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Research Article

Blow-Up Solutions and Global Existence for Quasilinear Parabolic Problems with Robin Boundary Conditions

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We study the blow-up and global solutions for a class of quasilinear parabolic problems with Robin boundary conditions. By constructing auxiliary functions and using maximum principles, the sufficient conditions for the existence of blow-up solution, an upper bound for the "blow-up time," an upper estimate of the "blow-up rate," the sufficient conditions for the existence of global solution, and an upper estimate of the global solution are specified.

1. Introduction

In this paper, we are going to investigate the blow-up and global solutions of the following quasilinear parabolic problem with Robin boundary conditions:

$$\begin{aligned} &(g(u))_t = \nabla\cdot(a(u,t)\,b(x)\,\nabla u) + h(t)\,f(u) \\ &\qquad\qquad\qquad \text{in } D\times(0,T), \\ &\frac{\partial u}{\partial n} + \gamma u = 0 \quad \text{on } \partial D\times(0,T), \\ &u(x,0) = u_0(x) > 0 \quad \text{in } \overline{D}, \end{aligned} \tag{1}$$

where $D\subset\mathbb{R}^N$ $(N\geq 2)$ is a bounded domain with smooth boundary ∂D, $\partial/\partial n$ represents the outward normal derivative on ∂D, γ is a positive constant, u_0 is the initial value, T is the maximal existence time of u, and $\overline{D}$ is the closure of D. Set $\mathbb{R}^+ := (0,+\infty)$. We assume, throughout the paper, that g is a $C^3(\mathbb{R}^+)$ function, $g'(s) > 0$ for any $s\in\mathbb{R}^+$, a is a positive $C^2(\mathbb{R}^+\times\overline{\mathbb{R}^+})$ function, b is a positive $C^1(\overline{D})$ function, h is a positive $C^1(\overline{\mathbb{R}^+})$ function, f is a positive $C^2(\mathbb{R}^+)$ function, and $u_0(x)$ is a positive $C^2(\overline{D})$ function. Under the above assumptions, the classical parabolic equation theory [1] assures that there exists a unique classical solution $u(x,t)$ with some $T>0$ for problem (1) and the solution is positive over $\overline{D}\times[0,T)$. Moreover, by regularity theorem [2], $u(x,t)\in C^3(D\times(0,T))\cap C^2(\overline{D}\times[0,T))$.

Many authors have studied the blow-up and global solutions of nonlinear parabolic problems (see, for instance, [3–14]). Some special cases of the problem (1) have been treated already. Enache [15] investigated the following problem:

$$\begin{aligned} &u_t = \nabla\cdot(a(u)\,\nabla u) + f(u) \quad \text{in } D\times(0,T), \\ &\frac{\partial u}{\partial n} + \gamma u = 0 \quad \text{on } \partial D\times(0,T), \\ &u(x,0) = h(x) \geq 0 \quad \text{in } \overline{D}, \end{aligned} \tag{2}$$

where $D\subset\mathbb{R}^N$ $(N\geq 2)$ is a bounded domain with smooth boundary ∂D. Some conditions on nonlinearities and the initial data were established to guarantee that $u(x,t)$ is global existence or blows up at some finite T. In addition, an upper bound and a lower bound for T were derived. Zhang [16] dealt with the following problem:

$$\begin{aligned} &u_t = \nabla\cdot(a(u)\,b(x)\,\nabla u) + h(t)\,f(u) \\ &\qquad\qquad\qquad \text{in } D\times(0,T), \\ &\frac{\partial u}{\partial n} + \gamma u = 0 \quad \text{on } \partial D\times(0,T), \\ &u(x,0) = u_0(x) > 0 \quad \text{in } \overline{D}, \end{aligned} \tag{3}$$

where $D \subset \mathbb{R}^N$ $(N \geq 2)$ is a bounded domain with smooth boundary ∂D. By constructing auxiliary functions and using maximum principles, the sufficient conditions characterized by functions a, b, h, f, and u_0 were given for the existence of blow-up solution. Ding [17] considered the following problem:

$$\begin{aligned}
&(g(u))_t = \nabla \cdot (a(u) \nabla u) + f(u) \\
&\qquad\qquad \text{in } D \times (0, T), \\
&\frac{\partial u}{\partial n} + \gamma u = 0 \quad \text{on } \partial D \times (0, T), \\
&u(x, 0) = u_0(x) > 0 \quad \text{in } \overline{D},
\end{aligned} \tag{4}$$

where $D \subset \mathbb{R}^N$ $(N \geq 2)$ is a bounded domain with smooth boundary ∂D. By constructing some appropriate auxiliary functions and using a first-order differential inequality technique, the sufficient conditions were specified for the existence of blow-up and global solutions. For the blow-up solution, a lower bound on blow-up time is also obtained. Some authors also discussed blow-up phenomena for parabolic problems with Robin boundary conditions and obtained a lot of interesting results (see [18–24] and the references cited therein).

