Research Article

Existence and Regularity for Boundary Cauchy Problems with Infinite Delay

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The aim of this work is to investigate a class of boundary Cauchy problems with infinite delay. We give some sufficient conditions ensuring the uniqueness, existence, and regularity of solutions. For illustration, we apply the result to an age dependent population equation, which covers some special cases considered in some recent papers.

1. Introduction

Consider the following problem:

$$
\frac{\partial}{\partial t}u(t,a) = -\frac{\partial}{\partial a}u(t,a) - \mu(t,a)u(t,a), \quad t \geq 0, \ a \geq 0,
$$

$$
u(t,0) = f(t), \quad t \geq 0,
$$

$$
u(0,a) = \varphi(a), \quad a \geq 0,
$$

(1)

where $u(t,a)$ represents the density of the population of age a at time t, μ is the death rate, and $f(t)$ is the number of newborns at time t. Such models were introduced by Lotka in 1925 and have been studied by many authors. For a detailed discussion, we refer the reader to [1, 2].

The problem (1) can be transformed into the following abstract boundary Cauchy problem:

$$
\frac{d}{dt}u(t) = \mathscr{A}u(t) + \Phi(t)u(t), \quad t \geq 0,
$$

$$
Lu(t) = f(t) \in \partial\mathscr{X}, \quad t \geq 0,
$$

$$
u(0) = x \in \mathscr{X},
$$

(2)

where $\mathscr{A}$ is an unbounded operator on a Banach space $(\mathscr{X}, \|\cdot\|)$ of functions on $[0,\infty)$ with domain $D(\mathscr{A})$, $u(t) = u(t,\cdot) \in D(\mathscr{A})$ for each $t \geq 0$, $L : D(\mathscr{A}) \to \partial\mathscr{X}$ is the operator defined by $L(v(\cdot)) = v(0)$ for $v(\cdot) \in D(\mathscr{A})$, and $\partial\mathscr{X} := \{v(0); v(\cdot) \in \mathscr{X}\}$ is a "boundary space." For each $t \geq 0$, $\Phi(t)$ is a bounded linear operator from $\mathscr{X}$ to $\mathscr{X}$.

Equation (2) can be further transformed into a Cauchy problem. To do this, suppose that the domain $D \equiv D(\mathscr{A})$ of $\mathscr{A}$ and $\partial\mathscr{X}$ are Banach spaces such that D is dense and continuously embedded in $\mathscr{X}$. $\mathscr{A} \in B(D,\mathscr{X})$ and $L \in B(D,\partial\mathscr{X})$. We make the following hypothesis.

(S1) $A := \mathscr{A}|_{\ker(L)}$ generates a C_0-semigroup $T(\cdot)$ on $\mathscr{X}$ where $\ker(L)$ denotes the kernel of L.

(S2) L is a surjection from D to $\partial\mathscr{X}$. $L|_{\ker(\lambda-\mathscr{A})}$ has a continuous inverse for any $\lambda \in \rho(A)$ (the resolvent set of A).

If assumptions (S1) and (S2) hold, then the operator $P_\lambda := (L|_{\ker(\lambda-\mathscr{A})})^{-1}$ is continuous from $\partial\mathscr{X}$ to $\mathscr{X}$, and for all $y \in \partial\mathscr{X}$ the operator P_λ satisfies

$$
(\lambda - \mathscr{A})P_\lambda y = 0, \qquad LP_\lambda y = y.
$$

(3)

At least formally, we can rewrite (2) as

$$
\frac{d}{dt}u(t) = A(u(t) - P_\lambda f(t)) + \lambda P_\lambda f(t) + \Phi(t)u(t), \quad t \geq 0,
$$

$$
u(0) = x \in \mathscr{X}.
$$

(4)

It is easy to see that (4) is a form of the following abstract Cauchy problem:

$$\frac{d}{dt}u(t) = A\left(u(t) + CG_1(t, u(t))\right) + G_2(t, u(t)), \quad t \geq 0,$$

$$u(0) = x \in X,$$

$$(5)$$

where A is the infinitesimal generator of a C_0-semigroup on a general Banach space X, C is a bounded linear operator satisfying certain conditions, and $G_1, G_2 : \mathbb{R}^+ \times X \to X$.

In this way, the problem of solving (1) or (2) is transformed to that of solving (5). Equations of the form like (5) were considered in [3–5]. An important tool used is the multiplicative perturbation which was first studied by Desch and Schappacher [3] in 1989 for C_0-semigroup. In recent years, this type of perturbations has been further developed and applied by many authors (cf., e.g., Engel and Nagel [6], Piskarëv and Shaw [7]). In this paper, our proof will also be based on an application of the multiplicative perturbation theorem.

Equation (2) has been considered in [8, 9] for the cases $f(t) = \int_0^\infty \beta(t, a)u(t, a)da$ and $f(t) = \int_0^\infty \int_{-r}^\infty \beta(a, \delta)u(t + \delta, a)d\delta\, da$, respectively. Suppose that $\mathcal{B}$ is a linear space of functions from $(-\infty, 0]$ to $\mathcal{X}$. Then these two cases can be viewed as a function from $\mathcal{B}$ to $\mathcal{X}$. That says that $f(t)$ depends on the "history" of u. Thus, for such functions f, (2) becomes a retarded Cauchy problem.

The following abstract retarded Cauchy problem has been considered by many authors (see [10–13] and the references therein):

$$\frac{d}{dt}u(t) = Au(t) + F(t, u_t), \quad t \geq 0,$$

$$u_0 = \varphi \in \mathcal{P},$$

$$(6)$$

where A generates a C_0-semigroup $T(\cdot)$ on X, $\mathcal{P}$ is a linear space of functions from $(-\infty, 0]$ to X satisfying some axiom which will be described later, F is a function from $[0, \infty) \times \mathcal{P}$ to X, and, for a solution function $u : \mathbb{R} \to X$ and for every $t \geq 0$, the function $u_t : (-\infty, 0] \to X$, defined by

$$u_t(\theta) = u(t + \theta), \quad \text{for } \theta \in (-\infty, 0], \quad (7)$$

is required to belong to $\mathcal{P}$.

The theory of partial differential equations with infinite delay has attracted widespread attention. In [14–16], the variation-of-constant formula

$$u(t) = T(t)\varphi(0) + \int_0^t T(t - s)F(s, u_s)\, ds, \quad t \geq 0,$$

$$u_0 = \varphi \in \mathcal{P}$$

$$(8)$$

is used to study existence of solutions, regularity, existence of periodic solution, and stability for (6) when the delay is finite. In [10], a similar argument is used to solve (6) when an operator A (not necessarily densely defined) satisfies the Hille-Yosida condition (maybe nondensely defined) and the delay is infinite. For a detailed discussion about infinite delay equations, we refer the reader to [13].

The main purpose of this paper is to consider the following more general boundary Cauchy problem with infinite delay:

$$\frac{d}{dt}u(t) = \mathscr{A}u(t) + G_1(t, u_t), \quad t \geq 0,$$

$$Lu(t) = G_2(t, u_t), \quad t \geq 0, \quad (9)$$

$$u_0 = \varphi \in \mathscr{P},$$

where G_1 is a function from $\mathbb{R}^+ \times \mathscr{P}$ to $\mathscr{X}$ and G_2 is a function from $\mathbb{R}^+ \times \mathscr{P}$ to $\partial\mathscr{X}$. G_1 and G_2 may be nonlinear. This abstract boundary delay problem has been studied by Piazzera [8] in some special cases. The case without delay also has been studied in [9]. Similar to the way that (2) is transformed into the form of (5), we can transform (9) into the following generalized retarded abstract Cauchy problem with delay:

$$\frac{d}{dt}u(t) = A\left(u(t) + CF_1(t, u_t)\right) + F_2(t, u_t), \quad t \geq 0,$$

$$u_0 = \varphi \in \mathscr{P},$$

$$(10)$$

where F_1, F_2 are functions from $\mathbb{R}^+ \times \mathscr{P}$ to X. It is a generalization of (5) (and hence of (2)) as well as of (6).

In Section 2, we show the uniqueness and existence of solution of (10). It will be solved by using a variation-of-constant formula similar to (8). The obtained result (Theorem 7) can be viewed as a partial generalization of [8, 9].

Then we apply Theorem 7 in Section 3 to investigate an age dependent population equation for the situation that the birth process depends on the past of the population, as the following system describes:

$$\frac{\partial}{\partial t}u(t, a) = -\frac{\partial}{\partial a}u(t, a) + f_1\left(t, \int_0^\infty u(t - \tau, \xi)\, d\xi, a\right),$$

$$t \geq 0, \ a \geq 0,$$

$$u(t, 0) = \int_0^\infty \int_{-\infty}^0 G(\theta, a)u(t + \theta, a)\, d\theta\, da + f_2(t), \quad t \geq 0,$$

$$u(t, \theta) = \omega(t, \theta), \quad t \leq 0, \ \theta \in \mathbb{R}^+.$$

$$(11)$$

This equation contains as particular cases those equations that are considered in the recent papers [8, 9].

Finally, we study in Section 4 regularity of mild solutions of (10). The property about equilibrium will be studied. The precise definition of equilibrium will be specified later. In [10], it is shown that the equilibrium of the solution semigroup associated with (6) is locally exponentially stable when its linearized solution around this equilibrium is exponentially bounded. We extend this result to a special case of (10).

2. Solutions to (10)

Let $(X, \|\cdot\|)$ be a Banach space. Throughout this paper, $(A, D(A))$ is the infinitesimal generator of C_0-semigroup $T(\cdot)$ on X with domain $D(A)$ and $\|T(t)\| \leq Me^{\omega t}$ for some $M \geq 1, \omega \in \mathbb{R}$, and $t \geq 0$. In this paper, we assume that the phase space

$(\mathscr{P}, \| \cdot \|_{\mathscr{P}})$ is a Banach space consisting of some functions from $(-\infty, 0]$ to X and satisfies the following axioms, which were introduced first by Hale and Kato in [17].

(A1) There exist a positive constant H and functions $M_1(\cdot)$, $M_2(\cdot) : \mathbb{R}^+ \to \mathbb{R}^+$, with M_1 continuous and M_2 locally bounded, such that, for any $\sigma \in \mathbb{R}$ and $a \geq 0$, if $x : (-\infty, \sigma + a] \to X$, $x_\sigma \in \mathscr{P}$ and $x(\cdot)$ is continuous on $[\sigma, \sigma + a]$, then for every $t \in [\sigma, \sigma + a]$ the following conditions hold:

(i) $x_t \in \mathscr{P}$,
(ii) $\|x(t)\| \leq H\|x_t\|_{\mathscr{P}}$,
(iii) $\|x_t\|_{\mathscr{P}} \leq M_1(t - \sigma)\sup_{\sigma \leq s \leq t}\|x(s)\| + M_2(t - \sigma)\|x_\sigma\|_{\mathscr{P}}$.

(A2) For each function $x(\cdot)$ in (A1), $t \mapsto x_t$ is a $\mathscr{P}$-valued continuous function on $[\sigma, \sigma + a]$.

