Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2014, Article ID 364675, 8 pages
http://dx.doi.org/10.1155/2014/364675

Research Article

Comparison of SUPG with Bubble Stabilization Parameters and the Standard SUPG

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Received 23 February 2014; Accepted 13 May 2014; Published 2 June 2014

Academic Editor: Gerassimos Barbatis

We study a streamline upwind Petrov-Galerkin method (SUPG) with bubble stabilization coefficients on quasiuniform triangular meshes. The new algorithm is a consistent Petrov-Galerkin method and shows similar numerical performances as the standard SUPG when the mesh Péclet number is greater than 1. Relationship between the new algorithm and the standard SUPG will be explored. Numerical experiments support these results.

1. Introduction

We consider the reaction-convection-diffusion problem in 2D

$$\begin{aligned} -\varepsilon\Delta u + \mathbf{b}\cdot\nabla u + cu &= f, \quad (x, y) \in \Omega, \\ u &= 0, \quad (x, y) \in \partial\Omega, \end{aligned} \tag{1}$$

where Ω is a bounded open domain in $\mathbb{R}^2$, with the boundary $\partial\Omega$, $\mathbf{b} \in (W_1^{\infty}(\Omega))^2$, $c \in L^{\infty}(\Omega)$ are given functions, and $\varepsilon \ll |\mathbf{b}|$ is a small positive parameter. For simplicity, we only consider the case that $\Omega = (0,1)^2$. In the following, suppose that there is a constant $\mu_0 > 0$ such that

$$c - \frac{1}{2}\nabla\cdot\mathbf{b} \geq \mu_0 > 0, \quad \forall (x, y) \in \Omega. \tag{2}$$

It is also assumed that f is sufficiently smooth. In the case of $\mathbf{b} = (b_1, b_2)^T > (0,0)^T$ ($b_1, b_2 \geq \beta$, where β is a positive constant), the solution of (1) typically has two exponential layers of width $O(\varepsilon \ln(1/\varepsilon))$ at the sides $x = 1$ and $y = 1$ of Ω. In the case of $\mathbf{b} = (b, 0)^T$ ($b \geq \beta > 0$), the solution of (1) typically has an exponential layer of width $O(\varepsilon)$ near the outflow boundary at $x = 1$ and two characteristic (or parabolic) layers of width $O(\sqrt{\varepsilon})$ near the characteristic boundaries at $y = 0$ and $y = 1$.

When the mesh Péclet number Pe is greater than 1, there exist global unphysical oscillations in numerical solutions of standard discretization schemes on general meshes. Hence, stabilized methods and/or a priori adapted meshes are widely used in order to get discrete solutions with satisfactory accuracy. An overview on these methods can be found in the survey [1, 2].

One of the most famous stabilized finite element methods is the streamline upwind Petrov-Galerkin method (SUPG). The SUPG proposed by Hughes and Brooks [3] is known to provide good stability properties and high accuracy. However, there are several drawbacks in SUPG, such as lacking discrete maximum principle and involving second derivatives and difficulties in determining the stabilization coefficients. Driven by these problems, many researchers were devoted to improving the SUPG. A lot of numerical methods were proposed, such as SOLD [4], nonlinear residual [5], and LPS [6]. Also, relations of SUPG and other numerical formulation, like residual-free bubble method [7] and variational multiscale method [8], were studied to seek possible directions of improvement.

By means of residual, the SUPG adds to the original bilinear form a term which introduces a suitable amount of artificial viscosity in the direction of streamlines. Also, the SUPG can be viewed as an inconsistent Petrov-Galerkin method, since its modified weighting function cannot apply to the diffusive term (see the details in Section 4).

For this reason, we are to analyze the SUPG with bubble stabilization coefficients in 2D and compare its numerical performance with the SUPG's. From theoretical analysis and numerical results, we find that the new scheme is classified into the consistent Petrov-Galerkin formulation (CPGF) and behaves as well as SUPG. Also, the standard finite element method (FEM), which shows excellent performances for $\mathrm{Pe} \leq 1$, can be classified into the CPGF and viewed as a special "SUPG with bubble stabilization coefficients" by taking $\delta_K = 0$ in (7). Thus, the FEM and our new scheme could be viewed as two reference numerical methods in the CPGF. This provides possibilities of constructing new numerical schemes between them in the CPGF. In fact, in our forthcoming works, we obtain a linear maximum-principle-preserving stabilized method in the CPGF by means of the FEM and the SUPG with bubble stabilization coefficients, which shows better numerical performances than the standard SUPG. Thus, our results in this paper can be viewed as a starting point to construct new numerical schemes in the CPGF.

The remainder of this paper is organized as follows. In Section 2 we formulate the problem and introduce notation and mesh. Theoretical results including stability analysis and energy norm estimates can be found in Section 3. Section 4 is devoted to the relationship between SUPG, standard finite element method, and our method. Finally, numerical experiments that illustrate our theoretical results are presented in Section 5.