As everyone knows, parabolic equation describes the process of heat conduction. Blow-up and global solutions for parabolic equations reflect the unsteady state and steady state of heat conduction process, respectively. In the problems (2) and (4), the heat conduction coefficient $a(u)$ depends only on the temperature variable u. In the problem (3), the heat conduction coefficient $a(u)b(x)$ depends on the temperature variable u and space variable x. However, in a lot of processes of heat conduction, heat conduction coefficient depends not only on the temperature variable u but also on the space variable x and the time variable t. Therefore, in this paper, we study the problem (1). It seems that the method of [15–17] is not applicable for the problem (1). In this paper, by constructing completely different auxiliary functions with those in [15–17] and technically using maximum principles, we obtain some existence theorems of blow-up solution, an upper bound of "blow-up time," an upper estimates of "blow-up rate," the existence theorems of global solution, and an upper estimate of the global solution. Our results extend and supplement those obtained in [15–17].

We proceed as follows. In Section 2, we study the blow-up solution of (1). Section 3 is devoted to the global solution of (1). A few examples are presented in Section 4 to illustrate the applications of the abstract results.

2. Blow-Up Solution

The main results for the blow-up solution are Theorems 1–3. For simplicity, we define the constant

$$\alpha := \min_{\overline{D}} \left\{ 1 + \frac{\nabla \cdot (a(u_0, 0) b(x) \nabla u_0)}{h(0) f(u_0)} \right\}. \tag{5}$$

In Theorems 1–3, the three cases $0 < \alpha < 1$, $\alpha = 1$, and $\alpha > 1$ are considered, respectively. In the first case, $0 < \alpha < 1$, we have the following conclusions.

Theorem 1. *Let u be a solution of the problem* (1). *Suppose the following.*

(i) Consider

$$0 < \alpha < 1. \tag{6}$$

(ii) For $(s, t) \in \mathbb{R}^+ \times \mathbb{R}^+$,

$$\begin{aligned}
&\left[\frac{1}{a(s,t)} \left(\frac{a(s,t) f(s)}{g'(s)} \right)_s \right]_s \geq 0, \qquad \left(\frac{a_s(s,t)}{a(s,t)} \right)_t \geq 0, \\
&\left(\frac{a(s,t)}{g'(s)} \right)_s \leq 0, \qquad \left(\frac{h(t)}{a(s,t)} \right)_t \geq 0.
\end{aligned} \tag{7}$$

(iii) For $s \in \mathbb{R}^+$,

$$(s g'(s))' \geq 0, \qquad \left(\frac{f(s)}{s g'(s)} \right)' \geq 0. \tag{8}$$

(iv) Consider

$$\int_{M_0}^{+\infty} \frac{g'(s)}{f(s)} ds < \alpha \int_0^{+\infty} h(t)\, dt, \quad M_0 := \max_{\overline{D}} u_0(x). \tag{9}$$

Then, the solution u of the problem (1) *must blow up in a finite time T, and*

$$\begin{aligned}
&T \leq P^{-1} \left(\frac{1}{\alpha} \int_{M_0}^{+\infty} \frac{g'(s)}{f(s)} ds \right), \\
&u(x,t) \leq H^{-1} \left(\alpha \int_t^T h(t)\, dt \right),
\end{aligned} \tag{10}$$

where

$$\begin{aligned}
&P(z) := \int_0^z h(t)\, dt, \quad z > 0, \\
&H(z) := \int_z^{+\infty} \frac{g'(s)}{f(s)} ds, \quad z > 0,
\end{aligned} \tag{11}$$

and P^{-1} and H^{-1} are the inverse functions of P and H, respectively.