The objective of this section is devoted to investigate well-posedness results for the Cauchy problem:

$$\frac{d}{dt}u(t) = A\left(u(t) + CF_1(t, u_t)\right) + F_2(t, u_t), \quad t \geq 0, \tag{12}$$
$$u_0 = \varphi \in \mathscr{P}.$$

Definition 1. Let A generate a C_0-semigroup $T(\cdot)$ on X. One says that $C \in B(X)$ satisfies condition (M) with respect to $T(\cdot)$ if there is a continuous, nondecreasing function $\sigma_C : [0, \infty) \to [0, \infty)$ with $\sigma_C(0) = 0$ such that $\int_0^t T(t - s)Ch(s)ds \in D(A)$ and

$$\left\| A\int_0^t T(t - s)Ch(s)ds \right\| \leq \sigma_C(t)\sup_{0 \leq s \leq t}\|h(s)\|, \tag{13}$$

for each $h \in C([0, t], X)$ and $t \geq 0$.

The following are important examples of operators that satisfy the condition in Definition 1:

(a) C is a bounded linear operator from X to $(D(A), \| \cdot \|_{D(A)})$, the Banach space $D(A)$ endowed with the graph norm $\|x\|_{D(A)} := \|x\| + \|Ax\|$.

(b) C is a bounded linear operator from X to $F(A)$, where $F(A)$ denotes the Favard space of A given by $F(A) = \{x \in X; \limsup_{t \to 0^+} \|T(t)x - x\| < \infty\}$ and endowed with the norm $\|x\|_{F(A)} := \limsup_{t \to 0^+} \|T(t)x - x\| + \|x\|$.

These conclusions can be found in [3, 4].

In the rest of this paper, we suppose that C satisfies condition (M) with respect to $T(\cdot)$ and the function σ_C satisfies the corresponding properties. Next, we make the hypotheses about F_i for $i = 1, 2$.

(H1) $F \in C([0, \infty) \times \mathscr{P}, X)$ is continuous and satisfies a Lipschitz condition; that is, there is a constant $L > 0$ such that

$$\|F(t, \psi_1) - F(t, \psi_2)\| \leq L\|\psi_1 - \psi_2\|_{\mathscr{P}}, \tag{14}$$

for $\psi_1, \psi_2 \in \mathscr{P}$ and $t \in \mathbb{R}^+$.

Definition 2. Let $T > 0$. A continuous function $u : (-\infty, T] \to X$ is called a mild solution of (12) on $[0, T]$ if u satisfies the following conditions:

(i) $u(t) = T(t)\varphi(0) + A\int_0^t T(t - s)CF_1(s, u_s)ds + \int_0^t T(t - s)F_2(s, u_s)ds$ for $t \in [0, T]$,

(ii) $u(t) = \varphi(t)$ for $t \in (-\infty, 0]$.

Definition 3. Let $T > 0$. A continuous function $u : (-\infty, T) \to X$ is called a classical solution of (12) on $[0, T]$ if u satisfies the following conditions:

(i) $u(t) + CF_1(t, u_t) \in D(A)$ for $t \in [0, T]$,

(ii) $u|_{[0,T]} \in C^1([0, T], X)$ and satisfies (12),

(iii) $u(t) = \varphi(t)$ for $t \in (-\infty, 0]$.

First, we show the uniqueness and existence of mild solutions to (12).

Theorem 4. *Suppose that F_1 and F_2 satisfy hypothesis (H1). Then (12) has a unique mild solution on $[0, T]$ for each $T > 0$.*

Proof. By assumptions on F_1 and F_2, there is a constant $L > 0$ independent of $t \in [0, T]$ such that

$$\sum_{i=1}^2 \|F_i(t, \phi_1) - F_i(t, \phi_2)\| \leq L\|\phi_1 - \phi_2\|_{\mathscr{P}}, \tag{15}$$

for $t \in [0, T]$ and $\phi_1, \phi_2 \in \mathscr{P}$. Moreover, we define the following real number:

$$M_T'' = LM_T'Me^{|\omega|T}, \tag{16}$$

where $M_T' = \max\{M_1', M_2'\}$, $M_i' = \sup_{0 \leq s \leq T}M_i(s)$, $i = 1, 2$, and $M_1(\cdot)$ and $M_2(\cdot)$ are the functions defined in hypothesis (A1). Note that M_T'' is independent of $t \in [0, T]$. Let $[0, r] \subset [0, T]$ and let

$$Z(r) := \left\{y : (-\infty, r] \longrightarrow X; y_0 \in \mathscr{P}, y|_{[0,r]} \in C([0, r], X)\right\} \tag{17}$$

be a Banach space equipped with the norm

$$\|y\|_{Z_r} := \|y_0\|_{\mathscr{P}} + \|y\|_{C([0,r],X)}. \tag{18}$$

Let

$$Z_\varphi(r) := \left\{y \in Z(r); y_0 = \varphi\right\}. \tag{19}$$

Then $Z_\varphi(r)$ is a closed subset of $Z(r)$.

Note that it follows from (A2) and (H1) that $F_i(s, v_s)$, $i = 1, 2$, are continuous in s on $[0, r]$. Then, since C, satisfies condition (M), we have $A\int_0^t T(t - s)CF_1(s, v_s)ds \in C([0, r], X)$ (see the proof of Theorem 2.2 in [7]). Thus we can define $S_r^\varphi : Z_\varphi(r) \to Z_\varphi(r)$ by

$(S_r^\varphi v)(t)$

$$:= \begin{cases} T(t)\varphi(0) + A\displaystyle\int_0^t T(t - s)CF_1(s, v_s)ds \\ \quad + \displaystyle\int_0^t T(t - s)F_2(s, v_s)ds, \quad t \in [0, r], \\ \varphi(t), \hspace{4.5cm} t \in (-\infty, 0]. \end{cases} \tag{20}$$

Note that the closed set $Z_\varphi(r)$ and the operator S_r^φ are dependent on r and φ. From the definition of S_r^φ, one can see that the fixed point of S_r^φ is a mild solution of (12) on $[0, r]$. Furthermore, if $r = T$ and S_T^φ has a unique fixed point, then the fixed point is the unique solution to (12) from the definition of S_T^φ and the proof is completed. So, it is sufficient to show that S_T^φ has a unique fixed point in $Z_\varphi(T)$. The unique fixed point will be found step by step. First, we show that there is an $r \in (0, T]$ such that S_r^φ has a unique fixed point. This fact will be shown by finding an $r \in (0, T]$ such that S_r^φ is a contraction. Suppose that $v_1, v_2 \in Z_\varphi(r)$. For $t \in [0, r]$, by the definition of S_r^φ, assumption of C and hypotheses (A1) and (A2), it follows that

$$\left\| \left(S_r^\varphi v_1 - S_r^\varphi v_2 \right)(t) \right\|$$

$$\leq \left\| A \int_0^t T(t - s) C \left(F_1(s, (v_1)_s) - F_1(s, (v_2)_s) \right) ds \right\|$$

$$+ \left\| \int_0^t T(t - s) \left(F_2(s, (v_1)_s) - F_2(s, (v_2)_s) \right) ds \right\|$$

$$\leq \sigma_C(t) \sup_{0 \leq s \leq t} \left\| F_1(s, (v_1)_s) - F_1(s, (v_2)_s) \right\| \qquad (21)$$

$$+ M t^{|\omega| t} \int_0^t \left\| F_2(s, (v_1)_s) - F_2(s, (v_2)_s) \right\| ds$$

$$\leq \sigma_C(t) L M_T' \|v_1 - v_2\|_{Z_r} + M e^{|\omega| t} L M_T' t \|v_1 - v_2\|_{Z_r}$$

$$\leq \left(\sigma_C(r) + r \right) M_T'' \|v_1 - v_2\|_{Z_r}.$$

So, by the assumption on σ_C, there exist $N \in \mathbb{N}$ and $r \in (0, T]$ such that $Nr = T$ and $(\sigma_C(t) + t) M_T'' < 1$ for each $t \in [0, r]$. On the other hand, $(S_r^\varphi v_1 - S_r^\varphi v_2)(t) = \varphi(t) - \varphi(t) = 0$ for all $t \leq 0$. It follows that S_r^φ is a contraction on $Z_\varphi(r)$. Hence S_r^φ has a unique fixed point $u^1 \in Z_\varphi(r)$ by the contraction mapping principle.

If $r = T$, then the proof is completed. Next, if $2r \leq T$, then the previous argument will be repeated. Let us define the function $\varsigma : (-\infty, 0]$ by

$$\varsigma(t) = \left(u^1 \right)_r (t) = \begin{cases} u^1(t + r), & t \in [-r, 0], \\ \varphi(t + r), & t \in (-\infty, -r]. \end{cases} \qquad (22)$$

Since $u^1 \in Z_\varphi(r)$ implies $u^1 : (-\infty, r] \to X$ with $u^1 \in C([0, r], X)$ and $(u^1)_0 = \varphi$, it follows from the hypotheses (A1)(i) that $\varsigma = (u^1)_r \in \mathcal{P}$.

Now, we can define the closed set $Z_\varsigma(r)$ of $Z(r)$ and define the operator from $Z_\varsigma(r)$ to $Z_\varsigma(r)$ by

$$\left(S_r^\varsigma v \right)(t)$$

$$= \begin{cases} T(t) \varsigma(0) + A \int_0^t T(t - s) C F_1(s + r, v_s) ds \\ \quad + \int_0^t T(t - s) F_2(s + r, v_s) ds, & t \in [0, r], \\ \varsigma(t), & t \in (-\infty, 0], \end{cases} \qquad (23)$$

for each $v \in Z_\varsigma(r)$ and $t \in [0, r]$. Repeating the previous argument, S_r^ς has a unique fixed point u^2 in $Z_\varsigma(r)$. Define $w : (-\infty, 2r]$ by

$$w(t) = \begin{cases} \varphi(t), & -\infty \leq t \leq 0 \\ u^1(t), & 0 \leq t \leq r, \\ u^2(h), & r \leq r + h = t \leq 2r. \end{cases} \qquad (24)$$

Then, we show that w is a fixed point of S_{2r}^φ on $Z_\varphi(2r)$. If $0 \leq t \leq r$, then $w(t) = u^1(t) = S_r^\varphi(u^1)(t)$, so that it follows from (20) that

$$w(t) = T(t) \varphi(0) + A \int_0^t T(t - s) C F_1(s, w_s) ds \qquad (25)$$

$$+ \int_0^t T(t - s) F_2(s, w_s) ds.$$

In particular, for $t = r$, it becomes

$$\varsigma(0) = u^1(r) = w(r)$$

$$= T(r) \varphi(0) + A \int_0^r T(r - s) C F_1(s, w_s) ds \qquad (26)$$

$$+ \int_0^r T(r - s) F_2(s, w_s) ds.$$

If $r \leq t \leq 2r$, let $t = r + h$; then by (26) one has

$$w(t) = u^2(h) = \left(S_r^\varsigma u^2 \right)(h)$$

$$= T(h) \varsigma(0) + A \int_0^h T(h - s) C F_1\left(s + r, \left(u^2 \right)_s \right) ds$$

$$+ \int_0^h T(h - s) F_2\left(r + s, \left(u^2 \right)_s \right) ds$$

$$= T(r + h) \varphi(0) + A \int_0^r T(r + h - s) C F_1(s, w_s) ds$$

$$+ \int_0^r T(r + h - s) F_2(s, w_s) ds$$

$$+ A \int_r^{h+r} T(h + r - s) C F_1(s, w_s) ds$$

$$+ \int_r^{h+r} T(r + h - s) F_2(s, (w)_s) ds$$

$$= T(t) \varphi(0) + A \int_0^t T(t - s) C F_1(s, w_s) ds$$

$$+ \int_0^t T(t - s) F_2(s, w_s) ds.$$

$$(27)$$

Hence w is a fixed point of S_{2r}^φ in $Z_\varphi(2r)$. Since u^1 and u^2 are the unique points in $Z_\varphi(r)$ and $Z_\varsigma(r)$, respectively, it follows that w is the unique fixed point in $Z_\varphi(2r)$. This argument can be repeated until $Nr = T$. At the end, we can find the unique fixed point of S_T^φ on $Z_\varphi(T)$. $\square$

Next, we want to give a sufficient condition for the existence of classical solution to (12). To do this, we need the differentiability of mild solutions. We give the following more restrictive conditions.

