

Research Article

On Critical Circle Homeomorphisms with Infinite Number of Break Points

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We prove that a critical circle homeomorphism with infinite number of break points without periodic orbits is conjugated to the linear rotation by a quasisymmetric map if and only if its rotation number is of bounded type. And we also prove that any two adjacent atoms of dynamical partition of a unit circle are comparable.

1. Introduction

Let $S^1 = \mathbb{R}/\mathbb{Z}$ with clearly defined orientation, metric, Lebesgue measure, and the operation of addition be the unit circle. Let $\pi : \mathbb{R} \rightarrow S^1$ denote the corresponding projection mapping that “winds” a straight line on the circle. An arbitrary homeomorphism f that preserves the orientation of the unit circle S^1 can “be lifted” on the straight line \mathbb{R} in the form of the homeomorphism $F : \mathbb{R} \rightarrow \mathbb{R}$ with property $F(x + 1) = F(x) + 1$ that is connected with f by relation $\pi \circ F = f \circ \pi$. This homeomorphism F is called the *lift* of the homeomorphism f and is defined up to an integer term. The most important arithmetic characteristic of the homeomorphism f of the unit circle S^1 is the *rotation number*:

$$\rho(f) = \lim_{n \rightarrow \infty} \frac{F^n(x)}{n} \pmod{1}, \quad (1)$$

where F is the lift of f with S^1 to \mathbb{R} . Here and below, for a given map F , F^n denotes its n th iterate. The rotation number is rational if and only if f has periodic points. Denjoy proved that if f is a circle diffeomorphism with irrational rotation number $\rho = \rho(f)$ and $\log Df$ is of bounded variation, then f is *topologically conjugate* to the pure rotation $f_\rho : x \rightarrow x + \rho \pmod{1}$; that is, there exists an essentially unique

homeomorphism φ of the circle with $\varphi \circ f = f_\rho \circ \varphi$ (see [1]). Since the conjugating map φ and the unique f -invariant measure μ_f are related by $\varphi(x) = \mu_f([0, x])$ (see [1]), regularity properties of the conjugating map φ imply corresponding properties of the density of the absolutely continuous invariant measure μ_f . The problem of relating the smoothness of φ to that of f has been studied extensively. In-depth results have been found; see [2–5].

Other classes of circle homeomorphisms are known to satisfy the conclusion of Denjoy’s theorem (see [6], Chapters I and IV, and [2], Chapter VI) and the study of the regularity of their conjugation maps arises naturally. Two of these classes are commonly referred to as the following.

1.1. Critical Circle Homeomorphisms. The orientation preserving circle homeomorphisms f , such that $f \in C^r$, $r \geq 3$, have finite number of critical points c_i , around which, in some C^r coordinate system, f has the form $x \rightarrow x^{p_i}$, where $p_i \geq 3$ are the odd integers. Such critical points we say are of *polynomial type* of order p_i .

1.2. P-Homeomorphisms. That is, orientation preserving circle homeomorphisms f are differentiable except in many countable points called *break points* admitting left and right derivatives (denoted by Df_- and Df_+ , resp.) such that

- (i) there exist some constants $0 < \gamma < \zeta < \infty$ such that $\gamma < Df(x) < \zeta$ for all $x \in S^1 \setminus BP(f)$, $\gamma < Df_+(b) < \zeta$ and $\gamma < Df_-(b) < \zeta$ for all $b \in BP(f)$, where $BP(f)$ denotes the set of the break points of f ;
- (ii) $\log Df$ has bounded variation. In this situation, $\log Df$, $\log Df_-$, $\log Df_+$ and $\log Df^{-1}$, $\log Df_+^{-1}$, $\log Df_-^{-1}$ have the same total variation denoted by $V = \text{Var } \log Df$.

The ratio $\sigma_f(b) = Df_-(b)/Df_+(b)$ is called *jump ratio* of f in b .

The existence of the conjugating map for the class critical circle homeomorphisms was proved by Yoccoz in [7] and for the class P -homeomorphisms the existence of conjugating map was proved by Herman in [2].