Proof. In order to discuss the blow-up solution by using maximum principles, we construct an auxiliary function

$$Q(x,t) := g'(u) u_t - \alpha h(t) f(u), \tag{12}$$

from which we have

$$\nabla Q = g'' u_t \nabla u + g' \nabla u_t - \alpha h f' \nabla u, \tag{13}$$

$$\begin{aligned}
\Delta Q = {}& g''' |\nabla u|^2 u_t + 2 g'' \nabla u \cdot \nabla u_t \\
&+ g'' u_t \Delta u + g' \Delta u_t - \alpha h f'' |\nabla u|^2 - \alpha h f' \Delta u,
\end{aligned} \tag{14}$$

$$
\begin{aligned}
Q_t &= \left[g'(u)u_t - \alpha h(t) f(u)\right]_t \\
&= \left[(g(u))_t - h(t) f(u) + (1-\alpha) h(t) f(u)\right]_t \\
&= \left[\nabla \cdot (a(u,t) b(x) \nabla u) + (1-\alpha) h(t) f(u)\right]_t \\
&= a_u b u_t \Delta u + a_t b \Delta u + ab\Delta u_t \\
&\quad + a_{uu} b |\nabla u|^2 u_t + a_{ut} b |\nabla u|^2 + 2a_u b \nabla u \cdot \nabla u_t \\
&\quad + a_u u_t \nabla b \cdot \nabla u + a_t \nabla b \cdot \nabla u \\
&\quad + a\nabla b \cdot \nabla u_t + (1-\alpha) h' f + (1-\alpha) h f' u_t .
\end{aligned} \tag{15}
$$

By (14) and (15), we have

$$
\begin{aligned}
&\frac{ab}{g'}\Delta Q - Q_t \\
&= \left(\frac{abg'''}{g'} - a_{uu} b\right)|\nabla u|^2 u_t + \left(2\frac{abg''}{g'} - 2a_u b\right)\nabla u \\
&\quad \cdot \nabla u_t + \left(\frac{abg''}{g'} - a_u b\right) u_t \Delta u \\
&\quad - \left(\alpha \frac{abhf''}{g'} + a_{ut} b\right)|\nabla u|^2 - \left(\alpha\frac{abhf'}{g'} + a_t b\right)\Delta u \\
&\quad - a_u u_t \nabla b \cdot \nabla u - a_t \nabla b \cdot \nabla u \\
&\quad - \mathrm{a}\nabla b \cdot \nabla u_t + (\alpha - 1) h' f + (\alpha - 1) h f' u_t .
\end{aligned} \tag{16}
$$

It follows from (1) that

$$
\Delta u = \frac{g'}{ab} u_t - \frac{a_u}{a}|\nabla u|^2 - \frac{1}{b}\nabla b \cdot \nabla u - \frac{hf}{ab}. \tag{17}
$$

Next, we substitute (17) into (16) to obtain

$$
\begin{aligned}
&\frac{ab}{g'}\Delta Q - Q_t \\
&= \left(\frac{abg'''}{g'} - a_{uu} b - \frac{a_u b g''}{g'} + \frac{(a_u)^2 b}{a}\right)|\nabla u|^2 u_t \\
&\quad + \left(2\frac{abg''}{g'} - 2a_u b\right)\nabla u \cdot \nabla u_t \\
&\quad + \left(g'' - \frac{a_u g'}{a}\right)(u_t)^2 - \frac{ag''}{g'} u_t \nabla b \cdot \nabla u \\
&\quad + \left(\frac{a_u hf}{a} - \frac{fg'' h}{g'} - hf' - \frac{a_t g'}{a}\right) u_t \\
&\quad + \left(\alpha\frac{a_u bhf'}{g'} - \alpha\frac{abhf''}{g'} + \frac{a_u a_t b}{a} - a_{ut} b\right)\nabla u|^2 \\
&\quad + \alpha\frac{ahf'}{g'}\nabla b \cdot \nabla u - a\nabla b \cdot \nabla u_t \\
&\quad + \alpha\frac{h^2 f f'}{g'} + \frac{a_t h f}{a} + (\alpha - 1) h' f .
\end{aligned} \tag{18}
$$

With (13), it has

$$
\nabla u_t = \frac{1}{g'}\nabla Q - \frac{g''}{g'} u_t \nabla u + \alpha\frac{hf'}{g'}\nabla u. \tag{19}
$$