(B) If (ϕ_n) is a Cauchy sequence in $\mathscr{P}$ and if (ϕ_n) converges compactly to ϕ on $(-\infty, 0]$ (i.e., for each compact subset E of $\mathbb{R}^-$, $f_n|_E$ is convergent uniformly to $\phi|_E$), then $\phi \in \mathscr{P}$ and $\|\phi_n - \phi\|_{\mathscr{P}} \to 0$, as $n \to \infty$.

(C) For a sequence (ϕ_n) in $\mathscr{P}$, if $\|\phi_n\|_{\mathscr{P}} \to 0$ as $n \to \infty$, then $\|\phi_n(\theta)\| \to 0$, as $n \to \infty$, for each $\theta \in (-\infty, 0]$.

(H2) $F : \mathbb{R} \times \mathscr{P} \to X$ is continuously differentiable and the derivatives $D_1 F$, $D_2 F$ satisfy the following Lipschitz conditions: there is a constant $L > 0$ such that

$$\|D_1 F(t, \psi_1) - D_1 F(t, \psi_2)\| \le L\|\psi_1 - \psi_2\|_{\mathscr{P}},$$
$$\|D_2 F(t, \psi_1) - D_2 F(t, \psi_2)\| \le L\|\psi_1 - \psi_2\|_{\mathscr{P}}, \tag{28}$$

for $t \in [0, \infty)$ and $\psi_1, \psi_2 \in \mathscr{P}$, where D_i denotes the derivative with respect to the ith variable.

The following lemmas are needed.

Lemma 5 (see [13]). *Let $\mathscr{P}$ satisfy axiom (B) and let $f : [0, a] \to \mathscr{P}, a > 0$, be a continuous function such that $f(t)(\theta)$ is continuous for $(t, \theta) \in [0, a] \times (-\infty, 0]$. Then*

$$\left[\int_0^a f(t)\, dt\right](\theta) = \int_0^a f(t)(\theta)\, dt, \tag{29}$$

for $\theta \in (-\infty, 0]$.

Lemma 6 (see [18]). *Let $\mathscr{P}$ satisfy axiom (C) and let $f : [0, a] \to \mathscr{P}, a > 0$, be a continuous function. Then for all $\theta \in (-\infty, a]$, the function $f(\cdot)(\theta)$ is continuous and*

$$\left[\int_0^a f(t)\, dt\right](\theta) = \int_0^a f(t)(\theta)\, dt \tag{30}$$

for $\theta \in (-\infty, 0]$.

Theorem 7. *Let $\mathscr{P}$ satisfy axiom (B) or (C). Assume that F_1 and F_2 satisfy assumptions (H1) and (H2). In addition, assume that $\varphi \in \mathscr{P}$ is continuously differentiable with $\varphi' \in \mathscr{P}$, $\varphi(0) + CF_1(0, \varphi) \in D(A)$, and $\varphi'(0) = A(\varphi(0) + CF_1(0, \varphi)) + F_2(0, \varphi)$. If $u(\cdot, \varphi)$ is the unique mild solution of (12) on $[0, T]$, then u is continuously differentiable on $[0, T]$. Furthermore, $u(\cdot, \varphi)$ is a classical solution of (12) on $[0, T]$.*

Proof. Consider the following equation:

$$y(t) = T(t)\left[A(\varphi(0) + CF_1(0, \varphi)) + F_2(0, \varphi)\right]$$

$$+ A\int_0^t T(t - s)C\left(D_1 F_1(s, u_s) + D_2 F_1(s, u_s) y_s\right) ds$$

$$+ \int_0^t T(t - s)\left(D_1 F_2(s, u_s) + D_2 F_2(s, u_s) y_s\right) ds,$$

$$t \in [0, T],$$

$$y_0 = \varphi' \in \mathscr{P}. \tag{31}$$

A similar argument as in the proof of Theorem 4 shows that there is a unique solution y to (31) on $[0, T]$. Define the function z by

$$z(t) = \begin{cases} \varphi(0) + \displaystyle\int_0^t y(s)\, ds, & t \in [0, T], \\[2mm] \varphi(t), & t \in (-\infty, 0]. \end{cases} \tag{32}$$

We first show that if there is an $r > 0$ such that $u = z$ on $[0, r]$, then u is a classical solution of (12) on $[0, r]$. In fact, in this case, u is a differentiable mild solution. Denote $G_i(s) = F_i(s, u_s)$ for $i = 1, 2$. It is easy to see that G_i is continuously differentiable. Using integration by parts, we can write

$$A\int_0^t T(t - s)CG_1(s)\, ds$$

$$= -A\int_0^{t-s} T(r)CG_1(s)\, dr\Big|_{s=0}^{s=t}$$

$$+ A\int_0^t \int_0^{t-s} T(r)CG_1'(s)\, dr\, ds \tag{33}$$

$$= \int_0^t T(t - s)CG_1'(s)\, ds - CG_1(t) + T(t)CG_1(0).$$

So, from the definition of mild solution, it follows that $u(t) + CG_1(t) = T(t)(\varphi(0) + CG_1(0)) + \int_0^t T(t-s)CG_1'(s)ds + \int_0^t T(t-s)G_2(s)ds$. Furthermore, by the assumption, $\varphi(0) + CG_1(0) \in D(A)$, and Definition 1, we see that $u(t) + CG_1(t) \in D(A)$ for each $t \ge 0$. Hence,

$$A(u(t) + CG_1(t))$$

$$= A\left[T(t)(\varphi(0) + CG_1(0)) + \int_0^t T(t - s)CG_1'(s)\, ds\right.$$

$$\left. + \int_0^t T(t - s)G_2(s)\, ds\right]$$

$$= T(t)A(\varphi(0) + CG_1(0)) + A\int_0^t T(t - s)CG_1'(s)\, ds$$

$$+ A\int_0^t T(t - s)G_2(s)\, ds$$

$$= T(t)A(\varphi(0) + CG_1(0)) + A\int_0^t T(t - s)CG_1'(s)\, ds$$

$$+ \int_0^t T(t - s)G_2'(s)\, ds + T(t)G_2(0) - G_2(t). \tag{$*$}$$

On the other hand, we see that

$$u(t) = A\left[\int_0^t T(r)\varphi(0)\, dr + \int_0^t T(t - s)CG_1(s)\, ds\right]$$

$$+ \varphi(0) + \int_0^t T(t - s)G_2(s)\, ds. \tag{34}$$

So,

$$u(t) - \left[\varphi(0) + \int_0^t T(t-s)G_2(s)\,ds\right]$$
$$= A\left[\int_0^t T(r)\varphi(0)\,dr + \int_0^t T(t-s)CG_1(s)\,ds\right]. \tag{35}$$

Differentiating both sides, we obtain

$$u'(t) - T(t)G_2(0) - \int_0^t T(t-s)G_2'(s)\,ds$$
$$= T(t)A(\varphi(0) + G_1(0)) + A\int_0^t T(t-s)CG_1'(s)\,ds. \tag{$**$}$$

Finally, comparing ($*$) with ($**$), we see that u is a classical solution on $[0, r]$.

Next, we show that there does exist an $r > 0$ such that $u = z$ on $[0, r]$. Recall the integrated semigroup $S(\cdot)$ generated by A; that is, $S(t)x = \int_0^t T(s)x\,ds$ for each $x \in X$. One can obtain that

$$\int_0^t T(s)\left[A(\varphi(0) + CF_1(0,\varphi)) + F_2(0,\varphi)\right]ds$$
$$= \int_0^t T(s)\varphi'(0)\,ds$$
$$= S(t)\varphi'(0),$$

$$\int_0^t \int_0^s T(s-r)C^{2-i}(D_1F_i(r,u_r) + D_2F_i(r,u_r)y_r)\,dr\,ds$$
$$= \int_0^t \int_r^t T(s-r)C^{2-i}(D_1F_i(r,u_r) + D_2F_i(r,u_r)y_r)\,ds\,dr$$
$$= \int_0^t S(t-r)C^{2-i}(D_1F_i(r,u_r) + D_2F_i(r,u_r)y_r)\,dr, \tag{36}$$

for $i = 1, 2$. Here C^0 denotes the identity map. Therefore, by the closedness of A and the assumption on C, z becomes

$$z(t) = \varphi(0) + S(t)\varphi'(0)$$
$$+ \sum_{i=1}^2 A^{2-i}\int_0^t S(t-s)C^{2-i}$$
$$\times (D_1F(s,u_s) + D_2F(s,u_s)y_s)\,ds, \tag{37}$$

for $t \in [0, T]$, where A^0 denotes the identity map. By Lemma 5 or Lemma 6, we obtain

$$z_t = \varphi + \int_0^t y_s\,ds \quad \text{for } t \in [0, T]. \tag{38}$$

By the elementary properties of $S(\cdot)$,

$$S(t)\varphi'(0) = T(t)(\varphi(0) + CF_1(0,\varphi))$$
$$- (\varphi(0) + CF_1(0,\varphi)) + S(t)F_2(0,\varphi)$$
$$= T(t)\varphi(0) - \varphi(0) + AS(t)CF_1(0,\varphi)$$
$$+ S(t)F_2(0,\varphi)$$
$$= T(t)\varphi(0) - \varphi(0) + \sum_{i=1}^2 A^{2-i}S(t)C^{2-i}F_i(0,\varphi), \tag{39}$$

for $t \in [0, T]$. Moreover, using integration by parts and simple computation, one can derive that

$$A^{2-i}\int_0^t T(t-s)C^{2-i}F_i(s,z_s)\,ds$$
$$= A^{2-i}S(t)C^{2-i}F_i(0,\varphi)$$
$$+ A^{2-i}\int_0^t S(t-s)C^{2-i}(D_1F_i(s,z_s) + D_2F_i(s,z_s)y_s)\,ds, \tag{40}$$

for $t \in [0, T]$ and $i = 1, 2$. Consequently, by (36)–(40), z satisfies

$$z(t) = T(t)\varphi(0) + \sum_{i=1}^2 A^{2-i}S(t)C^{2-i}F_i(0,\varphi)$$
$$+ \sum_{i=1}^2 A^{2-i}\left[\int_0^t S(t-s)C^{2-i}\right.$$
$$\left. \times (D_1F_i(s,u_s) + D_2F_i(s,u_s)y_s)\,ds\right]$$
$$= T(t)\varphi(0)$$
$$+ \sum_{i=1}^2 A^{2-i}\left[-\int_0^t S(t-s)C^{2-i}\right.$$
$$\times (D_1F_i(s,z_s) + D_2F_i(s,z_s)y_s)\,ds$$
$$+ \int_0^t T(t-s)C^{2-i}F_i(s,z_s)\,ds$$
$$+ \int_0^t S(t-s)C^{2-i}$$
$$\left. \times (D_1F_i(s,u_s) + D_2F_i(s,u_s)y_s)\,ds\right], \tag{41}$$