The singularity of the conjugating map for critical circle homeomorphisms was shown by Graczyk and Świątek in [8]. They proved that if f is C^3 smooth circle homeomorphism with finitely many critical points of polynomial type and an irrational rotation number of bounded type, then the conjugating map φ is a singular function. For the P -homeomorphisms, the situation is different; that is, in this case, the conjugating map can be singular or absolutely continuous. Indeed, in the works [9–11], it was shown that the conjugating map is singular. The deeper result in this area was obtained by Dzhaliilov et al. [12]. They proved that if f is piecewise-smooth P -homeomorphism with finite number of break points and the product of jump ratios at these break points is nontrivial, then the conjugating map is a singular function. But in the works [9, 13], it was shown that if f is piecewise-smooth P -homeomorphism with finite number of break points having the (\mathcal{D}) -property (see for the definition [13]) and the product of the jump ratios on each orbit is equal to 1, then the conjugating map is an absolutely continuous function. Now, we discuss the symmetric property of a given function.

Definition 1. A homeomorphism $g : \mathbb{R} \rightarrow \mathbb{R}$ is called Q -quasisymmetric if and only if, for every real x and $\delta \neq 0$,

$$\frac{|g(x + \delta) - g(x)|}{|g(x) - g(x - \delta)|} \leq Q. \tag{2}$$

The criteria of quasisymmetry of the conjugating map of the critical circle homeomorphisms were obtained by Świątek in [14]. Due to [14], if the circle homeomorphism with an irrational rotation number is analytic and has finitely many critical points, then the conjugating map is quasisymmetric if and only if the rotation number is of bounded type.

The quasisymmetric property of the conjugating map of P -homeomorphisms is also different from the case of critical circle homeomorphisms. More precisely, if the rotation number of P -homeomorphism is irrational of bounded type, then conjugating map is quasisymmetric, but there is a P -homeomorphism with irrational rotation number of unbounded type such that the conjugating map is quasisymmetric. In this paper, we introduce a new class of circle homeomorphisms with the aid of the above two classes. Our aim in this work is to show the existence of conjugating map

for this new class and study the quasisymmetric property of this conjugating map. Now, we introduce our class.

Let f be a circle homeomorphism.

- (a) f has one critical point polynomial type of order 3; that is, there is a point $x = x_{cr} \in S^1$ such that $Df(x_{cr}) = D^2 f(x_{cr}) = 0$ and $D^3 f(x_{cr}) \neq 0$.
- (b) $f \in C^3(U_\epsilon(x_{cr}))$ for some ϵ -neighborhood of the point x_{cr} .
- (c) f is a P -homeomorphism on the set $S^1 \setminus U_\epsilon(x_{cr})$.

Note that all the above results were obtained for the class P -homeomorphisms with finite number of break points, but in our work it is not necessary for the number of break points to be finite. Now, we state our main results.

Theorem 2. *Suppose that a circle homeomorphism f satisfies the conditions (a)–(c) and the rotation number $\rho(f)$ is irrational. Then, there exists circle homeomorphism $H : S^1 \rightarrow S^1$, such that the functional equation*

$$H(x + \rho(f)) = f(H(x)) \tag{3}$$

is satisfied. Moreover, H is quasisymmetric if and only if $\rho(f)$ is of bounded type.

The proof of Theorem 2 is based on the method of cross-ratio distortion estimates. Note that the cross-ratio estimates were used in dynamical systems for the first time by Yoccoz [7] and later by Świątek [14]. In fact, the proof of Theorem 2 follows closely that of Świątek [14]. Our second result below is also proved by using cross-ratio estimates.