Substitute (19) into (18) to get

$$
\begin{aligned}
&\frac{ab}{g'}\Delta Q + \left[2b\left(\frac{a}{g'}\right)_u \nabla u + \frac{a}{g'}\nabla b\right]\cdot\nabla Q - Q_t \\
&= \left(\frac{abg'''}{g'} - a_{uu} b + \frac{a_u b g''}{g'} + \frac{(a_u)^2 b}{a} - 2\frac{ab\left(g''\right)^2}{\left(g'\right)^2}\right) u_t \\
&\quad \times |\nabla u|^2 \\
&\quad + \left(2\alpha\frac{abhf' g''}{(g')^2} - \alpha\frac{a_u bhf'}{g'} - \alpha\frac{abhf''}{g'} + \frac{a_u a_t b}{a} - a_{ut} b\right) \\
&\quad \times |\nabla u|^2 + \left(g'' - \frac{a_u g'}{a}\right)(u_t)^2 \\
&\quad + \left(\frac{a_u hf}{a} - \frac{fg'' h}{g'} - hf' - \frac{a_t g'}{a}\right) u_t + \alpha\frac{h^2 f f'}{g'} \\
&\quad + \frac{a_t h f}{a} + (\alpha - 1) h' f .
\end{aligned} \tag{20}
$$

In view of (12), we have

$$
u_t = \frac{1}{g'} Q + \alpha\frac{hf}{g'}. \tag{21}
$$

Substituting (21) into (20), we get

$$
\begin{aligned}
&\frac{ab}{g'}\Delta Q + \left[2b\left(\frac{a}{g'}\right)_u \nabla u + \frac{a}{g'}\nabla b\right]\cdot\nabla Q \\
&\quad + \left\{ab\left[\frac{1}{a}\left(\frac{a}{g'}\right)_u\right]_u |\nabla u|^2 + \frac{1}{a}\left(\frac{a}{g'}\right)_u \right. \\
&\quad \left. \times \left[Q + (2\alpha - 1) hf\right] + \frac{hf'}{g'} + \frac{a_t}{a}\right\} Q - Q_t \\
&= -abh\left\{\alpha\left[\frac{1}{a}\left(\frac{af}{g'}\right)_u\right]_u + \frac{1}{h}\left(\frac{a_u}{a}\right)_t\right\}|\nabla u|^2 \\
&\quad - \alpha(\alpha - 1)\frac{h^2 f^2}{a}\left(\frac{a}{g'}\right)_u + (\alpha - 1) af\left(\frac{h}{a}\right)_t .
\end{aligned} \tag{22}
$$

The assumptions (6) and (7) imply that the right-hand side of (22) is nonpositive; that is,

$$
\frac{ab}{g'}\Delta Q + \left[2b\left(\frac{a}{g'}\right)_u \nabla u + \frac{a}{g'}\nabla b\right]\cdot\nabla Q
$$

$$+\left\{ab\left[\frac{1}{a}\left(\frac{a}{g'}\right)_u\right]_u |\nabla u|^2 + \frac{1}{a}\left(\frac{a}{g'}\right)_u \times [Q + (2\alpha - 1) hf] + \frac{hf'}{g'} + \frac{a_t}{a}\right\} \times Q - Q_t \leq 0, \text{ in } D \times (0,T). \tag{23}$$

Applying the maximum principle [25], it follows from (23) that Q can attain its nonpositive minimum only for $\overline{D} \times \{0\}$ or $\partial D \times (0,T)$. For $\overline{D} \times \{0\}$, (5) implies

$$\begin{aligned}
\min_{\overline{D}} Q(x,0) &= \min_{\overline{D}} \{g'(u_0)(u_0)_t - \alpha h(0) f(u_0)\} \\
&= \min_{\overline{D}} \{(g(u_0))_t - h(0) f(u_0) + (1-\alpha) h(0) f(u_0)\} \\
&= \min_{\overline{D}} \{\nabla \cdot (a(u_0,0) b(x) \nabla u_0) + (1-\alpha) h(0) f(u_0)\} \\
&= \min_{\overline{D}} \left\{h(0) f(u_0)\left[1 + \frac{\nabla \cdot (a(u_0,0) b(x) \nabla u_0)}{h(0) f(u_0)} - \alpha\right]\right\} \\
&= 0.
\end{aligned} \tag{24}$$

We claim that Q cannot take a negative minimum at any point $(x,t) \in \partial D \times (0,T)$. Indeed, if Q take a negative minimum at point $(x_0,t_0) \in \partial D \times (0,T)$, then

$$Q(x_0,t_0) < 0, \qquad \left.\frac{\partial Q}{\partial n}\right|_{(x_0,t_0)} < 0. \tag{25}$$