for $t \in [0, T]$. On the other hand, by hypotheses (H1) and (H2), there exists $L > 0$ such that

$$\sum_{i=1}^{2} \left\| F_i(t, \phi_1) - F_i(t, \phi_2) \right\| + \sum_{i=1}^{2} \left\| D_i F_1(t, \phi_1) - D_i F_1(t, \phi_2) \right\|$$

$$+ \sum_{i=1}^{2} \left\| D_i F_2(t, \phi_1) - D_i F_2(t, \phi_2) \right\| \le L \left\| \phi_1 - \phi_2 \right\|_{\mathscr{P}}, \tag{42}$$

for $\phi_1, \phi_2 \in \mathscr{P}$. Moreover, we define the following real number:

$$M_T'' = L M_T' M e^{|\omega| T}, \tag{43}$$

where $M_T' = \sup_{0 \le s \le T} M_2(s)$ and $M_2(\cdot)$ is the function defined in hypothesis (A1). Therefore, by (41) and definition of u, one can obtain that

$$\| u(t) - z(t) \|$$

$$\le \sum_{i=1}^{2} \left[\left\| A^{2-i} \int_0^t T(t-s) C^{2-i} \left(F_i(s, u_s) - F_i(s, z_s) \right) ds \right\| \right.$$

$$+ \left\| A^{2-i} \int_0^t S(t-s) C^{2-i} \left(D_1 F_i(s, u_s) \right.\right.$$

$$\left. - D_1 F_i(s, z_s) \right) ds \right\|$$

$$+ \left\| A^{2-i} \int_0^t S(t-s) C^{2-i} \right.$$

$$\left.\left. \times \left(D_2 F_i(s, u_s) y_s - D_2 F_i(s, z_s) y_s \right) ds \right\| \right]. \tag{44}$$

For $i = 1$, by (13), we have

$$\left\| A \int_0^t T(t-s) C \left(F_1(s, u_s) - F_1(s, z_s) \right) ds \right\|$$

$$\le \sigma_C(t) \sup_{0 \le s \le t} \left\| \left(F_1(s, u_s) - F_1(s, z_s) \right) \right\|,$$

$$\left\| A \int_0^t S(t-s) C \left(D_1 F_1(s, u_s) - D_1 F_1(s, z_s) \right) ds \right\|$$

$$= \left\| \int_0^t A \int_0^{t-s} T(r) C \left(D_1 F_1(s, u_s) - D_1 F_1(s, z_s) \right) dr \, ds \right\|$$

$$\le \int_0^t \| T(t-s) - I \| \left\| C \left(D_1 F_1(s, u_s) - D_1 F_1(s, z_s) \right) \right\| ds$$

$$\le 2t M e^{|\omega| T} \sup_{0 \le s \le t} \left\| C \left(D_1 F_1(s, u_s) - D_1 F_1(s, z_s) \right) \right\|, \tag{45}$$

and similarly

$$\left\| A \int_0^t S(t-s) C \left(D_2 F_1(s, u_s) y_s - D_2 F_1(s, z_s) y_s \right) ds \right\|$$

$$\le 2t M e^{|\omega| T} \sup_{0 \le s \le t} \left\| C \left(D_2 F_1(s, u_s) y_s - D_2 F_1(s, z_s) y_s \right) \right\|. \tag{46}$$

Therefore

$$\left\| A \int_0^t T(t-s) C \left(F_1(s, u_s) - F_1(s, z_s) \right) ds \right\|$$

$$+ \left\| A \int_0^t S(t-s) C \left(D_1 F_1(s, u_s) - D_1 F_1(s, z_s) \right) ds \right\|$$

$$+ \left\| A \int_0^t S(t-s) C \left(D_2 F_1(s, u_s) y_s - D_2 F_1(s, z_s) y_s \right) ds \right\|$$

$$\le \sigma_C(t) M_T'' \sup_{0 \le s \le t} \| u(s) - z(s) \|$$

$$+ 2t M_T'' \| C \| \left(1 + \sup_{0 \le s \le T} \| y_s \|_{\mathscr{P}} \right) \sup_{0 \le s \le t} \| u(s) - z(s) \|$$

$$= M_T'' \left[\sigma_C(t) + 2t \| C \| \left(1 + \sup_{0 \le s \le T} \| y_s \|_{\mathscr{P}} \right) \right]$$

$$\times \sup_{0 \le s \le t} \| u(s) - z(s) \|$$

$$= K \sup_{0 \le s \le t} \| u(s) - z(s) \|, \tag{47}$$

where

$$K := M_T'' \left[\sigma_C(r) + 2r \| C \| \left(1 + \max_{0 \le s \le T} \| y \|_{\mathscr{P}} \right) \right]. \tag{48}$$

By the assumption on σ_C, we can choose $N \in \mathbb{N}$ and $r > 0$ such that $Nr = T$ and $K < 1$.

For $i = 2$,

$$\left\| \int_0^t T(t-s) \left(F_2(s, u_s) - F_2(s, z_s) \right) ds \right\|$$

$$+ \left\| \int_0^t S(t-s) \left(D_1 F_2(s, u_s) - D_1 F_2(s, z_s) \right) ds \right\|$$

$$+ \left\| \int_0^t S(t-s) \left(D_2 F_2(s, u_s) y_s - D_2 F_2(s, z_s) y_s \right) ds \right\|$$

$$\le \left(M_T'' + M_T'' T + M_T'' T \times \max_{0 \le s \le T} \| y_s \|_{\mathscr{P}} \right)$$

$$\times \int_0^t \sup_{0 \le \zeta \le s} \| u(\zeta) - z(\zeta) \| ds. \tag{49}$$

Consequently, from (44), (47), and (49), one can derive that

$$\sup_{0 \le \zeta \le t} \|u(\zeta) - z(\zeta)\|$$

$$\le \frac{1}{1-K} \left(M_T'' + M_T'' T + M_T'' T \times \max_{0 \le s \le T} \|y_s\|_{\mathscr{P}} \right) \quad (50)$$

$$\times \int_0^t \sup_{0 \le \zeta \le s} \|u(\zeta) - z(\zeta)\| \, ds,$$

for $0 \le t \le r$. By a standard argument and using Gronwall's inequality, we get $u = z$ on $[0, r]$. So, we have derived that u is continuously differentiable on $[0, r]$, and hence u is a classical solution of (12) on $[0, r]$.

If $r = T$, then the proof is completed. If $r \le t = r + h \le 2r$, from the definition (Definition 2(i)) of u we see that

$$u(r + h)$$

$$= T(r+h)\varphi(0) + A \int_0^{r+h} T(r+h-s) CF_1(s, u_s) \, ds$$

$$+ \int_0^{r+h} T(r+h-s) F_2(s, u_s) \, ds$$

$$= T(r+h)\varphi(0) + A \int_0^r T(r+h-s) CF_1(s, u_s) \, ds$$

$$+ A \int_r^{r+h} T(r+h-s) CF_1(s, u_s) \, ds$$

$$+ \int_0^r T(r+h-s) F_2(s, u_s) \, ds$$

$$+ \int_r^{r+h} T(r+h-s) F_2(s, u_s) \, ds$$

$$= T(h) \left[T(r)\varphi(0) + A \int_0^r T(r-s) CF_1(s, u_s) \, ds \right.$$

$$\left. + \int_0^r T(r-s) F_2(s, u_s) \, ds \right]$$

$$+ A \int_r^{r+h} T(r+h-s) CF_1(s, u_s) \, ds$$

$$+ \int_r^{r+h} T(r+h-s) F_2(s, u_s) \, ds$$

$$= T(h) u(r) + A \int_0^h T(h-s) C(F_1(r+s, u_{r+s})) \, ds$$

$$+ \int_0^h T(h-s) F_2(s+r, u_{r+s}) \, ds.$$

$$(51)$$

It follows that $u(r + \cdot)$ is a mild solution of

$$\frac{d}{dt} x(t) = A(x(t) + CF_1(r+t, x_t)) + F_2(r+t, x_t),$$

$$t \in [0, r], \quad (52)$$

$$x_0 = u_r \in \mathscr{P}.$$

Repeating the previous argument, we can show that $u(r + \cdot)$ is differentiable on $[0, r]$. Hence, u is a classical solution of (12) on $[0, 2r]$. This argument can be repeated until $Nr = T$. At the end, one can show that u is a classical solution of (12) on $[0, T]$. □

3. Application to Age Dependent Population Equations

In this section, the results in the previous section will be applied to age dependent population equations.

Theorem 8 (see [13]). *Let $(X, \|\cdot\|_X)$ be a Banach space and let $\gamma > 0$ be a fixed number. Suppose that $\mathscr{P}$ denotes the space $\mathscr{P} := \{\phi : (-\infty, 0] \to X; \lim_{\theta \to -\infty} e^{\gamma\theta}\phi(\theta) \in X\}$ endowed with the norm $\|\phi\|_{\mathscr{P}} := \sup_{0 \ge \theta} e^{\gamma\theta}\|\phi(\theta)\|_X$; then $\mathscr{P}$ satisfies assumptions (A1), (A2), (B), and (C).*

Let $X = L^1(\mathbb{R}^+, \mathbb{R})$ with L^1 norm $\|\cdot\|$, and let $\mathscr{P}$ be the phase space $\{\phi : (-\infty, 0] \to X; \lim_{\theta \to -\infty} e^{\gamma\theta}\phi(\theta) \in X\}$ endowed with the norm $\|\phi\|_{\mathscr{P}} := \sup_{0 \ge \theta} e^{\gamma\theta}\|\phi(\theta)\|$ for a fixed $\gamma > 0$. Let us consider the following system:

$$\frac{\partial}{\partial t} u(t, a) = -\frac{\partial}{\partial a} u(t, a) + f_1\left(t, \int_0^\infty u(t-\tau, \xi) \, d\xi, a\right),$$

$$t \ge 0, \ a \ge 0,$$

$$u(t, 0) = \int_0^\infty \int_{-\infty}^0 G(\theta, a) u(t+\theta, a) \, d\theta \, da + f_2(t), \quad t \ge 0,$$

$$u(t, \theta) = \omega(t, \theta), \quad t \le 0, \ \theta \in \mathbb{R}^+,$$

$$(53)$$

where $G : (-\infty, 0] \times \mathbb{R}^+ \to \mathbb{R}$, $f_1 : \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}$, $f_2 : \mathbb{R}^+ \to \mathbb{R}$, and $\tau \ge 0$ is a fixed real number.

