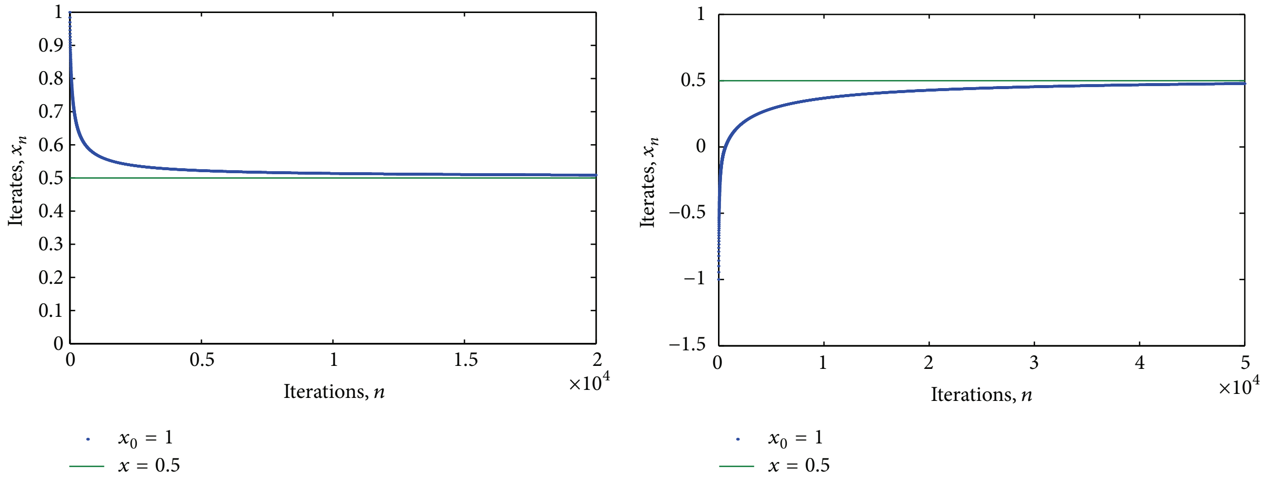

(a) (b)

Figure 1

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