It follows from (1) and (21) that

$$\begin{aligned}
\frac{\partial Q}{\partial n} &= g'' u_t \frac{\partial u}{\partial n} + g' \frac{\partial u_t}{\partial n} - \alpha h f' \frac{\partial u}{\partial n} \\
&= -\gamma g'' u u_t + g' \left(\frac{\partial u}{\partial n}\right)_t + \gamma \alpha h f' u \\
&= -\gamma g'' u u_t + g'(-\gamma u)_t + \gamma \alpha h f' u \\
&= -\gamma (u g')' u_t + \gamma \alpha h f' u \\
&= -\gamma (u g')' \left(\frac{1}{g'} Q + \alpha \frac{hf}{g'}\right) + \gamma \alpha h f' u \\
&= -\gamma \frac{(u g')'}{g'} Q + \gamma \alpha u^2 g' h \left(\frac{f}{u g'}\right) \quad \text{on } \partial D \times (0,T).
\end{aligned} \tag{26}$$

Next, by using (8) and the fact $Q(x_0,t_0) < 0$, it follows from (26) that

$$\left.\frac{\partial Q}{\partial n}\right|_{(x_0,t_0)} \geq 0, \tag{27}$$

which contradicts inequality (25). Thus, we know that the minimum of Q in $\overline{D} \times [0,T)$ is zero. Thus,

$$Q \geq 0 \quad \text{in } \overline{D} \times [0,T); \tag{28}$$

that is,

$$\frac{g'(u)}{f(u)} u_t \geq \alpha h(t). \tag{29}$$

At the point $x^* \in \overline{D}$, where $u_0(x^*) = M_0$, integrate (29) over $[0,t]$ to get

$$\int_0^t \frac{g'(u)}{f(u)} u_t dt = \int_{M_0}^{u(x^*,t)} \frac{g'(s)}{f(s)} ds \geq \alpha \int_0^t h(t)\, dt, \tag{30}$$

which shows that u must blow up in finite time. In fact, suppose u is a global solution of (1), then, for any $t > 0$, it follows from (30) that

$$\int_{M_0}^{+\infty} \frac{g'(s)}{f(s)} ds > \int_{M_0}^{u(x^*,t)} \frac{g'(s)}{f(s)} ds \geq \alpha \int_0^t h(t)\, dt. \tag{31}$$

Passing to the limit as $t \to +\infty$ in (31) yields

$$\int_{M_0}^{+\infty} \frac{g'(s)}{f(s)} ds \geq \alpha \int_0^{+\infty} h(t)\, dt, \tag{32}$$

which contradicts assumption (9). This shows that u must blow up in a finite time $t = T$. Furthermore, letting $t \to T$ in (30), we have

$$\lim_{t \to T} \int_{M_0}^{u(x^*,t)} \frac{g'(s)}{f(s)} ds \geq \lim_{t \to T} \alpha \int_0^t h(t)\, dt; \tag{33}$$

that is,

$$\frac{1}{\alpha} \int_{M_0}^{+\infty} \frac{g'(s)}{f(s)} ds \geq \int_0^T h(t)\, dt = P(T), \tag{34}$$

which implies that

$$T \leq P^{-1}\left(\frac{1}{\alpha} \int_{M_0}^{+\infty} \frac{g'(s)}{f(s)} ds\right). \tag{35}$$

By integrating inequality (29) over $[t,s]$ $(0 < t < s < T)$, for each fixed x, one gets

$$\begin{aligned}
H(u(x,t)) &\geq H(u(x,t)) - H(u(x,s)) \\
&= \int_{u(x,t)}^{u(x,s)} \frac{g'(s)}{f(s)} ds \geq \alpha \int_t^s h(t)\, dt.
\end{aligned} \tag{36}$$

Hence, by letting $s \to T$, we obtain

$$H(u(x,t)) \geq \alpha \int_t^T h(t)\, dt. \tag{37}$$

Since H is a decreasing function, we have

$$u(x,t) \leq H^{-1}\left(\alpha \int_t^T h(t)\, dt\right). \tag{38}$$

The proof is complete. □

In the second case, $\alpha = 1$, the following two assumptions $(\mathrm{i})_{\mathrm{a}}$ and $(\mathrm{ii})_{\mathrm{a}}$ can guarantee that inequality (23) holds.

$(\mathrm{i})_{\mathrm{a}}$ Consider

$$\alpha = 1. \tag{39}$$

$(\mathrm{ii})_{\mathrm{a}}$ For $(s,t) \in \mathbb{R}^+ \times \mathbb{R}^+$,

$$\left[\frac{1}{a(s,t)}\left(\frac{a(s,t) f(s)}{g'(s)}\right)_s\right]_s \geq 0, \qquad \left(\frac{a_s(s,t)}{a(s,t)}\right)_t \geq 0. \tag{40}$$

Hence, by repeating the proof of Theorem 1, we have the following results.