2. Mesh and Numerical Formulation

First we define a finite element space on triangular meshes

$$V_h := \left\{ v_h \in C\left(\overline{\Omega}\right) : v_h\big|_{\partial\Omega} = 0,\ v_h\big|_K \text{ is linear},\ \forall K \in \mathscr{T}_h \right\}, \tag{3}$$

where the term "linear" is to be understood in the usual isoparametric sense. Here we assume that the triangulations $\mathscr{T}_h$ on Ω are quasiuniform:

$$h_K > \alpha_0 h, \qquad \alpha_K > \alpha_0, \tag{4}$$

for any $K \in \mathscr{T}_h$, where α_0 is a positive constant and h_K, α_K, h denote, respectively, the diameter of K, the smallest angle of K, and the maximum of all diameters of triangles in $\mathscr{T}_h$.

Using the linear finite element space V_h, we can state the standard Galerkin discretisation of (1) which reads as follows.

Find $u_h \in V_h$ such that for all $v_h \in V_h$,

$$a_h(u_h, v_h) = f_h(v_h), \tag{5}$$

where $a(u, v) = \varepsilon(\nabla u, \nabla v) + (\mathbf{b} \cdot \nabla u + cu, v)$.

The SUPG consists in adding to the original bilinear form a term which introduces a suitable amount of artificial viscosity in the direction of streamlines. In this case, the SUPG reads as follows.

Find $u_h \in V_h$ such that for all $v_h \in V_h$,

$$a_h(u_h, v_h) = f_h(v_h), \tag{6}$$

where

$$\begin{aligned} a_h(u_h, v_h) &= a(u_h, v_h) \\ &\quad + \sum_K \left(-\varepsilon\Delta u_h + \mathbf{b} \cdot \nabla u_h + cu_h, \delta_K \mathbf{b} \cdot \nabla v_h\right)_K, \\ f_h(v_h) &= (f, v_h) + \sum_K \left(f, \delta_K \mathbf{b} \cdot \nabla v_h\right)_K. \end{aligned} \tag{7}$$

In (7) the term δ_K is defined as $\delta_K(x, y) := \delta_K = C_K^* h \lambda_1 \lambda_2 \lambda_3, (x, y) \in K$ in which C_K^* is a constant to be determined and λ_i $(i = 1, 2, 3)$ are the linear basis functions. Actually, δ_K is a bubble function. Moreover,

$$\lambda_1 \lambda_2 \lambda_3 \leq \frac{1}{27} \quad \text{in } K, \qquad \int_K \lambda_1 \lambda_2 \lambda_3 = \frac{S(K)}{60}, \tag{8}$$

where $S(K)$ represents the area of K.

Finally, we define a special energy norm (SD norm) associated with $a_h(\cdot, \cdot)$:

$$|||u_h|||^2 := \varepsilon |u_h|_1^2 + \sum_K \left\| \delta_K^{1/2} \mathbf{b} \cdot \nabla u_h \right\|_K^2 + \|u_h\|^2. \tag{9}$$

We denote by $\|\cdot\|_D$ the L^2 norm in $L^2(D)$; that is,

$$\|v\|_D^2 = (v, v)_D \quad \forall v \in L^2(D). \tag{10}$$

If $D = \Omega$, then we drop Ω from the notation.

3. Stability and Energy Estimates

Throughout this subsection, we assume C is some positive constant.

3.1. Stability Analysis. The stability properties are a consequence of the following.

Lemma 1. *Let the parameter C_K^* in δ_K satisfies*

$$C_K^* \leq \frac{27\mu_0}{hC_K^2} \tag{11}$$

for each $K \in \mathscr{T}_h$, where $C_K := \|c\|_{L^\infty, K}$. Then the discrete bilinear form is coercive; that is,

$$a_h(v_h, v_h) \geq \frac{A}{2} |||v_h|||^2, \tag{12}$$

where $A := \max\{\mu_0, 1\}$.

Proof. By divergence theorem we obtain

$$\begin{aligned} a_h(v_h, v_h) &= \varepsilon |v_h|_1^2 + \int_\Omega \left(c - \frac{1}{2}\nabla \cdot \mathbf{b}\right) v_h^2 \\ &\quad + \sum_K \left\| \delta_K^{1/2} \mathbf{b} \cdot \nabla v_h \right\|_K^2 + \sum_K \int_K c v_h \delta_K \mathbf{b} \cdot \nabla v_h. \end{aligned} \tag{13}$$

Obviously the sum of first three terms is greater than $A|||v_h|||^2$ from the condition of (2), and we only need to estimate the last term. Using Hölder inequality, we have

$$\begin{aligned}
&\left|\sum_K \int_K c v_h \delta_K \mathbf{b}\cdot\nabla v_h\right| \\
&\quad\le \sum_K \left(\int_K c^2 v_h^2 \delta_K\right)^{1/2} \left(\int_K (\mathbf{b}\cdot\nabla v_h)^2 \delta_K\right)^{1/2} \\
&\quad\le \frac{1}{2}\sum_K \left[\int_K c^2 v_h^2 \delta_K + \int_K (\mathbf{b}\cdot\nabla v_h)^2 \delta_K\right] \\
&\quad\le \frac{\mu_0}{2}\left\|v_h\right\|^2 + \frac{1}{2}\sum_K \left\|\delta_K^{1/2}\mathbf{b}\cdot\nabla v_h\right\|_K^2 \\
&\quad\le \frac{A}{2}|||v_h|||^2,
\end{aligned} \tag{14}$$

where (14) is based on the definition of δ_K, (8), and (11):

$$\begin{aligned}
\sum_K \int_K c^2 v_h^2 \delta_K &\le C_K^2 \sum_K \int_K C_K^* h \lambda_1 \lambda_2 \lambda_3 v_h^2 \\
&\le \sum_K \int_K \mu_0 v_h^2 = \mu_0 \left\|v_h\right\|^2.
\end{aligned} \tag{15}$$