Remark 9. The linear cases for f_1 and f_2 have been considered by many authors. In [8], Piazzera considers the case that $f_1 : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ is defined by $f_1(t, a) = \mu(a)u(t, a)$, $f_2 \equiv 0$, and $u(t, 0) = \int_0^\infty \int_{-r}^0 \beta(\theta, a)u(t+\theta, a) \, d\theta \, da$, where $\mu \in L^\infty_{\text{loc}}(\mathbb{R}^+, \mathbb{R}^+)$ and $\beta \in L^\infty([-r, 0] \times \mathbb{R}^+, \mathbb{R}^+)$. In [9], the authors consider the case that $f_1(t, a) = \mu(t, a)u(t, a)$ and $u(t, 0) = \int_0^\infty \beta(t, a)u(t, a) \, da$, where $\mu \in L^\infty(\mathbb{R}^+ \times \mathbb{R}^+, \mathbb{R}^+)$, $\beta(\cdot, a) \in C^1(\mathbb{R}^+, \mathbb{R}^+)$, and $\beta(t, \cdot) \in L^1(\mathbb{R}^+, \mathbb{R}^+)$.

In the first step, we rewrite (53) in operator theoretic form on the Banach space X. We define the operator

$$\mathscr{A}\varphi = -\varphi', \quad (54)$$

with domain $D(\mathscr{A}) := W^{1,1}(\mathbb{R}^+, \mathbb{R})$. The boundary operator $L : W^{1,1}(\mathbb{R}^+, \mathbb{R}) \to \mathbb{R}$ is defined by $L\phi = \phi(0)$. Then $A\varphi := (\mathscr{A}|_{\mathrm{Ker}(L)})\varphi = -\varphi'$ with domain $\{\phi \in W^{1,1} \mid \phi(0) = 0\}$ generates a C_0-semigroup $T(\cdot)$ on X. Before doing the next step, we give the following definition and theorem which characterize the condition (M) in some special cases.

Definition 10. Let B generate a C_0-semigroup $S(\cdot)$ on a Banach space $(Y, \|\cdot\|_Y)$. The Favard class $\mathrm{Fav}(B)$ is the space $\{x \mid x \in Y, \; \limsup_{t \to 0^+}(1/t)\|S(t)x - x\|_Y < \infty\}$.

It is known that $\mathrm{Fav}(B)$ becomes a Banach space if we define $\|x\|_{\mathrm{Fav}(B)} := \|x\|_Y + \limsup_{t \to 0^+}(1/t)\|S(t)x - x\|_Y$ for $x \in \mathrm{Fav}(B)$. The following theorem, which gives an important example of operators satisfying condition (M), can be found in [3, 7].

Theorem 11. *Let B generate a C_0-semigroup $S(\cdot)$ on a Banach space Y. If $C \in B(Y, \mathrm{Fav}(B))$, then C satisfies condition (M) with respect to $S(\cdot)$.*

Theorem 12 (see [3]). *Let $\lambda > 0$. Then $\lambda \in \sigma(A)$ and $P_\lambda := (L|_{\mathrm{Ker}(\lambda - \mathscr{A})})^{-1}$ is given by $x \mapsto e^{-\lambda\theta}x$ for $x \in \mathbb{R}$ and $\theta \geq 0$. Moreover, P_λ is a bounded linear operator from $\mathbb{R}$ to $\mathrm{Fav}(A)$.*

According to Theorem 12, we know that $\mathscr{A}$, X, and L satisfy assumptions (S1) and (S2). Now, we suppose that $\lambda > 0$ in the rest of this section.

For rewriting (53), we define the function

$$Q_h : [0, \infty) \times \mathscr{P} \longrightarrow X$$
$$(t, \varphi) \longmapsto h(t, \varphi)\chi_{[0,1]}, \tag{55}$$

where $\chi_{[0,1]}$ denotes the characterization function and $h : [0, \infty) \times \mathscr{P} \to \mathbb{R}$, and define the linear functional:

$$Q : X \longrightarrow \mathbb{R}$$
$$\phi \longmapsto \int_0^1 \phi(t)\,dt. \tag{56}$$

It is easy to see that $Q \in B(X, \mathbb{R})$ and $Q(Q_h(t, \varphi)) = h(t, \varphi)$ for $t \in [0, T]$. Furthermore, by Theorem 12, it follows that $P_\lambda Q \in B(X, \mathrm{Fav}(A))$. So, $P_\lambda Q$ satisfies condition (M) with respect to $T(\cdot)$ by Theorem 11. We introduce the following notations:

(i) $u(t)(\cdot) := u(t, \cdot)$, $\varphi(t)(\cdot) := \omega(t, \cdot)$, $v(t)(\cdot) := v(t, \cdot)$, and $C := P_\lambda Q$;

(ii) $G_1 : \mathbb{R}^+ \times \mathscr{P} \to X$ is defined by $G_1(t, v)(\cdot) := f_1(t, \int_0^\infty v(-\tau)(\xi)d\xi, \cdot)$ $(v \in \mathscr{P})$; here we assume that $f_1(t, \int_0^\infty \psi(a)da, \cdot) \in X$ for each $\psi \in X$ and $t \geq 0$;

(iii) $G_2 : \mathbb{R}^+ \times \mathscr{P} \to \mathbb{R}$ is defined by

$$G_2(t, v) := \int_0^\infty \int_{-\infty}^0 G(\theta, a)\,v(\theta)(a)\,d\theta\,da + f_2(t) \tag{57}$$
$$(v \in \mathscr{P}),$$

where $G : \mathbb{R}^- \times \mathbb{R}^+ \to \mathbb{R}$ and $f_2 : \mathbb{R}^+ \to \mathbb{R}$ are the functions in (53);

(iv) $F_1(t, v) := Q_{G_2}(t, v)$ $(v \in \mathscr{P})$ and $F_2(t, v) := \lambda P_\lambda G_2(t, v) + G_1(t, v)$.

Using these notations, we can rewrite (53) as

$$\frac{d}{dt}u(t) = A(u(t) - CF_1(t, u_t)) + F_2(t, u_t), \quad t \in \mathbb{R}^+,$$
$$u_0(t) = \varphi(t), \quad t \in \mathbb{R}^-. \tag{58}$$

Indeed, condition $u_0(t) = \varphi(t), t \in \mathbb{R}^-$, means $u(t, \theta) = \omega(t, \theta), t \in \mathbb{R}^-, \theta \in \mathbb{R}^+$. To see that the differential equation in (58) is equivalent to the first two lines of (53), we first suppose that $u(\cdot)$ is a solution of (58). By the definition of C and the definition of P_λ in Theorem 12, we see that

$$\frac{\partial}{\partial t}u(t, \cdot) = \frac{d}{dt}u(t) = A(u(t) - CF_1(t, u_t)) + F_2(t, u_t)$$
$$= A(u(t) - CF_1(t, u_t)) + \lambda P_\lambda G_2(t, u_t) + G_1(t, u_t)$$
$$= \mathscr{A}u(t) - P_\lambda Q Q_{G_2}(t, u_t)$$
$$\quad + \lambda P_\lambda G_2(t, u_t) + G_1(t, u_t)$$
$$= \mathscr{A}u(t) + (\lambda - \mathscr{A})P_\lambda G_2(t, u_t) + G_1(t, u_t)$$
$$= \mathscr{A}u(t) + G_1(t, u_t), \tag{59}$$

which is the equation in the first line of (53). Moreover, condition $u(t) - CF_1(t, u_t) \in D(A)$ means that

$$(u(t))(0) - (CF_1(t, u_t))(0)$$
$$= (u(t))(0) - P_\lambda G_2(t, u_t)(0) \tag{60}$$
$$= (u(t))(0) - e^{-\lambda 0}G_2(t, u_t) = 0,$$

and so $(u(t))(0) = G_2(t, u_t)$, that is, the second line of (53). Hence $(u(\cdot))(\cdot)$ is a solution of (53).

Conversely, it is easy to see that $(u(\cdot))(\cdot) = u(\cdot, \cdot)$ is a solution of (58) wherever $u(\cdot, \cdot)$ is a solution of (53).

In the rest of this section, we suppose that the following conditions on the functions f_1, f_2, ω, and G hold.

(I) Suppose that a function $f_1 : \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}$ satisfies the following conditions:

(a) For each $(t, s) \in \mathbb{R}^+ \times \mathbb{R}$, the function $f_1(t, s, \cdot) \in X$. f_1 is continuously differentiable with respect to the first and second variables.

(b) There are $L > 0$ and $\rho \in X$ such that

$$|f_1(t, x_1, a) - f_1(t, x_2, a)|$$
$$+ \sum_{i=1}^2 |D_i f_1(t, x_1, a) - D_i f_1(t, x_2, a)| \tag{61}$$
$$\leq L|x_1 - x_2|\rho(a),$$

for $t \in \mathbb{R}^+$, $a \geq 0$, and $x_1, x_2 \in \mathbb{R}$.

(II) f_2 is differentiable. There is a function $g : \mathbb{R}^+ \to \mathbb{R}^+$ such that $|G(t, a)| \le g(t)$ for all $a \ge 0$ and $t \le 0$, and $g(\cdot)e^{-\gamma \cdot}$ is integrable on $(-\infty, 0]$.

(III) (a) $\omega \in C^1(\mathbb{R}^- \times \mathbb{R}^+, \mathbb{R})$, $t \mapsto \omega(t, \cdot) \in \mathscr{P}$, $t \mapsto (\partial/\partial t)\omega(t, \cdot)$ is continuous in X and $\lim_{t \to -\infty} e^{\gamma t} \int_0^\infty |(\partial/\partial t)\omega(t, x)| dx < \infty$.

(b) ω satisfies

$$\omega(0, 0) = \int_0^\infty \int_{-\infty}^0 G(\theta, a)\, \omega(\theta, a)\, d\theta\, da + f_2(0),$$

$$\frac{\partial}{\partial t}\omega(0, a) = -\frac{\partial}{\partial a}\omega(0, a) + f_1\left(0, \int_0^\infty \omega(-\tau, \xi)\, d\xi, a\right),$$

(62)

for $a \in \mathbb{R}^+$.

Now, we are going to verify that all assumptions of Theorem 7 are satisfied.

Lemma 13. F_1 satisfies hypotheses (H1) and (H2).

Proof. From the definition of F_1 and assumption (II), it follows that

$$\|F_1(t, \psi_1) - F_1(t, \psi_2)\|$$
$$= \int_0^\infty |G_2(t, \psi_1)(\chi_{[0,1]}(s)) - G_2(t, \psi_2)(\chi_{[0,1]}(s))|\, ds$$
$$= |G_2(t, \psi_1) - G_2(t, \psi_2)|$$
$$\le \int_0^\infty \int_{-\infty}^0 |G(\theta, a)\, \psi_1(\theta)(a) - G(\theta, a)\, \psi_2(\theta)(a)|\, d\theta\, da$$
$$\le \int_{-\infty}^0 \int_0^\infty |G(\theta, a)\, e^{-\gamma \theta}|\, e^{\gamma \theta}\, |\psi_1(\theta)(a) - \psi_2(\theta)(a)|\, da\, d\theta$$
$$\le \int_{-\infty}^0 g(\theta)\, e^{-\gamma \theta} e^{\gamma \theta}\, \|\psi_1(\theta) - \psi_2(\theta)\|\, d\theta$$

(use assumption (II)),

$$\le \int_{-\infty}^0 g(\theta)\, e^{-\gamma \theta} d\theta \|\psi_1 - \psi_2\|_{\mathscr{P}},$$

(63)

for $\psi_1, \psi_2 \in \mathscr{P}$. Obviously, $\|D_1 F_1(t, \psi_1) - D_1 F_1(t, \psi_2)\| = 0$. Since

$$v \longmapsto \int_0^\infty \int_{-\infty}^0 G(\theta, a)\, v(\theta)(a)\, d\theta\, da$$

(64)

is a linear transformation, it follows that

$$\|D_2 F_1(t, \psi_1) - D_2 F_1(t, \psi_2)\| = 0.$$

(65)

Consequently, F_1 satisfies hypotheses (H1) and (H2). □

Lemma 14. The function G_1 satisfies hypotheses (H1) and (H2).