Theorem 2. *Let u be a solution of the problem* (1). *Suppose that $(i)_a$ and $(ii)_a$ hold and assumptions (iii) and (iv) of Theorem 1 hold. Then, the conclusions of Theorem 1 are valid.*

In the third case, $\alpha > 1$, the following two assumptions $(\mathrm{i})_{\mathrm{b}}$ and $(\mathrm{ii})_{\mathrm{b}}$ imply that inequality (23) holds.

$(\mathrm{i})_{\mathrm{b}}$ Consider

$$\alpha > 1. \tag{41}$$

$(\mathrm{ii})_{\mathrm{b}}$ For $(s,t) \in \mathbb{R}^+ \times \mathbb{R}^+$,

$$\begin{gathered}\left[\frac{1}{a(s,t)}\left(\frac{a(s,t) f(s)}{g'(s)}\right)_s\right]_s \geq 0, \qquad \left(\frac{a_s(s,t)}{a(s,t)}\right)_t \geq 0, \\ \left(\frac{a(s,t)}{g'(s)}\right)_s \geq 0, \qquad \left(\frac{h(t)}{a(s,t)}\right)_t \leq 0.\end{gathered} \tag{42}$$

Theorem 3. *Let u be a solution of the problem* (1). *Suppose that $(i)_b$ and $(ii)_b$ hold and assumptions (iii) and (iv) of Theorem 1 hold. Then, the results stated in Theorem 1 still hold.*

Remark 4. When

$$\int_0^{+\infty} h(t)\,dt = +\infty, \tag{43}$$

(9) implies that

$$\int_{M_0}^{+\infty} \frac{g'(s)}{f(s)} ds < +\infty, \quad M_0 = \max_{\overline{D}} u_0(x). \tag{44}$$

When

$$\int_0^{+\infty} h(t)\,dt < +\infty, \tag{45}$$

(9) implies that

$$\int_{M_0}^{+\infty} \frac{g'(s)}{f(s)} ds < +\infty, \qquad \int_{M_0}^{+\infty} \frac{g'(s)}{f(s)} ds < \alpha \int_0^{+\infty} h(t)\,dt. \tag{46}$$

3. Global Solution

We define the constant

$$\beta := \max_{\overline{D}} \left\{1 + \frac{\nabla\cdot(a(u_0,0)\, b(x)\, \nabla u_0)}{h(0)\, f(u_0)}\right\}. \tag{47}$$

The following Theorems 5–7 are the main results for the global solution. In Theorems 5–7, we study the three cases $0 < \beta < 1, \beta = 1$, and $\beta > 1$, respectively. In the first case, $0 < \beta < 1$, we have the following results.

Theorem 5. *Let u be a solution of the problem* (1). *Suppose the following.*

(i) Consider

$$0 < \beta < 1. \tag{48}$$

(ii) For $(s,t) \in \mathbb{R}^+ \times \mathbb{R}^+$,

$$\begin{gathered}\left[\frac{1}{a(s,t)}\left(\frac{a(s,t) f(s)}{g'(s)}\right)_s\right]_s \leq 0, \qquad \left(\frac{a_s(s,t)}{a(s,t)}\right)_t \leq 0, \\ \left(\frac{a(s,t)}{g'(s)}\right)_s \geq 0, \qquad \left(\frac{h(t)}{a(s,t)}\right)_t \leq 0.\end{gathered} \tag{49}$$

(iii) For $s \in \mathbb{R}^+$,

$$\left(sg'(s)\right)' \geq 0, \qquad \left(\frac{f(s)}{sg'(s)}\right)' \leq 0. \tag{50}$$

(iv) Consider

$$\int_{M_0}^{+\infty} \frac{g'(s)}{f(s)} ds \geq \beta \int_0^{+\infty} h(t)\,dt, \quad M_0 := \max_{\overline{D}} u_0(x). \tag{51}$$

Then, the solution u to the problem (1) *must be a global solution and*

$$u(x,t) \leq F^{-1}\left(\beta \int_0^t h(t)\,dt + F(u_0(x))\right), \qquad (x,t) \in \overline{D} \times \overline{\mathbb{R}^+}, \tag{52}$$

where

$$F(z) := \int_{m_0}^{z} \frac{g'(s)}{f(s)} ds, \quad z \geq m_0, \ m_0 := \min_{\overline{D}} u_0(x), \tag{53}$$

and F^{-1} is the inverse function of F.