Then the proof of the lemma is finished. □

3.2. Energy Norm Estimate. We denote by u^I the nodal piecewise linear interpolant to u over $\mathcal{T}_h$. From Lemma 1 and the fact of $a_h(u-u_h, u^I-u_h)=0$ one gets

$$\begin{aligned}
\frac{\mu_0}{2}\left|\left|\left|u^I-u_h\right|\right|\right|^2 &\le a_h\left(u^I-u_h, u^I-u_h\right) \\
&= a_h\left(u^I-u, u^I-u_h\right).
\end{aligned} \tag{16}$$

Denote $\eta := u^I - u$, $e := u^I - u_h$ and estimate the right-hand side of (16) term by term:

$$\varepsilon\left(\nabla\eta, \nabla e\right) \le \varepsilon^{1/2}|\eta|_1\,|||e|||, \tag{17}$$

$$\begin{aligned}
&\left(\mathbf{b}\cdot\nabla\eta + c\eta, e\right) \\
&\quad= \int_{\partial\Omega} \mathbf{b}\cdot\mathbf{n}\eta e - \int_\Omega \nabla\cdot\mathbf{b}\eta e - \int_\Omega (\mathbf{b}\cdot\nabla e)\,\eta + \int_\Omega c\eta e \\
&\quad= \int_\Omega (c-\nabla\cdot\mathbf{b})\,\eta e - \int_\Omega (\mathbf{b}\cdot\nabla e)\,\eta \\
&\quad\le C\left(\int_\Omega \eta^2\right)^{1/2}\left(\int_\Omega e^2\right)^{1/2} + \left(\int_\Omega (\mathbf{b}\cdot\nabla e)^2\right)^{1/2}\left(\int_\Omega \eta^2\right)^{1/2} \\
&\quad\le Ch^2\,|||e||| + Ch^{-1/2}\,|||e|||\,h^2 \le Ch^{3/2}\,|||e|||,
\end{aligned} \tag{18}$$

where we have used the standard interpolation results $\|\ \eta\ \|\le Ch^2$ (see [9]) and the first inequality of (18) is obtained by

$$\begin{aligned}
&\int_\Omega (\mathbf{b}\cdot\nabla e)^2 \le C\beta \int_\Omega (\nabla e)^2 \\
&\quad= Ch^{-1}\beta(\nabla e)^2 \sum_K \int_K \delta_K \\
&\quad\le Ch^{-1}\sum_K \int_K \delta_K(\mathbf{b}\cdot\nabla e)^2 \le Ch^{-1}|||e|||^2, \\
&\sum_K \left(\varepsilon\Delta u + \mathbf{b}\cdot\nabla\eta + c\eta, \delta_K \mathbf{b}\cdot\nabla e\right) \\
&\quad\le \sum_K \int_K \left(C\varepsilon + \mathbf{b}\cdot\nabla\eta + c\eta\right)\delta_K^{1/2}\delta_K^{1/2}\mathbf{b}\cdot\nabla e \\
&\quad\le \sum_K \left\|(C\varepsilon + \mathbf{b}\cdot\nabla\eta + c\eta)\delta_K^{1/2}\right\|_K \left\|\delta_K^{1/2}\mathbf{b}\cdot\nabla e\right\|_K \\
&\quad\le \left(\sum_K \left\|\delta_K^{1/2}\left(C\varepsilon + \mathbf{b}\cdot\nabla\eta + c\eta\right)\right\|_K^2\right)^{1/2} \left(\sum_K \left\|\delta_K^{1/2}\mathbf{b}\cdot\nabla e\right\|_K^2\right)^{1/2} \\
&\quad\le Ch^2\,|||e|||\,.
\end{aligned} \tag{19}$$

Combining all of these estimates, one gets

$$|||u-u_h||| \le Ch^{3/2}. \tag{20}$$

4. Comparison of SUPG with Bubble Stabilization Coefficients and the Standard SUPG

Consider the bilinear form of SUPG:

$$\begin{aligned}
&\varepsilon\left(\nabla u_h, \nabla v_h\right) + \left(\mathbf{b}\cdot\nabla u_h + cu_h, v_h\right) \\
&\qquad + \sum_K \delta_K\left(-\varepsilon\Delta u_h + \mathbf{b}\cdot\nabla u_h + cu_h, \mathbf{b}\cdot\nabla v_h\right)_K \\
&\quad= \left(f, v_h\right) + \sum_K \left(f, \delta_K \mathbf{b}\cdot\nabla v_h\right)_K,
\end{aligned} \tag{21}$$

where δ_K is constant in K.