Theorem 3. *Suppose that a circle homeomorphism f satisfies the conditions (a)–(c) and the rotation number $\rho(f)$ is irrational. Then, there exists universal constant $C = C(f) > 1$ such that any two adjacent atoms I_1 and I_2 of a dynamical partition $\mathbf{P}_n(x_{cr}, f)$ (see, for the definition, below) are C -comparable; that is,*

$$C^{-1} |I_2| \leq |I_1| \leq C |I_2|. \tag{4}$$

2. Dynamical Partition

We will assume that the rotation number $\rho = \rho(f)$ is irrational throughout this paper. We use the continued fraction representation $\rho = 1/(k_1 + 1/(k_2 + \dots)) := [k_1, k_2, \dots, k_n, \dots]$ of the rotation number, which is understood as a limit of the sequence of convergents $p_n/q_n = [k_1, k_2, \dots, k_n]$. The sequence of positive integers k_n with $n \geq 1$, which are called *partial quotients*, are uniquely determined for each ρ . The coprimes p_n and q_n satisfy the recurrence relations $p_n = k_n p_{n-1} + p_{n-2}$ and $q_n = k_n q_{n-1} + q_{n-2}$ for $n \geq 1$, where we set, for convenience, $p_{-1} = 0$, $q_{-1} = 1$ and $p_0 = 1$, $q_0 = k_1$. For $\xi \in S^1$ we define the n th *fundamental segment* $I_0^n = I_0^n(\xi)$ as the circle arc $[\xi, f^{q_n}(\xi)]$ if n is even and $[f^{q_n}(\xi), \xi]$ if n is odd. We denote two sets of closed intervals of order $n : q_n$ “long” intervals: $\{I_i^{n-1} = f^i(I_0^{n-1}), 0 \leq i < q_n\}$ and q_{n-1} “short” intervals: $\{I_j^n = f^j(I_0^n), 0 \leq j < q_{n-1}\}$. The long and short

intervals are mutually disjoint except for the endpoints and cover the whole circle. The partition obtained by the above construction will be denoted by $\mathbf{P}_n = \mathbf{P}_n(\xi, f)$ and called the n th dynamical partition of S^1 . Obviously, the partition \mathbf{P}_{n+1} is a refinement of the partition \mathbf{P}_n . Indeed, the short intervals are members of \mathbf{P}_{n+1} and each long interval $I_i^{n-1} \in \mathbf{P}_n$, $0 \leq i < q_n$, is partitioned into $k_{n+1} + 1$ intervals belonging to \mathbf{P}_{n+1} such that

$$I_i^{n-1} = I_i^{n+1} \cup \bigcup_{s=0}^{k_{n+1}-1} I_{i+q_{n-1}+sq_n}^n. \tag{5}$$

3. Cross-Ratio Inequality

Now, we equip S^1 with the usual metric $|x - y| = \inf\{|\tilde{x} - \tilde{y}|\}$, where \tilde{x}, \tilde{y} range over the lifts of $x, y \in S^1$, resp.}. Our main analytic tools are ratio and cross-ratio distortions. Let $a, b, c, d \in S^1$ be four points of the circle which preserve orientation; that is, $a < b < c < d < a$ on the circle; we define a ratio of three points of a, b, c by

$$R(a, b, c) = \frac{|b - a|}{|c - a|}, \tag{6}$$

and we define a cross-ratio of four points of a, b, c, d by

$$Cr(a, b, c, d) = \frac{|b - a|}{|c - a|} \cdot \frac{|d - c|}{|d - b|}. \tag{7}$$

The distortions Dr and Dcr of the ratio and the cross-ratio by the function f are defined by

$$Dr(a, b, c; f) = \frac{R(f(a), f(b), f(c))}{R(a, b, c)}, \tag{8}$$

$$Dcr(a, b, c, d; f) = \frac{Cr(f(a), f(b), f(c), f(d))}{Cr(a, b, c, d)},$$

respectively. It is clear that

$$Dcr(a, b, c, d; f) = Dr(a, b, c; f) \cdot Dr(d, c, b; f). \tag{9}$$

Notice that the ratio and the cross-ratio distortions have the following properties.