Proof. In order to study the global solution by using maximum principles, we construct an auxiliary function

$$G(x,t) := g'(u)\, u_t - \beta h(t)\, f(u). \tag{54}$$

Substituting Q and α with G and β in (22), respectively, gives

$$\begin{aligned}&\frac{ab}{g'}\Delta G + \left[2b\left(\frac{a}{g'}\right)_u \nabla u + \frac{a}{g'}\nabla b\right]\cdot\nabla G \\ &\quad + \left\{ab\left[\frac{1}{a}\left(\frac{a}{g'}\right)_u\right]_u |\nabla u|^2 + \frac{1}{a}\left(\frac{a}{g'}\right)_u \right. \\ &\quad \left. \times \left[G + (2\beta - 1)\, hf\right] + \frac{hf'}{g'} + \frac{a_t}{a}\right\} G - G_t\end{aligned}$$

$$
\begin{aligned}
&= -abh\left\{\beta\left[\frac{1}{a}\left(\frac{af}{g'}\right)_u\right]_u + \frac{1}{h}\left(\frac{a_u}{a}\right)_t\right\}|\nabla u|^2 \\
&\quad - \beta(\beta-1)\frac{h^2f^2}{a}\left(\frac{a}{g'}\right)_u + (\beta-1)af\left(\frac{h}{a}\right)_t.
\end{aligned} \tag{55}
$$

Assumptions (48) and (49) guarantee that the right side in equality (55) is nonnegative; that is,

$$
\begin{aligned}
&\frac{ab}{g'}\Delta G + \left[2b\left(\frac{a}{g'}\right)_u\nabla u + \frac{a}{g'}\nabla b\right]\cdot\nabla G \\
&\quad + \left\{ab\left[\frac{1}{a}\left(\frac{a}{g'}\right)_u\right]_u|\nabla u|^2 + \frac{1}{a}\left(\frac{a}{g'}\right)_u\right. \\
&\quad \left.\times\left[G + (2\beta-1)hf\right] + \frac{hf'}{g'} + \frac{a_t}{a}\right\}G - G_t \ge 0 \\
&\qquad \text{in } D\times(0,T).
\end{aligned} \tag{56}
$$

It follows from (47) that

$$
\begin{aligned}
&\max_{\overline{D}} G(x,0) \\
&\quad = \max_{\overline{D}}\left\{g'(u_0)(u_0)_t - \beta h(0)f(u_0)\right\} \\
&\quad = \max_{\overline{D}}\left\{(g(u_0))_t - h(0)f(u_0) + (1-\beta)h(0)f(u_0)\right\} \\
&\quad = \max_{\overline{D}}\left\{\nabla\cdot(a(u_0,0)b(x)\nabla u_0) + (1-\beta)h(0)f(u_0)\right\} \\
&\quad = \max_{\overline{D}}\left\{h(0)f(u_0)\left[1 + \frac{\nabla\cdot(a(u_0,0)b(x)\nabla u_0)}{h(0)f(u_0)} - \beta\right]\right\} \\
&\quad = 0.
\end{aligned} \tag{57}
$$

Replacing Q and α with G and β in (26), respectively, we have

$$
\frac{\partial G}{\partial n} = -\gamma\frac{(ug')'}{g'}G + \gamma\beta u^2g'h\left(\frac{f}{ug'}\right)' \quad \text{on } \partial D\times(0,T). \tag{58}
$$

Combining (56)–(58) with (50) and applying the maximum principles again, it follows that the maximum of G in $\overline{D}\times[0,T)$ is zero. Thus,

$$
G \le 0 \quad \text{in } \overline{D}\times[0,T), \tag{59}
$$

$$
\frac{g'(u)}{f(u)}u_t \le \beta h(t). \tag{60}
$$

For each fixed $x\in\overline{D}$, integration of (60) from 0 to t yields

$$
\int_0^t \frac{g'(u)}{f(u)}u_t\,dt = \int_{u_0(x)}^{u(x,t)}\frac{g'(s)}{f(s)}ds \le \beta\int_0^t h(t)\,dt, \tag{61}
$$

which implies that u must be a global solution. Actually, if u blows up at a finite time T, then