It can be rewritten in the form

$$\begin{aligned}
&\varepsilon\left(\nabla u_h, \nabla v_h\right) + \left(\mathbf{b}\cdot\nabla u_h + cu_h, v_h + \tilde{\delta}\mathbf{b}\cdot\nabla v_h\right) \\
&\quad= \left(f, v_h + \tilde{\delta}\mathbf{b}\cdot\nabla v_h\right),
\end{aligned} \tag{22}$$

where $\tilde{\delta}|_K = \delta_K$.

Notice that (22) does not correspond to a consistent Petrov-Galerkin formulation in general except in the case when $\mathbf{b} \equiv \mathbf{C}$ and v_h is linear. Clearly, when δ_K is a bubble function,

$$\begin{aligned}\varepsilon\left(\nabla u_h, \nabla\left(\tilde{\delta}\mathbf{b}\cdot\nabla v_h\right)\right) &= \sum_K \varepsilon(\nabla u_h, \nabla(\delta_K \mathbf{b}\cdot\nabla v_h))_K \\ &= \varepsilon\sum_K \int_{\partial K} \nabla u_h\cdot\mathbf{n}\delta_K\mathbf{b}\cdot\nabla v_h \\ &\quad - \varepsilon\sum_K\int_K (\nabla\cdot\nabla u_h)\,\delta_K\mathbf{b}\cdot\nabla v_h \\ &= \varepsilon\sum_K \int_{\partial K} \nabla u_h\cdot\mathbf{n}\delta_K\mathbf{b}\cdot\nabla v_h \equiv 0,\end{aligned} \tag{23}$$

since δ_K vanishes on the boundary of K. Thus (22) can be written as

$$\begin{aligned}&\varepsilon\left(\nabla u_h, \nabla\left(v_h + \tilde{\delta}\mathbf{b}\cdot\nabla v_h\right)\right) + \left(\mathbf{b}\cdot\nabla u_h + c u_h, v_h + \tilde{\delta}\mathbf{b}\cdot\nabla v_h\right) \\ &\quad = \left(f, v_h + \tilde{\delta}\mathbf{b}\cdot\nabla v_h\right).\end{aligned} \tag{24}$$

Then SUPG with bubble stabilization coefficients is classified into the consistent Petrov-Galerkin formulation.

Moreover, in general, piecewise constants δ_K in SUPG make the test function $v_h+\delta_K\mathbf{b}\cdot\nabla v_h$ discontinuous. However, the test functions are continuous in the case of bubble stabilization coefficients and the consequent test space W_h is contained in $H_0^1(\Omega)$. In this case, SUPG with bubble stabilization coefficients reads as follows.

Find $u_h \in V_h$, such that for all $w_h \in W_h$,

$$a(u_h, w_h) = (f, w_h), \tag{25}$$

where

$$W_h := \left\{w_h \in C\left(\overline{\Omega}\right) : w_h|_K = v_h|_K + \delta_K\mathbf{b}\cdot\nabla v_h|_K, v_h \in V_h\right\}. \tag{26}$$

On the other hand, SUPG with bubble stabilization coefficients is gradually close to the FEM in the same space $H_0^1(\Omega)$ as $C_K^* \to 0$ for any $\in \mathcal{T}_h$.

In a word, SUPG with bubble stabilization coefficients inherits the advantages of SUPG and constructs the relation between FEM and SUPG in consistent Petrov-Galerkin formulations.

5. Numerical Experiments

In this section we give numerical results that appear to support our theoretical results. Errors and convergence rates for our numerical scheme are presented. All calculations are carried out by using Intel visual Fortran 11. The discrete problems are solved by using a version of Pardiso solver (see [10, 11]).

For the computations we have chosen $\delta_K = 60.0h\lambda_1\lambda_2\lambda_3$ in SUPG with bubble stabilization coefficients and $\delta_K = 1.0h$ in SUPG. We set $\Omega = [0,1]^2$ and calculate the errors and convergence rates in the subdomain Ω_C away from layers for Tables 1–16. For parabolic Problems 1 and 2, $\Omega_C = [0,1/2]\times[1/3,2/3]$. For exponential Problems 3 and 4, $\Omega_C = [0,1/2]\times[0,1/2]$. The errors in Tables 17–20 are calculated in the whole domain Ω. In the following we only list the results of the case of $\varepsilon = 10^{-2}$ and $\varepsilon = 10^{-8}$ since the comparison results between the two methods of other cases like $\varepsilon = 10^{-4}$, $\varepsilon = 10^{-6}$, and $\varepsilon = 10^{-10}$ are similar.