- (i) The ratio and the cross-ratio are invariant by translation:

$$R(a + t, b + t, c + t) = R(a, b, c), \tag{10}$$

$$Cr(a + t, b + t, c + t, d + t) = Cr(a, b, c, d), \quad t \in S^1.$$

- (ii) The ratio and the cross-ratio distortions are multiplicative with respect to composition for two functions F and G on S^1 ; we have

$$Dr(a, b, c; F \circ G) = Dr(G(a), G(b), G(c); F) \cdot Dr(a, b, c; G),$$

$$Dcr(a, b, c, d; F \circ G) = Dcr(G(a), G(b), G(c), G(d); F) \cdot Dcr(a, b, c, d; G). \tag{11}$$

Now, let us formulate the following theorem, which plays an important role during studying the properties of dynamical partitions.

Theorem 4. *Suppose that a homeomorphism f satisfies the conditions (a)–(c). Consider a system of quadruples $\{(a_i, b_i, c_i, d_i), a_i < b_i < c_i < d_i < a_i, 1 \leq i \leq m\}$ on the circle S^1 . Suppose that the system of intervals $\{(a_i, d_i), 1 \leq i \leq m\}$ covers each point of the circle at most p times. Then, there exists a constant $Q_1 = Q_1(p, f)$ such that the following inequality holds:*

$$\prod_{i=1}^m Dcr(a_i, b_i, c_i, d_i; f) \leq Q_1. \tag{12}$$

Inequality (12) is called the *cross-ratio inequality*. This inequality was proved for the critical circle homeomorphisms by Świątek [15]. Now, we prove the following three important lemmas to be used in the proof of the main results.

Lemma 5. *Suppose that a homeomorphism f satisfies the conditions (a)–(c). Consider a system of quadruples $\{(a_i, b_i, c_i, d_i), a_i < b_i < c_i < d_i < a_i, 1 \leq i \leq m\}$ on the circle S^1 . Suppose that the system of intervals $\{(a_i, d_i), 1 \leq i \leq m\}$ covers each point of the set $S^1 \setminus U_\epsilon(x_{cr})$ at most p times. Then, there exists a constant $Q_2 = Q_2(f, p)$ such that the following inequality holds:*

$$\prod_{i=1}^m Dcr(a_i, b_i, c_i, d_i; f) \leq Q_2. \tag{13}$$

Proof. Since f is a P -homeomorphism on the set $S^1 \setminus U_\epsilon(x_{cr})$ for any quadruples (a_i, b_i, c_i, d_i) satisfying assumption of lemma, we have

$$\begin{aligned} |\log Dr(a_i, b_i, c_i; f)| &\leq \sup_{x, y \in [a_i, c_i]} |\log Df(x) - \log Df(y)| \\ &\leq \text{Var}_{[a_i, c_i]} \log Df. \end{aligned} \tag{14}$$

Using this inequality, we get

$$\begin{aligned} \left| \log \prod_{i=1}^m Dcr(a_i, b_i, c_i, d_i; f) \right| &\leq \sum_{i=1}^m |\log Dcr(a_i, b_i, c_i, d_i; f)| \\ &\leq 2 \sum_{i=1}^m \text{Var}_{[a_i, d_i]} \log Df. \end{aligned} \tag{15}$$

By assumption, the system of intervals $\mathfrak{F} := \{(a_i, d_i), 1 \leq i \leq m\}$ covers each point of the set $S^1 \setminus U_\epsilon(x_{cr})$ at most p times. Now, we describe this system of intervals as a union of subsystems of \mathfrak{F}_j , $j \leq p$ in the following way: first, we take $[a_1, d_1]$ as an element of \mathfrak{F}_1 and then consider the intersection $[a_1, d_1] \cap [a_2, d_2]$; if this intersection is empty, then we count the interval $[a_2, d_2]$ an element of \mathfrak{F}_1 ; otherwise, we count an element of \mathfrak{F}_2 . Next, consider $\mathfrak{F}_1 \cap [a_3, d_3]$ (here and below, it is considered the intersection with each element