Proof. Suppose that $\phi_1, \phi_2 \in X$ and $t \ge 0$. From assumption (I)(a), it follows that G_1 is differentiable with respect to the first variable. By assumption (I)(b) and the definition of norm $\|\cdot\|_{\mathscr{P}}$, it follows that

$$\|G_1(t, \phi_1) - G_1(t, \phi_2)\| + \|D_1 G_1(t, \phi_1) - D_1 G_1(t, \phi_2)\|$$
$$\le L \left| \int_0^\infty \phi_1(-\tau)(\xi)\, d\xi - \int_0^\infty \phi_2(-\tau)(\xi)\, d\xi \right| \times \|\rho\|$$
$$\le L \|\rho\|\, \|\phi_1(-\tau) - \phi_2(-\tau)\|$$
$$\le L e^{\gamma \tau} \|\rho\| \times \|\phi_1 - \phi_2\|_{\mathscr{P}}.$$

(66)

So, hypotheses (H1) and (H2) hold.

Let $\phi \in \mathscr{P}$ be a fixed element of $\mathscr{P}$. Define $\mathscr{L}_{t,\phi} : \mathscr{P} \to X$ by

$$\mathscr{L}_{t,\phi}\psi := G_1(t, \phi)(\cdot) \int_0^\infty \psi(-\tau)(\xi)\, d\xi,$$

(67)

for $\psi \in \mathscr{P}$. Obviously, $\mathscr{L}_{t,\phi}$ is well defined and linear. We show that $D_2 G_1(t, \phi) = \mathscr{L}_{t,\phi}$. Let $\psi \in \mathscr{P}$ with $\|\phi - \psi\|_{\mathscr{P}} \ne 0$. Because of assumption (I)((a) and (b)) and the Mean Value Theorem, for each pair $(t, a) \in \mathbb{R}^+ \times \mathbb{R}^+$, there exists $\varsigma_{t,a}$ between $\int_0^\infty \phi(-\tau)(\xi)d\xi$ and $\int_0^\infty \psi(-\tau)(\xi)d\xi$ such that

$$\frac{\|G_1(t, \psi) - G_1(t, \phi) - \mathscr{L}_{t,\phi}(\psi - \phi)\|}{\|\phi - \psi\|_{\mathscr{P}}}$$
$$= \frac{1}{\|\phi - \psi\|_{\mathscr{P}}}$$
$$\times \left[\left\| f_1\left(t, \int_0^\infty \phi(-\tau)(\xi)\, d\xi, a\right) \right. \right.$$
$$- f_1\left(t, \int_0^\infty \psi(-\tau)(\xi)\, d\xi, a\right)$$
$$- D_2 f_1\left(t, \int_0^\infty \phi(-\tau)(\xi)\, d\xi, a\right)$$
$$\left. \left. \times \left(\int_0^\infty \psi(-\tau)(\xi)\, d\xi - \int_0^\infty \phi(-\tau)(\xi)\, d\xi\right) \right\| \right]$$
$$= \frac{1}{\|\phi - \psi\|_{\mathscr{P}}}$$
$$\times \left\| \left[D_2 f_1(t, \varsigma_{t,a}, a) - D_2 f_1\left(t, \int_0^\infty \phi(-\tau)(\xi)\, d\xi, a\right) \right] \right\|$$
$$\times \left| \int_0^\infty \phi(-\tau)(\xi)\, d\xi - \int_0^\infty \psi(-\tau)(\xi)\, d\xi \right|$$
$$\le \frac{1}{\|\phi - \psi\|_{\mathscr{P}}} \left[\int_0^\infty \left| \varsigma_{t,a} - \int_0^\infty \phi(-\tau)(\xi)\, d\xi \right| \rho(a)\, da \right]$$
$$\times L e^{\gamma \tau} \|\phi - \psi\|_{\mathscr{P}}$$

$$\le Le^{\gamma\tau}\left|\int_0^\infty \psi(-\tau)(\xi)\,d\xi - \int_0^\infty \phi(-\tau)(\xi)\,d\xi\right| \times \|\rho\|$$

$$= Le^{\gamma\tau}\|\psi(-\tau) - \phi(-\tau)\|\,\|\rho\| \le Le^{\gamma\tau}\|\psi - \phi\|_{\mathscr{P}}\,\|\rho\|. \tag{68}$$

Hence $\|G_1(t,\phi) - G_1(t,\psi) - \mathscr{L}_{t,\phi}(\phi - \psi)\|/\|\phi - \psi\|_{\mathscr{P}} \to 0$ as $\|\phi - \psi\|_{\mathscr{P}} \to 0$. So, we obtain that $D_2 G_1(t,\phi) = \mathscr{L}_{t,\phi}$. Moreover, from assumption (I)(b) and the definition of $\mathscr{P}$, it is easy to see that

$$\begin{aligned} &\|D_2 G_1(t,\phi_1) - D_2 G_1(t,\phi_2)\| \\ &\quad = \left\|\mathscr{L}_{t,\phi_1} - \mathscr{L}_{t,\phi_2}\right\| \\ &\quad \le Le^{\gamma\tau}\|\rho\| \times \|\phi_1 - \phi_2\|_{\mathscr{P}}. \end{aligned} \tag{69}$$

Consequently, G_1 satisfies hypothesis (H2). $\qquad\square$

Lemma 15. *F_2 satisfies hypotheses (H1) and (H2).*

Proof. In view of Lemma 14, it suffices to show that the operator $\lambda P_\lambda G_2$ satisfies hypotheses (H1) and (H2). Suppose that $t \ge 0$ and $\psi_1, \psi_2 \in \mathscr{P}$. Using the estimate in the proof of Lemma 13, we see that

$$\begin{aligned} &\|\lambda P_\lambda G_2(t,\psi_1) - \lambda P_\lambda G_2(t,\psi_2)\| \\ &\quad \le \left\|\lambda e^{-\lambda\cdot}G_2(t,\psi_1) - \lambda e^{-\lambda\cdot}[G_2(t,\psi_2)]\right\| \\ &\quad \le \left\|\lambda e^{-\lambda\cdot}\right\| \times |G_2(t,\psi_1) - G_2(t,\psi_2)| \\ &\quad = |G_2(t,\psi_1) - G_2(t,\psi_2)| \\ &\quad \le \int_{-\infty}^0 g(\theta)e^{-\gamma\theta}\,d\theta\|\psi_1 - \psi_2\|_{\mathscr{P}}. \end{aligned} \tag{70}$$

Next, from the definition of G_2, it is easy to see that

$$\|D_1(\lambda P_\lambda G_2(t,\psi_1)) - D_1(\lambda P_\lambda G_2(t,\psi_2))\| = 0. \tag{71}$$

Since

$$v \longmapsto \int_0^\infty \int_{-\infty}^0 G(\theta,a)\,v(\theta)(a)\,d\theta\,da \tag{72}$$

is a linear transformation, it follows that $\|D_2(\lambda P_\lambda G_2(t,\psi_1)) - D_2(\lambda P_\lambda G_2(t,\psi_2))\| = 0$. So, $\lambda P_\lambda G_2$ satisfies hypotheses (H1) and (H2). The proof is completed. $\qquad\square$

Lemma 16. *$\varphi \in \mathscr{P}$ is continuously differentiable with $\varphi' \in \mathscr{P}$, $\varphi(0) - CF_1(0,\varphi) \in D(A)$, and $\varphi'(0) = A(\varphi(0) - CF_1(0,\varphi)) + F_2(0,\varphi)$.*

Proof. Let $t \ge 0$. By the assumption of (III)(a) and the Mean Value Theorem, we know that there is an ξ_s between t and $t+h$ such that

$$\begin{aligned} &\left\|\frac{\phi(t+h) - \phi(t)}{h} - \phi'(t)\right\| \\ &\quad = \int_0^\infty \frac{|\omega(t+h,s) - \omega(t,s) - (\partial/\partial t)\omega(t,s)h|}{|h|}\,ds \\ &\quad = \int_0^\infty \frac{|[(\partial/\partial t)\omega(\xi_s,s) - (\partial/\partial t)\omega(t,s)]h|}{|h|}\,ds \\ &\quad = \int_0^\infty \left|\frac{\partial}{\partial t}\omega(\xi_s,s) - \frac{\partial}{\partial t}\omega(t,s)\right|\,ds. \end{aligned} \tag{73}$$

The continuity of $t \mapsto (\partial/\partial t)\omega(t,\cdot)dx$ in X implies that the last term goes to 0 as $h \to 0$. So one can see that φ is continuously differentiable in X with $\varphi'(t)(\cdot) = (\partial/\partial t)\omega(t,\cdot)$. Moreover, $\phi' \in \mathscr{P}$ by assumption (III)(a) and the definition of $\mathscr{P}$. Next, from assumption (III)(b), one can derive that

$$\begin{aligned} &\varphi(0) - CF_1(0,\varphi) \\ &\quad = \omega(0,s) - e^{-\lambda s}\left[\int_0^\infty \int_{-\infty}^0 G(\theta,a)\,\omega(\theta,a)\,d\theta\,da + f_2(0)\right]. \end{aligned} \tag{74}$$

Hence, by (III)(a), this implies that $\varphi(0) - CF_1(0,\varphi) \in D(A)$. Finally, by using assumption (III)(b) one can derive that

$$\begin{aligned} &(A(\varphi(0) - CF_1(0,\varphi)) + F_2(0,\varphi))(s) \\ &\quad = -\frac{\partial}{\partial s}\left[\omega(0,s) - e^{-\lambda s}\left(\int_0^\infty \int_{-\infty}^0 G(\theta,a)\right.\right. \\ &\qquad\qquad\qquad\qquad\qquad \left.\left.\times\,\omega(\theta,a)\,d\theta\,da + f_2(0)\right)\right] \\ &\qquad + \lambda e^{-\lambda s}\left[\int_0^\infty \int_{-\infty}^0 G(\theta,a)\,\omega(\theta,a)\,d\theta\,da + f_2(0)\right] \\ &\qquad + f_1\left(0,\int_0^\infty w(-\tau,\xi)\,d\xi,s\right) \\ &\quad = -\frac{\partial}{\partial s}\omega(0,s) + f_1\left(0,\int_0^\infty w(-\tau,\xi)\,d\xi,s\right) \\ &\quad = \frac{\partial}{\partial t}\omega(0,s) = \varphi'(0)(s). \end{aligned} \tag{75}$$

The proof is completed. $\qquad\square$

Consequently, in view of Lemmas 13–16, we can apply Theorem 7 to obtain the following theorem.