Problem 1. One has

$$\begin{aligned}-\varepsilon\Delta u + (2-x)\,u_x + \frac{3}{2}u = f \quad (x,y)\in\Omega, \\ u = 0 \quad (x,y)\in\partial\Omega,\end{aligned} \tag{27}$$

where f is chosen such that the solution u is

$$\begin{aligned}u(x,y) &= \left(\sin\frac{\pi}{2}x - \frac{e^{-((1-x)/\varepsilon)} - e^{-(1/\varepsilon)}}{1-e^{-(1/\varepsilon)}}\right) \\ &\quad\times \frac{\left(1-e^{-(y/\sqrt{\varepsilon})}\right)\left(1-e^{-((1-y)/\sqrt{\varepsilon})}\right)}{1-e^{-1/\sqrt{\varepsilon}}}.\end{aligned} \tag{28}$$

Problem 2. One has

$$\begin{aligned}-\varepsilon\Delta u + u_x + u = x(1-x) + y(1-y), \quad (x,y)\in\Omega, \\ u = 0 \quad (x,y)\in\partial\Omega.\end{aligned} \tag{29}$$

Problem 3. One has

$$\begin{aligned}-\varepsilon\Delta u + 2u_x + 3u_y + u = f \quad (x,y)\in\Omega, \\ u = 0 \quad (x,y)\in\partial\Omega,\end{aligned} \tag{30}$$

where f is chosen such that the solution u is

$$\begin{aligned}u(x,y) &= 2\sin x\left(1-\exp\left(\frac{-2(1-x)}{\varepsilon}\right)\right) \\ &\quad\times y^2\left(1-\exp\left(\frac{-(1-y)}{\varepsilon}\right)\right).\end{aligned} \tag{31}$$

Problem 4. One has

$$\begin{aligned}-\varepsilon\Delta u + 2u_x + 3u_y + u = x(1-x) + y(1-y) \quad (x,y)\in\Omega, \\ u = 0 \quad (x,y)\in\partial\Omega.\end{aligned} \tag{32}$$

From the above tables it is shown that the errors and convergence rates of SUPG with bubble stabilization coefficients and standard SUPG in the maximum norm, in the L^2 norm, and in the SD norm are similar not only in the subdomain away from layers but also in the global sense. These results illustrate that SUPG with bubble stabilization coefficients also has good stability properties and high accuracy as standard SUPG.

TABLE 1: $\varepsilon = 10^{-8}$, Problem 1 (comparison of error).

N	L^∞ norm		L^2 norm		SD norm	
	Bubble SUPG	SUPG	Bubble SUPG	SUPG	Bubble SUPG	SUPG
6	4.10×10^{-2}	4.13×10^{-2}	2.59×10^{-2}	2.60×10^{-2}	2.59×10^{-2}	2.6×10^{-2}
12	6.30×10^{-3}	6.35×10^{-3}	3.32×10^{-3}	3.34×10^{-3}	3.33×10^{-3}	3.34×10^{-3}
24	3.53×10^{-4}	3.45×10^{-4}	1.32×10^{-4}	1.33×10^{-4}	1.32×10^{-4}	1.33×10^{-4}
48	1.14×10^{-4}	1.13×10^{-4}	2.00×10^{-5}	2.00×10^{-5}	2.00×10^{-5}	2.01×10^{-5}
96	3.11×10^{-5}	3.10×10^{-5}	4.74×10^{-6}	4.75×10^{-6}	4.75×10^{-6}	4.75×10^{-6}
192	8.15×10^{-6}	8.13×10^{-6}	1.16×10^{-6}	1.16×10^{-6}	1.16×10^{-6}	1.16×10^{-6}

TABLE 2: $\varepsilon = 10^{-8}$, Problem 1 (comparison of convergence rate).

N	L^∞ norm		L^2 norm		SD norm	
	Bubble SUPG	SUPG	Bubble SUPG	SUPG	Bubble SUPG	SUPG
6	2.70	2.70	2.95	2.96	2.96	2.96
12	4.16	4.20	4.66	4.65	4.66	4.65
24	1.63	1.61	2.73	2.73	2.73	2.73
48	1.87	1.87	2.07	2.08	2.07	2.08
96	1.93	1.93	2.03	2.04	2.03	2.03

TABLE 3: $\varepsilon = 10^{-2}$, Problem 1 (comparison of error).

N	L^∞ norm		L^2 norm		SD norm	
	Bubble SUPG	SUPG	Bubble SUPG	SUPG	Bubble SUPG	SUPG
6	3.63×10^{-2}	3.63×10^{-2}	2.75×10^{-2}	2.73×10^{-2}	3.02×10^{-2}	3.01×10^{-2}
12	9.03×10^{-3}	9.07×10^{-3}	4.63×10^{-3}	4.64×10^{-3}	6.23×10^{-3}	6.23×10^{-3}
24	9.72×10^{-4}	9.85×10^{-4}	4.71×10^{-4}	4.74×10^{-4}	1.36×10^{-3}	1.36×10^{-3}
48	3.96×10^{-4}	3.98×10^{-4}	1.63×10^{-4}	1.64×10^{-4}	6.40×10^{-4}	6.39×10^{-4}
96	2.58×10^{-4}	2.59×10^{-4}	8.24×10^{-5}	8.25×10^{-5}	3.20×10^{-4}	3.20×10^{-4}
192	1.47×10^{-4}	1.47×10^{-4}	4.19×10^{-5}	4.19×10^{-5}	1.61×10^{-4}	1.61×10^{-4}

TABLE 4: $\varepsilon = 10^{-2}$, Problem 1 (comparison of convergence rate).