of \mathfrak{F}_1); if it is empty, we count $[a_3, d_3]$ an element of \mathfrak{F}_1 ; otherwise, we check the intersection $\mathfrak{F}_2 \cap [a_3, d_3]$. Again, if $\mathfrak{F}_2 \cap [a_3, d_3]$ is empty, we count $[a_3, d_3]$ an element of \mathfrak{F}_2 ; otherwise, we count $[a_3, d_3]$ an element of \mathfrak{F}_3 . Continuing this process, we get all $\mathfrak{F}_j, j \leq p$. By construction of subsystems $\mathfrak{F}_j, j \leq p$ of \mathfrak{F} , the elements of each subsystem do not intersect with each other and $\mathfrak{F} = \bigcup_j \mathfrak{F}_j, j \leq p$. Therefore

$$\sum_{i=1}^m \text{Var} \log Df = \sum_j \sum_{[a_i, d_i] \in \mathfrak{F}_j} \text{Var} \log Df \leq pV, \quad (16)$$

where $V = \text{Var}_{S^1 \setminus U_\epsilon(x_{cr})} \log Df$. □

We will use the following definition and fact to formulate the next lemma.

Definition 6. Let g be a C^3 function such that $Dg \neq 0$. The Schwarzian derivative of g is defined by

$$\mathcal{S}g = \frac{D^3 g}{Dg} - \frac{3}{2} \left(\frac{D^2 g}{Dg} \right)^2. \quad (17)$$

Fact (see [6]). If $\mathcal{S}f < 0$ on interval I , then

$$Dcr(a, b, c, d; f) < 1, \quad (18)$$

for any quadruples $a, b, c, d \in I$.

Lemma 7. *Suppose that a circle homeomorphism f satisfies the conditions (a)–(c). Then, there is $\epsilon_0 \in (0, \epsilon/2)$, such that $f(x)$ satisfies the following conditions:*

- (i) $\mathcal{S}f < 0$ in $U_{2\epsilon_0}(x_{cr}) \setminus \{x_{cr}\}$;
- (ii) *there exist constants $0 < Q_3 \leq Q_4$, such that $Q_3(x - x_{cr})^2 \leq Df(x) \leq Q_4(x - x_{cr})^2$ for all $x \in U_{2\epsilon_0}(x_{cr})$.*

Proof. Consider the following function:

$$v(x) = \frac{D^3 f(x)}{2}, \quad x \in U_\epsilon(x_{cr}). \quad (19)$$

By using Taylor's formula, we have

$$Df(x) = v(x_1)(x - x_{cr})^2, \quad D^2 f(x) = 2v(x_2)(x - x_{cr}), \quad (20)$$

where $|x_i - x_{cr}| \leq |x - x_{cr}|, i = 1, 2$. Since $v(x)$ is a nonzero, continuous function and if we take $Q_3 = \inf_{x \in U_\epsilon(x_{cr})} v(x), Q_4 = \sup_{x \in U_\epsilon(x_{cr})} v(x)$, then the second assertion of this lemma follows. In order to prove the first assertion of this lemma we calculate the Schwarzian derivative of the function $f(x)$ as follows:

$$\mathcal{S}f(x) = \frac{2}{(x - x_{cr})^2} \left[\frac{v(x)}{v(x_1)} - 3 \left(\frac{v(x_2)}{v(x_1)} \right)^2 \right]. \quad (21)$$

Since $v(x)$ is nonzero and continuous on $U_\epsilon(x_{cr})$, there is a sufficient small $\epsilon_0 \in (0, \epsilon/2)$ such that the ratios $v(x)/v(x_1)$ and $v(x_2)/v(x_1)$ are uniformly close to 1 for any $x, x_1, x_2 \in U_{2\epsilon_0}(x_{cr})$. It ensures that $\mathcal{S}f(x) < 0, x \in U_{2\epsilon_0}(x_{cr})$. □

Lemma 8. *Suppose that a circle homeomorphism f satisfies the conditions (a)–(c). Let $a, b, c, d \in S^1$ and $a < b < c < d < a$. Then, there is a constant $Q_5 > 1$ such that*

$$Dcr(a, b, c, d; f) \leq Q_5. \quad (22)$$

Proof. By the property of the cross-ratio distortion, we know that $Dcr(a, b, c, d; f) = Dr(a, b, c; f) \cdot Dr(d, c, b; f)$. Now, to prove this lemma, it is enough to show that $Dr(a, b, c; f)$ is bounded above. The second ratio distortion $Dr(d, c, b; f)$ is analogously estimated. Denote $\mathcal{H} = S^1 \setminus U_{\epsilon_0}(x_{cr})$. Consider the following three cases:

(1) $(a, c) \subset \mathcal{H}$, (2) $(a, c) \subset U_{2\epsilon_0}(x_{cr})$, and (3) neither (1) nor (2).