Theorem 17. *Under assumptions (I)–(III), (58) admits a unique classical solution.*

4. Solution Semigroups and Regularity

In this section, the regularity of the mild solution for

$$\frac{d}{dt} u(t) = A\left(u(t) + CF_1(u_t)\right) + F_2(u_t), \quad t \geq 0, \tag{76}$$

$$u_0 = \varphi \in \mathscr{P}$$

will be found. Throughout this section, we suppose that F_1 and F_2 satisfy the following condition:

(H3) $F : \mathscr{P} \to X$ satisfies a Lipschitz condition; that is, there is a constant $L > 0$ such that

$$\|F(\psi_1) - F(\psi_2)\| \leq L\|\psi_1 - \psi_2\|_{\mathscr{P}}, \tag{77}$$

for $\psi_1, \psi_2 \in \mathscr{P}$.

By Theorem 4, we know that (76) has a unique mild solution $u(\cdot, \varphi)$ on $[0, \infty)$ for each $\varphi \in \mathscr{P}$. Hence, we can define the nonlinear operator $U(t)$ on $\mathscr{P}$ by

$$U(t)\varphi = u_t(\cdot, \varphi) = u(t + \cdot, \varphi), \tag{78}$$

for each $\varphi \in \mathscr{P}$ and $t \geq 0$.

Theorem 18. *Under hypotheses (A_1) and (H_3), $U(\cdot)$ is a nonlinear strongly continuous semigroup on $\mathscr{P}$; that is,*

(i) $U(0) = I$, where I denotes the identity map,

(ii) $U(t + s) = U(t)U(s)$ for each $t, s \geq 0$,

(iii) $t \mapsto U(t)\varphi$ is a continuous function.

Furthermore,

(iv) *for each $t \geq 0$, $\varphi \mapsto U(t)\varphi$ is a continuous function,*

(v) *for each $t \geq 0$ and $\theta \in (-\infty, 0]$, the following translation property holds:*

$$(U(t)\varphi)(\theta) = \begin{cases} (U(t + \theta)\varphi)(0), & t + \theta \geq 0, \\ \varphi(t + \theta), & t + \theta \leq 0, \end{cases} \tag{79}$$

(vi) *there exist constants N_1 and N_2 such that*

$$\|U(t)\psi_1 - U(t)\psi_2\|_{\mathscr{P}} \leq N_1 e^{N_2 t}\|\psi_1 - \psi_2\|_{\mathscr{P}}, \tag{80}$$

for $\psi_1, \psi_2 \in \mathscr{P}$ and $t \geq 0$.

Proof. (i), (ii), and (v) are easy to see from the definition of $U(\cdot)$. (iii) is obtained from hypothesis (A2) and the definition of $U(t)\varphi$. (iv) follows from (vi). Hence it remains to show (vi). By assumption (H3) on F_1 and F_2, there is a constant $L > 0$ such that

$$\sum_{i=1}^{2}\left(\|F_i(\psi_1) - F_i(\psi_2)\|\right) \leq L\|\psi_1 - \psi_2\|_{\mathscr{P}}, \tag{81}$$

for $\psi_1, \psi_2 \in \mathscr{P}$. Moreover, we define the following real number:

$$M_t'' = LM_t'Me^{|\omega|t}, \tag{82}$$

where $M_t' = \sup_{0 \leq s \leq t} \max\{M_1(s), M_2(s)\}$ and $M_1(\cdot)$ and $M_2(\cdot)$ are the functions defined in (A1). Let $\psi_1, \psi_2 \in \mathscr{P}$. Use v_1 and v_2 to denote $U(\cdot)\psi_1$ and $U(\cdot)\psi_2$, respectively. By assumption (13) on C and hypotheses (A1) and (H3), it follows that

$$\|v_1(t) - v_2(t)\|$$

$$\leq \|T(t)\psi_1(0) - T(t)\psi_2(0)\|$$

$$+ \left\|A\int_0^t T(t - s) C\left(F_1((v_1)_s) - F_1((v_2)_s)\right) ds\right\|$$

$$+ \left\|\int_0^t T(t - s)\left(F_2((v_1)_s) - F_2((v_2)_s)\right) ds\right\|$$

$$\leq HMe^{|\omega|t}\|\psi_1 - \psi_2\|_{\mathscr{P}}$$

$$+ \sigma_C(t)\sup_{0 \leq s \leq t}\|F_1((v_1)_s) - F_1((v_2)_s)\| \tag{83}$$

$$+ Me^{|\omega|t}\int_0^t\|F_2((v_1)_s) - F_2((v_2)_s)\| ds$$

$$\leq HMe^{\omega t}\|\psi_1 - \psi_2\|_{\mathscr{P}}$$

$$+ \sigma_C(t)L\sup_{0 \leq s \leq t}\|(v_1)_s - (v_2)_s\|_{\mathscr{P}}$$

$$+ Me^{|\omega|t}L\int_0^t\sup_{0 \leq \zeta \leq s}\|(v_1)_\zeta - (v_2)_\zeta\|_{\mathscr{P}} ds,$$

and so

$$\|(v_1)_t - (v_2)_t\|_{\mathscr{P}}$$

$$\leq M_1(t)\sup_{0 \leq s \leq t}\|v_1(s) - v_2(s)\| + M_2(t)\|(v_1)_0 - (v_2)_0\|_{\mathscr{P}}$$

$$\leq M_1(t)\left[HMe^{\omega t}\|\psi_1 - \psi_2\|_{\mathscr{P}}\right.$$

$$+ \sigma_C(t)L\sup_{0 \leq s \leq t}\|(v_1)_s - (v_2)_s\|_{\mathscr{P}}$$

$$\left. + Me^{|\omega|t}L\int_0^t\sup_{0 \leq \zeta \leq s}\|(v_1)_\zeta - (v_2)_\zeta\|_{\mathscr{P}} ds\right]$$

$$+ M_2(t)\|\psi_1 - \psi_2\|_{\mathscr{P}}$$

$$\leq (H + 1)M_t'Me^{\omega t}\|\psi_1 - \psi_2\|_{\mathscr{P}}$$

$$+ \sigma_C(t)M_t'L\sup_{0 \leq s \leq t}\|(v_1)_s - (v_2)_s\|_{\mathscr{P}}$$

$$+ M_t''\int_0^t\sup_{0 \leq \zeta \leq s}\|(v_1)_\zeta - (v_2)_\zeta\|_{\mathscr{P}} ds, \tag{84}$$

for $t \in [0, T]$. We can choose ε so small that $K := \sigma_C(\varepsilon) L M'_\varepsilon < 1$ and it follows that

$$\sup_{0 \leq s \leq t} \|(v_1)_s - (v_2)_s\|_{\mathscr{P}}$$

$$\leq \frac{1}{1-K} (H+1) M'_\varepsilon M e^{|\omega|\varepsilon} \|\psi_1 - \psi_2\|_{\mathscr{P}} \tag{85}$$

$$+ \frac{1}{1-K} M''_\varepsilon \int_0^t \sup_{0 \leq \zeta \leq s} \|(v_1)_\zeta - (v_2)_\zeta\|_{\mathscr{P}} ds,$$

for $0 \leq t \leq \varepsilon$. By Gronwall's inequality, it follows that

$$\sup_{0 \leq s \leq \varepsilon} \|U(t)\psi_1 - U(t)\psi_2\|_{\mathscr{P}} = \sup_{0 \leq s \leq \varepsilon} \|(v_1)_s - (v_2)_s\|_{\mathscr{P}}$$

$$\leq K' \|\psi_1 - \psi_2\|_{\mathscr{P}}, \tag{86}$$

where

$$K' = \frac{1}{1-K} (H+1) M'_\varepsilon M e^{|\omega|\varepsilon} e^{[(1/(1-K))M''_\varepsilon]\varepsilon}. \tag{87}$$

Since, by (ii) and (86),

$$\|U(t)\psi_1 - U(t)\psi_2\|_{\mathscr{P}}$$

$$= \|U(\varepsilon)(U(t-\varepsilon)\psi_1) - U(\varepsilon)(U(t-\varepsilon)\psi_2)\|_{\mathscr{P}}$$

$$\leq K' \|(U(t-\varepsilon)\psi_1) - (U(t-\varepsilon)\psi_2)\|_{\mathscr{P}}$$

$$\leq (K')^n \|(U(t-n\varepsilon)\psi_1) - (U(t-n\varepsilon)\psi_2)\|_{\mathscr{P}} \tag{88}$$

$$\leq K'(K')^n \|\psi_1 - \psi_2\|_{\mathscr{P}}$$

$$= K' e^{n \ln K'} \|\psi_1 - \psi_2\|_{\mathscr{P}}$$

$$\leq K' e^{\omega_1 t} \|\psi_1 - \psi_2\|_{\mathscr{P}},$$

for $n\varepsilon \leq t \leq (n+1)\varepsilon$, where $\omega_1 = \varepsilon^{-1} \ln K'$, then $N_1 := K'$ and $N_2 := \omega_1$ are desired constants. The proof is completed. □

Now, we will focus on the stability near an equilibrium of the nonlinear semigroup $U(\cdot)$ on $\mathscr{P}$. The following assumption is needed.

(H4) $F : \mathscr{P} \to X$ is continuously Fréchet differentiable with respect to $(\mathscr{P}, \|\cdot\|_{\mathscr{P}})$ and $F(0) = 0$.

Suppose that F_1 and F_2 satisfy hypothesis (H4) with $F'_1(0) = L_1$ and $F'_2(0) = L_2$ being linear operators on $\mathscr{P}$. Then, by Theorem 4, the equation

$$\frac{d}{dt} u(t) = A(u(t) + CL_1(u_t)) + L_2(u_t), \quad t \geq 0, \tag{89}$$

$$u_0 = \varphi \in \mathscr{P}$$

has a unique mild solution. Let $V(\cdot)$ denote the solution semigroup on $\mathscr{P}$ associated with (89). Then $V(\cdot)$ is a C_0-semigroup.