N	L^∞ norm		L^2 norm		SD norm	
	Bubble SUPG	SUPG	Bubble SUPG	SUPG	Bubble SUPG	SUPG
6	2.01	2.00	2.57	2.56	2.28	2.27
12	3.22	3.20	3.30	3.29	2.20	2.20
24	1.30	1.31	1.53	1.53	1.09	1.09
48	0.62	0.62	0.99	0.99	1.00	1.00
96	0.82	0.82	0.98	0.98	1.00	1.00

TABLE 5: $\varepsilon = 10^{-8}$, Problem 2 (comparison of error).

N	L^∞ norm		L^2 norm		SD norm	
	Bubble SUPG	SUPG	Bubble SUPG	SUPG	Bubble SUPG	SUPG
6	4.99×10^{-3}	5.01×10^{-3}	5.80×10^{-3}	5.82×10^{-3}	8.81×10^{-3}	9.23×10^{-3}
12	6.04×10^{-4}	6.04×10^{-4}	5.63×10^{-4}	5.64×10^{-4}	1.33×10^{-3}	1.42×10^{-3}
24	2.37×10^{-4}	2.36×10^{-4}	5.54×10^{-5}	5.54×10^{-5}	2.93×10^{-4}	3.16×10^{-4}
48	6.59×10^{-5}	6.59×10^{-5}	1.43×10^{-5}	1.43×10^{-5}	1.02×10^{-4}	1.10×10^{-4}
96	1.73×10^{-5}	1.73×10^{-5}	3.65×10^{-6}	3.65×10^{-6}	3.60×10^{-5}	3.88×10^{-5}

TABLE 6: $\varepsilon = 10^{-8}$, Problem 2 (comparison of convergence rate).

N	L^∞ norm		L^2 norm		SD norm	
	Bubble SUPG	SUPG	Bubble SUPG	SUPG	Bubble SUPG	SUPG
6	3.05	3.05	3.37	3.37	2.72	2.70
12	1.35	1.35	3.34	3.35	2.18	2.17
24	1.84	1.84	1.95	1.95	1.52	1.52
48	1.93	1.93	1.97	1.97	1.51	1.51

Table 7: $\varepsilon = 10^{-2}$, Problem 2 (comparison of error).

N	L^∞ norm		L^2 norm		SD norm	
	Bubble SUPG	SUPG	Bubble SUPG	SUPG	Bubble SUPG	SUPG
6	6.67×10^{-3}	6.69×10^{-3}	6.12×10^{-3}	6.13×10^{-3}	9.36×10^{-3}	9.75×10^{-3}
12	1.32×10^{-3}	1.32×10^{-3}	9.23×10^{-4}	9.25×10^{-4}	1.94×10^{-3}	2.03×10^{-3}
24	1.72×10^{-4}	1.72×10^{-4}	1.07×10^{-4}	1.07×10^{-4}	4.55×10^{-4}	4.69×10^{-4}
48	1.09×10^{-4}	1.09×10^{-4}	4.56×10^{-5}	4.57×10^{-5}	1.98×10^{-4}	2.01×10^{-4}
96	7.14×10^{-5}	7.14×10^{-5}	2.45×10^{-5}	2.45×10^{-5}	9.28×10^{-5}	9.39×10^{-5}

Table 8: $\varepsilon = 10^{-2}$, Problem 2 (comparison of convergence rate).

N	L^∞ norm		L^2 norm		SD norm	
	Bubble SUPG	SUPG	Bubble SUPG	SUPG	Bubble SUPG	SUPG
6	2.34	2.34	2.73	2.73	2.27	2.26
12	2.94	2.94	3.11	3.12	2.09	2.11
24	0.66	0.67	1.22	1.22	1.20	1.22
48	0.60	0.60	0.90	0.90	1.09	1.10

Table 9: $\varepsilon = 10^{-8}$, Problem 3 (comparison of error).

N	L^∞ norm		L^2 norm		SD norm	
	Bubble SUPG	SUPG	Bubble SUPG	SUPG	Bubble SUPG	SUPG
6	2.70×10^{-1}	2.70×10^{-1}	5.66×10^{-2}	5.66×10^{-2}	3.27×10^{-1}	3.27×10^{-1}
12	9.15×10^{-2}	9.15×10^{-2}	1.51×10^{-2}	1.51×10^{-2}	6.40×10^{-2}	6.43×10^{-2}
24	6.60×10^{-3}	6.60×10^{-3}	8.04×10^{-4}	8.04×10^{-4}	6.35×10^{-3}	6.71×10^{-3}
48	1.62×10^{-5}	1.62×10^{-5}	1.09×10^{-5}	1.09×10^{-5}	2.13×10^{-3}	2.26×10^{-3}
96	3.84×10^{-6}	3.85×10^{-6}	2.56×10^{-6}	2.56×10^{-6}	7.53×10^{-4}	7.80×10^{-4}
192	9.51×10^{-7}	9.52×10^{-7}	6.38×10^{-7}	6.38×10^{-7}	2.66×10^{-4}	2.83×10^{-4}

Table 10: $\varepsilon = 10^{-8}$, Problem 3 (comparison of convergence rate).