If $(a, c) \subset \mathcal{H}$, then

$$Dr(a, b, c; f) = \frac{c - a}{b - a} \cdot \frac{f(b) - f(a)}{f(c) - f(a)} \leq \frac{L}{\ell}, \quad (23)$$

where $L = \sup_{x \in \mathcal{H}} Df(x)$ and $\ell = \inf_{x \in \mathcal{H}} Df(x)$. It is clear that $\ell > 0$. Let $(a, c) \subset U_{2\epsilon_0}(x_{cr})$. Using Lemma 7, we get

$$\begin{aligned} Dr(a, b, c; f) &= \frac{c - a}{b - a} \cdot \frac{f(b) - f(a)}{f(c) - f(a)} \\ &\leq \frac{Q_4}{Q_3} \cdot \frac{(c - x_{cr}) - (a - x_{cr})}{(b - x_{cr}) - (a - x_{cr})} \\ &\quad \times \frac{(b - x_{cr})^3 - (a - x_{cr})^3}{(c - x_{cr})^3 - (a - x_{cr})^3} \\ &= \frac{Q_4}{Q_3} \cdot Dr(A, B, C; p), \end{aligned} \quad (24)$$

where $A = a - x_{cr}, B = b - x_{cr}, C = c - x_{cr}$, and $p(x) = x^3$. It is easy to check that $Dr(rA, rB, rC; p) = Dr(A, B, C; p)$ for any $r \in \mathbb{R}^1 \setminus \{0\}$. Without any loss of generality, one can assume that $A = 1$. Then,

$$Dr(1, B, C; p) = \frac{B^2 + B + 1}{C^2 + C + 1}. \quad (25)$$

Here, there can be four cases: (1') $A = 1 < B < C$, (2') $A = 1 > 0 > B > C$, (3') $A = 1 > B > 0 > C$, and (4') $A = 1 > B > C > 0$. It is easy to check that for all cases (1')–(4') the ratio distortion $Dr(1, B, C; p)$ is bounded above. Now, consider case (3). It is clear that $c - a > \epsilon_0$. From this it is implied that $(f(c) - f(a))/(c - a) \geq \text{const} > 0$. Hence, from $(f(b) - f(a))/(b - a) \leq \text{const}$, (23), and (24), we get the proof of Lemma 8. □

Proof of Theorem 4. Let $\mathcal{N}_m = \{1, 2, 3, \dots, m\}$. We divide the set \mathcal{N}_m into three disjoint subsets $\mathcal{D}_1, \mathcal{D}_2$, and \mathcal{D}_3 in the following way:

- (i) $\mathcal{D}_1 = \{i : i \in \mathcal{N}_m \text{ and } (a_i, d_i) \subset U_{2\epsilon_0}(x_{cr})\}$;
- (ii) $\mathcal{D}_2 = \{i : i \in \mathcal{N}_m \text{ and } (a_i, d_i) \subset \mathcal{H}\}$;
- (iii) $\mathcal{D}_3 = \mathcal{N}_m \setminus (\mathcal{D}_1 \cup \mathcal{D}_2)$.