Theorem 19. *Suppose that F_1 and F_2 satisfy hypotheses (H3) and (H4) with $F'_1(0) = L_1$ and $F'_2(0) = L_2$. Then the Fréchet derivative at 0 of the nonlinear semigroup $U(\cdot)$, associated with (76), is equal to the semigroup $V(\cdot)$ associated with (89).*

Proof. First, we show that, for any $T > 0, U(t)$ is differentiable with respect to $\mathscr{P}$ for each $t \in [0, T]$. First, since $V(\cdot)$ is a C_0-semigroup, there exists a constant M_V such that $\|T(t)\| \leq M_V$ for $t \in [0, T]$. Fix a $t \in [0, T]$ and let $\varepsilon > 0$ be arbitrary. Since the uniqueness of solution and assumption $F(0) = 0$ imply $U(t)0 = 0$, it is sufficient to find a $\delta > 0$ such that

$$\|U(t)\varphi - V(t)\varphi\|_{\mathscr{P}} \leq \varepsilon \|\varphi\|_{\mathscr{P}}, \tag{90}$$

for each $\|\varphi\|_{\mathscr{P}} < \delta$. By assumptions of F_1 and F_2, there is a constant $L > 1$ such that

$$\sum_{i=1}^2 (\|F_i(\phi_1) - F_i(\phi_2)\|) \leq L \|\phi_1 - \phi_2\|_{\mathscr{P}}, \tag{91}$$

for $\phi_1, \phi_2 \in \mathscr{P}$. Moreover, we define the following real number:

$$M''_t = LM'_1 M e^{|\omega|t}, \tag{92}$$

where $M'_1 = \sup_{0 \leq s \leq T} M_1(\cdot)$ and $M_1(\cdot)$ is the function defined in (A1). Let u and v denote the mild solutions of (76) and (89), respectively. Let $r > 0$. For all $0 \leq \tau \leq t \leq r$, we have

$$\|U(\tau)\varphi - V(\tau)\varphi\|_{\mathscr{P}}$$

$$\leq M_1(\tau) \sup_{0 \leq s \leq \tau} \|u(s) - v(s)\| \leq M'_1 \sup_{0 \leq s \leq t} \|u(s) - v(s)\|$$

$$\leq M'_1 \sup_{0 \leq s \leq t} \left\{ \left\| A \int_0^s T(t-\varsigma) \right. \right.$$

$$\times C[F_1(U(\varsigma)\varphi) - L_1(V(\varsigma)\varphi)] d\varsigma \Big\|$$

$$+ \left\| \int_0^s T(t-\varsigma) \right.$$

$$\left. \left. \times (F_2(U(\varsigma)\varphi) - L_2(V(\varsigma)\varphi)) d\varsigma \right\| \right\}$$

$$\leq M'_1 \sigma_C(t) \sup_{0 \leq s \leq t} \|F_1(U(s)\varphi) - L_1(V(s)\varphi)\|$$

$$+ M'_1 M e^{|\omega|t} \int_0^t \|F_2(U(s)\varphi) - L_2(V(s)\varphi)\| ds$$

$$\leq M'_1 \sigma_C(t) \sup_{0 \leq s \leq t} \{\|F_1(U(s)\varphi) - F_1(V(s)\varphi)\|$$

$$+ \|F_1(V(s)\varphi) - L_1(V(s)\varphi)\|\}$$

$$+ M'_1 M e^{|\omega|t} \left(\int_0^t \|F_2(U(s)\varphi) - F_2(V(s)\varphi)\| ds \right.$$

$$\left. + \int_0^t \|F_2(V(s)\varphi) - L_2(V(s)\varphi)\| ds \right)$$

$$\leq M_r'' \left(\sigma_C \left(r \right) + r \right) \sup_{0 \leq s \leq t} \left\| U \left(s \right) \varphi - V \left(s \right) \varphi \right\|_{\mathscr{P}}$$

$$+ M_r'' \left[\sigma_C \left(r \right) \sup_{0 \leq s \leq t} \left\| F_1 \left(V \left(s \right) \varphi \right) - L_1 \left(V \left(s \right) \varphi \right) \right\| \right.$$

$$\left. + \int_0^t \sup_{0 \leq \eta \leq s} \left\| F_2 \left(V \left(\eta \right) \varphi \right) - L_2 \left(V \left(\eta \right) \varphi \right) \right\| ds \right].$$

$$(93)$$

We can choose r so small that $K := M_r''(\sigma_C(r)+r) < 1$. Hence, the last inequality implies that

$$\sup_{0 \leq s \leq t} \left\| U \left(s \right) \varphi - V \left(s \right) \varphi \right\|_{\mathscr{P}}$$

$$\leq \frac{M_r''}{1 - K} \sigma_C \left(r \right) \sup_{0 \leq s \leq t} \left\| F_1 \left(V \left(s \right) \varphi \right) - L_1 \left(V \left(s \right) \varphi \right) \right\|$$

$$+ \frac{M_r''}{1 - K} \int_0^t \sup_{0 \leq \eta \leq s} \left\| F_2 \left(V \left(\eta \right) \varphi \right) - L_2 \left(V \left(\eta \right) \varphi \right) \right\| ds,$$

$$(94)$$

for $t \in [0, r]$.

Since F_1 is differentiable at 0 and $F_1(0) = 0$ with Fréchet derivative L_1, from the definition of Fréchet derivative and linearity of $V(s)$, for each $\varepsilon' > 0$, there exists a $\delta > 0$ such that

$$\left\| F_1 \left(V \left(s \right) \varphi \right) - L_1 \left(V \left(s \right) \varphi \right) \right\|$$

$$= \left\| F_1 \left(V \left(s \right) \varphi \right) - F_1 \left(V \left(s \right) 0 \right) - L_1 \left(\left(V \left(s \right) \varphi - V \left(s \right) 0 \right) \right) \right\|$$

$$\leq \varepsilon' \left\| V \left(s \right) \varphi - V \left(s \right) 0 \right\|_{\mathscr{P}} = \varepsilon' M_V \left\| \varphi \right\|_{\mathscr{P}},$$

$$(95)$$

for $\|\varphi\|_{\mathscr{P}} < \delta$ and $s \in [0, t]$, and so

$$\sup_{0 \leq s \leq t} \left\| F_1 \left(V \left(s \right) \varphi \right) - L_1 \left(V \left(s \right) \varphi \right) \right\| \leq \varepsilon' M_V \left\| \varphi \right\|_{\mathscr{P}}, \quad (96)$$

for $t \in [0, r]$. Let $\varepsilon' = (2M_r'')^{-1}(1/(1 + \sigma_C(r)))((1 - K)\varepsilon/M_V)$. Then for given ε there exists a $\delta > 0$ such that

$$\sup_{0 \leq s \leq t} \left\| F_1 \left(V \left(s \right) \varphi \right) - L_1 \left(V \left(s \right) \varphi \right) \right\|$$

$$\leq \left(2M_r'' \right)^{-1} \frac{1}{1 + \sigma_C \left(r \right)} \left(1 - K \right) \varepsilon \left\| \varphi \right\|_{\mathscr{P}},$$

$$(97)$$

for $\|\varphi\|_{\mathscr{P}} < \delta$ and $t \in [0, r]$. Similarly,

$$\int_0^t \sup_{0 \leq \eta \leq s} \left\| F_2 \left(V \left(\eta \right) \varphi \right) - L_2 \left(V \left(\eta \right) \varphi \right) \right\| ds$$

$$\leq \left(2M_r'' \right)^{-1} \left(1 - K \right) \varepsilon \left\| \varphi \right\|_{\mathscr{P}},$$

$$(98)$$

for $\|\varphi\|_{\mathscr{P}} < \delta$ and $t \in [0, r]$. Consequently, $\|U(t)\varphi - V(t)\varphi\|_{\mathscr{P}} \leq \varepsilon\|\varphi\|_{\mathscr{P}}$ for $\|\varphi\|_{\mathscr{P}} < \delta$ and $t \in [0, r]$; that is, $\varphi \mapsto U(t)\varphi$ is Fréchet differentiable for $t \in [0, r]$.

If $r \leq t = r + h \leq 2r$, then, by (ii) and (iv) of Theorem 18, it follows that

$$\left\| U \left(t \right) \varphi - V \left(t \right) \varphi \right\|_{\mathscr{P}}$$

$$= \left\| U \left(r \right) U \left(h \right) \varphi - U \left(r \right) V \left(h \right) \varphi \right.$$

$$\left. + U \left(r \right) V \left(h \right) \varphi - V \left(r \right) V \left(h \right) \varphi \right\|_{\mathscr{P}}$$

$$\leq \left\| U \left(r \right) U \left(h \right) \varphi - U \left(r \right) V \left(h \right) \varphi \right\|_{\mathscr{P}} \quad (99)$$

$$+ \left\| U \left(r \right) V \left(h \right) \varphi - V \left(r \right) V \left(h \right) \varphi \right\|_{\mathscr{P}}$$

$$\leq N_1 e^{N_2 r} \left\| U \left(h \right) \varphi - V \left(h \right) \varphi \right\|_{\mathscr{P}} + \varepsilon \left\| V \left(h \right) \varphi \right\|_{\mathscr{P}}$$

$$\leq N_1 e^{N_2 r} \varepsilon \left\| \varphi \right\|_{\mathscr{P}} + \varepsilon M_V \left\| \varphi \right\|_{\mathscr{P}},$$

for $\varphi \in \mathscr{P}$ with $\|\varphi\|_{\mathscr{P}} < \delta$. Thus, for given $\varepsilon_1 > 0$, letting $\varepsilon = (N_1 e^{N_2 r} + M_V)^{-1} \varepsilon_1$, we have shown that

$$\left\| U \left(t \right) \varphi - V \left(t \right) \varphi \right\|_{\mathscr{P}} \leq \varepsilon_1 \left\| \varphi \right\|_{\mathscr{P}}, \quad (100)$$

for $\|\varphi\|_{\mathscr{P}} < \delta$. So, the mapping $\varphi \mapsto U(t)\varphi$ is Fréchet differentiable for $t \in [0, 2r]$ and the Fréchet derivative equals the map $\varphi \mapsto V(t)\varphi$. Repeating this argument, we can get the conclusion. □

Definition 20 (see [19]). Let $U(\cdot)$ be a strongly continuous semigroup on a Banach space $(W, \| \cdot \|_W)$. A point $w \in W$ is called an equilibrium of $U(\cdot)$ if $U(t)w = w$ for all $t \geq 0$. An equilibrium $w \in W$ is said to be exponentially stable if there exist $\delta > 0$, $\nu > 0$, and $N > 1$ such that

$$\left\| U \left(t \right) x - w \right\|_W \leq N e^{-\nu t} \|x - w\|_W, \quad (101)$$

for $t \geq 0$ and $\|x - w\|_W \leq \delta$.

When $U(\cdot)$ is a linear semigroup, this definition reduces to the usual definition of exponential stability of C_0-semigroups: $\|U(t)\| \leq N e^{-\nu t}$.

Theorem 21 (see [19]). *Let $U(\cdot)$ be a nonlinear strongly continuous semigroup in a Banach space W. Assume that $w \in W$ is an equilibrium of $U(\cdot)$ such that $U(t)$ is Fréchet differentiable at w for each $t \geq 0$, with $V(t)$ the Fréchet derivative at w of $U(t)$. Then, $V(\cdot)$ is a strongly continuous semigroup of bounded linear operators on W. Moreover, if $V(\cdot)$ is exponentially stable, then w is an exponentially stable equilibrium of $U(\cdot)$.*

Since (H4) implies that 0 is an equilibrium of the semigroup $U(\cdot)$ in Theorem 19, by Theorem 21, we have the following consequence.

Theorem 22. *Suppose that F_1 and F_2 satisfy hypotheses (H3) and (H4). If $V(\cdot)$ is exponentially stable on $\mathscr{P}$, that is, there exist constants $\beta \leq 1$ and $\omega_1 > 0$ such that $\|V(t)\varphi\|_{\mathscr{P}} \leq \beta e^{-\omega_1 t}\|\varphi\|_{\mathscr{P}}$ for $t \geq 0$ and $\varphi \in \mathscr{P}$, then zero is an exponentially stable equilibrium of $U(\cdot)$ on $\mathscr{P}$; that is, there exist $\delta > 0$, $\nu > 0$, and $N > 1$ such that*

$$\left\| U \left(t \right) \varphi \right\|_{\mathscr{P}} \leq N e^{-\nu t} \left\| \varphi \right\|_{\mathscr{P}}, \quad (102)$$

for $t \geq 0$ and $\|\varphi\|_{\mathscr{P}} \leq \delta$.

Remark 23. Theorems 19 and 22 for the special case that $C = 0$ and A satisfies the Hille-Yosida condition can be found in [10].

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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