N	L^∞ norm		L^2 norm		SD norm	
	Bubble SUPG	SUPG	Bubble SUPG	SUPG	Bubble SUPG	SUPG
6	1.56	1.56	1.91	1.91	2.35	2.35
12	3.79	3.79	4.23	4.23	3.33	3.26
24	8.67	8.67	6.21	6.21	1.58	1.57
48	2.08	2.07	2.08	2.08	1.50	1.50
96	2.01	2.02	2.01	2.01	1.50	1.50

Table 11: $\varepsilon = 10^{-2}$, Problem 3 (comparison of error).

N	L^∞ norm		L^2 norm		SD norm	
	Bubble SUPG	SUPG	Bubble SUPG	SUPG	Bubble SUPG	SUPG
6	1.66×10^{-1}	1.48×10^{-1}	3.65×10^{-2}	3.30×10^{-2}	2.14×10^{-1}	1.96×10^{-1}
12	4.07×10^{-2}	3.28×10^{-2}	7.07×10^{-3}	5.83×10^{-3}	3.65×10^{-2}	3.38×10^{-2}
24	2.59×10^{-3}	2.23×10^{-3}	3.06×10^{-4}	2.63×10^{-4}	6.13×10^{-3}	6.49×10^{-3}
48	1.41×10^{-4}	1.42×10^{-4}	4.02×10^{-5}	4.02×10^{-5}	2.19×10^{-3}	2.31×10^{-3}
96	7.77×10^{-5}	7.77×10^{-5}	2.01×10^{-5}	2.01×10^{-5}	7.91×10^{-4}	8.35×10^{-4}
192	3.85×10^{-5}	3.85×10^{-5}	9.98×10^{-6}	9.98×10^{-6}	2.92×10^{-4}	3.07×10^{-4}

Table 12: $\varepsilon = 10^{-2}$, Problem 3 (comparison of convergence rate).

N	L^∞ norm		L^2 norm		SD norm	
	Bubble SUPG	SUPG	Bubble SUPG	SUPG	Bubble SUPG	SUPG
6	2.03	2.18	2.37	2.50	2.55	2.54
12	3.97	3.88	4.53	4.47	2.57	2.38
24	4.20	3.97	2.93	2.71	1.49	1.49
48	0.86	0.87	1.00	1.00	1.47	1.47
96	1.01	1.01	1.01	1.01	1.44	1.44

Table 13: $\varepsilon = 10^{-8}$, Problem 4 (comparison of error).

N	L^∞ norm		L^2 norm		SD norm	
	Bubble SUPG	SUPG	Bubble SUPG	SUPG	Bubble SUPG	SUPG
6	1.24×10^{-2}	1.24×10^{-2}	4.31×10^{-3}	4.31×10^{-3}	2.11×10^{-2}	2.11×10^{-2}
12	8.76×10^{-3}	8.76×10^{-3}	2.35×10^{-3}	2.35×10^{-3}	9.09×10^{-3}	9.09×10^{-3}
24	1.54×10^{-3}	1.54×10^{-3}	3.20×10^{-4}	3.20×10^{-4}	2.04×10^{-3}	2.04×10^{-3}
48	1.06×10^{-4}	1.06×10^{-4}	1.78×10^{-5}	1.78×10^{-5}	6.65×10^{-4}	6.65×10^{-4}
96	2.59×10^{-5}	2.59×10^{-5}	4.06×10^{-6}	4.06×10^{-6}	2.35×10^{-4}	2.35×10^{-4}

Table 14: $\varepsilon = 10^{-8}$, Problem 4 (comparison of convergence rate).

N	L^∞ norm		L^2 norm		SD norm	
	Bubble SUPG	SUPG	Bubble SUPG	SUPG	Bubble SUPG	SUPG
6	0.50	0.50	0.87	0.87	1.22	1.22
12	2.51	2.51	2.88	2.88	2.16	2.16
24	3.85	3.85	4.16	4.16	1.62	1.62
48	2.04	2.04	2.13	2.13	1.50	1.50

Table 15: $\varepsilon = 10^{-2}$, Problem 4 (comparison of error).

N	L^∞ norm		L^2 norm		SD norm	
	Bubble SUPG	SUPG	Bubble SUPG	SUPG	Bubble SUPG	SUPG
6	1.21×10^{-2}	1.21×10^{-2}	4.23×10^{-3}	4.23×10^{-3}	2.08×10^{-2}	2.08×10^{-2}
12	8.77×10^{-3}	8.77×10^{-3}	2.34×10^{-3}	2.34×10^{-3}	9.07×10^{-3}	9.07×10^{-3}
24	1.62×10^{-3}	1.62×10^{-3}	3.56×10^{-4}	3.56×10^{-4}	2.05×10^{-3}	2.05×10^{-3}
48	1.14×10^{-4}	1.14×10^{-4}	2.59×10^{-5}	2.59×10^{-5}	6.63×10^{-4}	6.63×10^{-4}
96	2.73×10^{-5}	2.73×10^{-5}	8.23×10^{-6}	8.23×10^{-6}	2.41×10^{-4}	2.41×10^{-4}

Table 16: $\varepsilon = 10^{-2}$, Problem 4 (comparison of convergence rate).