We estimate each product $\prod_{i \in \mathcal{D}_s} Dcr(a_i, b_i, c_i, d_i; f)$, $s = 1, 2, 3$ separately. Let $\mathcal{D}_1^1 = \{i : i \in \mathcal{N}_m \text{ and } x_{cr} \in (a_i, d_i)\}$. Then, by Lemma 8, we have

$$\prod_{i \in \mathcal{D}_1^1} Dcr(a_i, b_i, c_i, d_i; f) \leq Q_5^p. \tag{26}$$

Denote $\mathcal{D}_1^2 = \mathcal{D}_1 \setminus \mathcal{D}_1^1$; that is, $\mathcal{D}_1^2 = \{i : i \in \mathcal{D}_1 \text{ and } x_{cr} \notin (a_i, d_i)\}$. By Lemma 7, we have

$$\prod_{i \in \mathcal{D}_1^2} Dcr(a_i, b_i, c_i, d_i; f) < 1. \tag{27}$$

Let $i \in \mathcal{D}_2$. Then, using Lemma 5, we get

$$\prod_{i \in \mathcal{D}_2} Dcr(a_i, b_i, c_i, d_i; f) \leq Q_2. \tag{28}$$

Let $i \in \mathcal{D}_3$. Note that the number of intervals does not exceed $2p$. This and Lemma 8 imply that

$$\prod_{i \in \mathcal{D}_3} Dcr(a_i, b_i, c_i, d_i; f) \leq Q_5^{2p}. \tag{29}$$

Hence, from (26) to (29), the assertion of Theorem 4 follows. \square

4. Proof of Main Theorems

Proof of Theorem 2. The proof of Theorem 2 follows from assertion of Theorem 4 together with the following proposition which was proved by Świątek in [14]. \square

Proposition 9. *Let f be a circle homeomorphism with irrational rotation number ρ . Assume that f satisfies the cross-ratio inequality with bound Q_1 . Then, there is a circle homeomorphism $H : S^1 \rightarrow S^1$, which conjugates f to the linear rotation f_ρ . Furthermore, H is quasymmetric if ρ is of bounded type. If f has at least one critical point of polynomial type, then H is quasymmetric if and only if ρ is of bounded type.*

Before we prove Theorem 3, we formulate two lemmas and use them to prove this theorem. Note that these lemmas also were obtained by Świątek [14].

Lemma 10. *Let f be a circle homeomorphism with irrational rotation number. Assume that f satisfies the cross-ratio inequality with bound Q_1 . Then, there is a constant $Q_6 = Q_6(Q_1, f) \geq 1$ such that for every $x \in S^1$*

$$Q_6^{-1} |I_0^n(x)| \leq |I_0^n(f^{-q_n}(x))| \leq Q_6 |I_0^n(x)|. \tag{30}$$

Lemma 11. *Let f be a circle homeomorphism with irrational rotation number. Assume that f satisfies the cross-ratio inequality with bound Q_1 . Then, there is a constant $Q_7 = Q_7(Q_1, f) > 0$ such that*

$$|I_0^n(x_{cr})| \geq Q_7 |I_0^{n-1}(x_{cr})|. \tag{31}$$

Proof of Theorem 3. It is clear that any two adjacent atoms of $\mathbf{P}_n = \mathbf{P}_n(x_{cr}, f)$ can be of the following three types:

- (I) $I_i^{n-1}, I_{i+q_{n-1}}^{n-1}$,
- (II) I_i^n, I_i^{n-1} ,
- (III) $I_i^n, I_{i+q_n-q_{n-1}}^{n-1}$.

In (I), the adjacent atoms are I and $f^{q_{n-1}}(I)$, and in this case by Lemma 10 these intervals are Q_6 -comparable. Consider case (II). Using the property of dynamical partition, it is easy to see that (it suffices to prove it for the linear rotation f_ρ , which follows from arithmetical properties of ρ).

$$I_0^{n-1}(x) \supset \bigcup_{s=1}^{k_{n+1}-1} I_0^n(f^{-sq_n}(x)) \tag{33}$$

for any $x \in S^1$. This together with Lemma 10 implies that

$$|I_0^{n-1}(x)| \geq |I_0^n(f^{-q_n}(x))| \geq Q_6^{-1} |I_0^n(x)|, \tag{34}$$

for any $x \in S^1$. If we take $f^i(x_{cr})$ instead of x , we get

$$|I_i^{n-1}| \geq Q_6^{-1} |I_i^n|. \tag{35}$$

Now, we show an opposite inequality to (35). Let us take I_i^n, I_i^{n-1} , and $f^{q_{n-1}}(I_i^{n-1})$. Applying $f^{q_{n-1}}$ to these intervals, we have $f^{q_n}(I_0^n)$, $f^{q_n}(I_0^{n-1})$, and $f^{q_n+q_{n-1}}(I_0^{n-1})$, respectively. Due to Lemma 10, the intervals $f^{q_n}(I_0^n)$ and I_0^n are Q_6 -comparable. Besides that, by Lemma 11, we have

$$|I_0^n| \geq Q_7 |I_0^{n-1}|. \tag{36}$$

From (34) and (36), it follows that the intervals I_0^n and I_0^{n-1} are $Q_8 = \max\{Q_6, Q_7^{-1}\}$ comparable. Since the intervals I_0^n and I_0^{n-1} are Q_8 -comparable and $I_0^n \subset f^{q_n}(I_0^{n-1}) \subset I_0^n \cup I_0^{n-1}$, the intervals I_0^n and $f^{q_n}(I_0^{n-1})$ are $1 + Q_8$ -comparable. From this it is implied that the intervals $f^{q_n}(I_0^n)$ and $f^{q_n}(I_0^{n-1})$ are $Q_6(1 + Q_8)$ comparable. Hence, $f^{q_n}(I_0^n)$, $f^{q_n}(I_0^{n-1})$, and $f^{q_n+q_{n-1}}(I_0^{n-1})$ are $Q_6^2(1 + Q_8)$ comparable.

Since the intervals $I_{i+q_{n-1}}^{n-1}, I_i^{n-1}$, and I_i^n are adjacent on S^1 and the system of intervals $\{f^s(I_{i+q_{n-1}}^{n-1} \cup I_i^{n-1} \cup I_i^n)\}, 0 \leq s \leq q_n - i$, covers each point of the circle at most three times, by the cross-ratio inequality (12), we have

$$\frac{|f^{q_n-i}(I_i^n)|}{|I_i^n|} \cdot \frac{|I_i^n| + |I_i^{n-1}|}{|f^{q_n-i}(I_i^n)| + |f^{q_n-i}(I_i^{n-1})|} \cdot \frac{|f^{q_n-i}(I_{i+q_{n-1}}^{n-1})|}{|I_{i+q_{n-1}}^{n-1}|} \cdot \frac{|I_{i+q_{n-1}}^{n-1}| + |I_i^{n-1}|}{|f^{q_n-i}(I_{i+q_{n-1}}^{n-1})| + |f^{q_n-i}(I_i^{n-1})|}$$

$$= \prod_{j=0}^{q_n-i-1} Dcr(f^{i+2q_{n-1}+j}(x_{cr}), f^{i+q_{n-1}+j}(x_{cr}),$$

$$f^{i+j}(x_{cr}), f^{i+q_n+j}(x_{cr}); f)$$

$$\leq Q_1(3, f). \tag{37}$$

Since the intervals $f^{q_n}(I_0^n)$, $f^{q_n}(I_0^{n-1})$, $f^{q_n+q_{n-1}}(I_0^{n-1})$ are $Q_6^2(1 + Q_8)$ -comparable and the intervals $f^{i+q_{n-1}}(I_0^{n-1})$, $f^i(I_0^{n-1})$ are Q_6 -comparable, and by inequality (37) for constant $Q_9 = Q_9(Q_1, Q_6, Q_8) > 1$, we have

$$\frac{|f^i(I_0^n)|}{|f^i(I_0^{n-1})|} \geq \frac{|f^i(I_0^n)|}{|f^i(I_0^n)| + |f^i(I_0^{n-1})|} \geq Q_9^{-1}. \quad (38)$$

Hence, case (II) follows from (35) and (38). Case (III). By using Lemma 10 twice, we get the comparability of the following intervals: $I_{i+q_n-q_{n-1}}^{n-1}$, $I_{i+q_n}^{n-1}$ and $I_{i+q_n}^{n-1}$, I_i^{n-1} . By case (II), the intervals I_i^{n-1} and I_i^n are comparable and it gives the proof of case (III). Theorem 3 is proved. \square

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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