N	L^∞ norm		L^2 norm		SD norm	
	Bubble SUPG	SUPG	Bubble SUPG	SUPG	Bubble SUPG	SUPG
6	0.47	0.47	0.85	0.85	1.20	1.20
12	2.44	2.44	2.72	2.72	2.14	2.14
24	3.82	3.82	3.78	3.78	1.63	1.63
48	2.06	2.06	1.65	1.65	1.46	1.46

Table 17: $\varepsilon = 10^{-8}$, Problem 1 (comparison of global error).

N	L^∞ norm		L^2 norm		SD norm	
	Bubble SUPG	SUPG	Bubble SUPG	SUPG	Bubble SUPG	SUPG
6	3.18×10^{-1}	3.19×10^{-1}	3.72×10^{-1}	3.73×10^{-1}	1.44	1.44
12	3.31×10^{-1}	3.32×10^{-1}	2.64×10^{-1}	2.64×10^{-1}	1.42	1.42
24	3.38×10^{-1}	3.38×10^{-1}	1.86×10^{-1}	1.86×10^{-1}	1.41	1.41
48	3.41×10^{-1}	3.41×10^{-1}	1.31×10^{-1}	1.31×10^{-1}	1.41	1.41
96	3.42×10^{-1}	3.42×10^{-1}	9.29×10^{-2}	9.29×10^{-2}	1.41	1.41
192	3.43×10^{-1}	3.43×10^{-1}	6.54×10^{-2}	6.54×10^{-2}	1.41	1.41

Table 18: $\varepsilon = 10^{-8}$, Problem 2 (comparison of global error).

N	L^∞ norm		L^2 norm		SD norm	
	Bubble SUPG	SUPG	Bubble SUPG	SUPG	Bubble SUPG	SUPG
6	6.60×10^{-2}	6.60×10^{-2}	3.13×10^{-2}	3.14×10^{-2}	6.61×10^{-2}	7.02×10^{-2}
12	6.56×10^{-2}	6.56×10^{-2}	2.16×10^{-2}	2.16×10^{-2}	5.78×10^{-2}	6.17×10^{-2}
24	6.55×10^{-2}	6.55×10^{-2}	1.52×10^{-2}	1.52×10^{-2}	5.42×10^{-2}	5.81×10^{-2}
48	6.55×10^{-2}	6.55×10^{-2}	1.08×10^{-2}	1.08×10^{-2}	5.26×10^{-2}	5.65×10^{-2}
96	6.54×10^{-2}	6.54×10^{-2}	7.63×10^{-3}	7.63×10^{-3}	5.18×10^{-2}	5.58×10^{-2}

Table 19: $\varepsilon = 10^{-8}$, Problem 3 (comparison of global error).

N	L^∞ norm		L^2 norm		SD norm	
	Bubble SUPG	SUPG	Bubble SUPG	SUPG	Bubble SUPG	SUPG
6	1.16	1.16	4.84×10^{-1}	4.84×10^{-1}	1.47	1.48
12	1.25	1.25	3.73×10^{-1}	3.73×10^{-1}	1.20	1.20
24	1.28	1.28	2.66×10^{-1}	2.66×10^{-1}	1.41	1.41
48	1.29	1.29	1.88×10^{-1}	1.88×10^{-1}	1.62	1.62
96	1.30	1.30	1.32×10^{-1}	1.32×10^{-3}	1.75	1.75

Table 20: $\varepsilon = 10^{-8}$, Problem 4 (comparison of global error).

N	L^∞ norm		L^2 norm		SD norm	
	Bubble SUPG	SUPG	Bubble SUPG	SUPG	Bubble SUPG	SUPG
6	2.53×10^{-2}	2.53×10^{-2}	1.26×10^{-2}	1.26×10^{-2}	7.82×10^{-2}	7.82×10^{-2}
12	2.97×10^{-2}	2.97×10^{-2}	1.27×10^{-2}	1.27×10^{-2}	7.18×10^{-2}	7.18×10^{-2}
24	2.87×10^{-2}	2.87×10^{-2}	9.44×10^{-3}	9.44×10^{-3}	6.66×10^{-2}	6.66×10^{-2}
48	2.79×10^{-2}	2.79×10^{-2}	6.60×10^{-3}	6.60×10^{-3}	6.43×10^{-2}	6.43×10^{-2}
96	2.76×10^{-2}	2.76×10^{-2}	4.64×10^{-3}	4.64×10^{-3}	6.34×10^{-2}	6.34×10^{-2}

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

This work was partially supported by Promotive Research Fund for Excellent Young and Middle-aged Scientists of Shandong Province (Grant no. BS2012SF012) and Independent Innovation Foundation of Shandong University (Grant no. 2012TS194).

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