$$
\lim_{t\to T^-} u(x,t) = +\infty. \tag{62}
$$

Letting $t\to T^-$ in (61), we have

$$
\begin{aligned}
&\int_{u_0(x)}^{+\infty}\frac{g'(s)}{f(s)}ds \le \beta\int_0^T h(t)\,dt < \beta\int_0^{+\infty} h(t)\,dt, \\
&\int_{M_0}^{+\infty}\frac{g'(s)}{f(s)}ds = \int_{M_0}^{u_0(x)}\frac{g'(s)}{f(s)}ds + \int_{u_0(x)}^{+\infty}\frac{g'(s)}{f(s)}ds \\
&\qquad \le \int_{u_0(x)}^{+\infty}\frac{g'(s)}{f(s)}ds < \beta\int_0^{+\infty}h(t)\,dt,
\end{aligned} \tag{63}
$$

which contradicts with assumption (51). This shows that u is global. Moreover, it follows from (61) that

$$
\begin{aligned}
\int_{u_0(x)}^{u(x,t)}\frac{g'(s)}{f(s)}ds &= \int_{m_0}^{u(x,t)}\frac{g'(s)}{f(s)}ds - \int_{m_0}^{u_0(x)}\frac{g'(s)}{f(s)}ds \\
&= F(u(x,t)) - F(u_0(x)) \le \beta\int_0^t h(t)\,\mathrm{d}t.
\end{aligned} \tag{64}
$$

Since F is an increasing function, we have

$$
u(x,t) \le F^{-1}\left(\beta\int_0^t h(t)\,dt + F(u_0(x))\right). \tag{65}
$$

The proof is complete. □

In the second case $\beta = 1$ and the third case $\beta > 1$, we have the following results.

Theorem 6. *Let u be a solution of the problem* (1). *Suppose that assumptions $(i)_c$ and $(ii)_c$ hold.*

$(i)_c$ *Consider*

$$
\beta = 1. \tag{66}
$$

$(ii)_c$ *For $(s,t)\in\mathbb{R}^+\times\mathbb{R}^+$,*

$$
\left[\frac{1}{a(s,t)}\left(\frac{a(s,t)f(s)}{g'(s)}\right)_s\right]_s \le 0, \qquad \left(\frac{a_s(s,t)}{a(s,t)}\right)_t \le 0. \tag{67}
$$

And assumptions (iii) and (iv) of Theorem 5 hold. Then, the results of Theorem 5 are valid.

Theorem 7. *Let u be a solution of the problem* (1). *Suppose that assumptions $(i)_d$ and $(ii)_d$ hold.*

$(\mathrm{i})_\mathrm{d}$ *Consider*

$$
\beta > 1. \tag{68}
$$

$(\mathrm{ii})_\mathrm{d}$ *For $(s,t)\in\mathbb{R}^+\times\mathbb{R}^+$,*

$$
\begin{aligned}
&\left[\frac{1}{a(s,t)}\left(\frac{a(s,t)f(s)}{g'(s)}\right)_s\right]_s \le 0, \qquad \left(\frac{a_s(s,t)}{a(s,t)}\right)_t \le 0, \\
&\left(\frac{a(s,t)}{g'(s)}\right)_s \le 0, \qquad \left(\frac{h(t)}{a(s,t)}\right)_t \ge 0.
\end{aligned} \tag{69}
$$

And assumptions (iii) and (iv) of Theorem 5 hold. Then, the conclusions stated in Theorem 5 still hold.

Remark 8. When

$$\int_0^{+\infty} h(t)\,dt = +\infty, \tag{70}$$

(51) implies that

$$\int_{M_0}^{+\infty} \frac{g'(s)}{f(s)}ds = +\infty, \quad M_0 = \max_{\overline{D}} u_0(x). \tag{71}$$

When

$$\int_0^{+\infty} h(t)\,dt < +\infty, \tag{72}$$

(51) implies that

$$\int_{M_0}^{+\infty} \frac{g'(s)}{f(s)}ds = +\infty \tag{73}$$

or

$$\int_{M_0}^{+\infty} \frac{g'(s)}{f(s)}ds < +\infty, \quad \int_{M_0}^{+\infty} \frac{g'(s)}{f(s)}ds \geq \beta \int_0^{+\infty} h(t)\,dt. \tag{74}$$

4. Applications

When $g(u) \equiv u$, $a(u,t) \equiv a(u)$, $b(x) \equiv 1$ and $h(t) \equiv 1$ or $g(u) \equiv u$ and $a(u,t) \equiv a(u)$ or $a(u,t) \equiv a(u), b(x) \equiv 1$ and $h(t) \equiv 1$, the conclusions of Theorems 1–3 and 5–7 still hold true. In this sense, our results extend and supplement the results of [15–17].

In what follows, we present several examples to demonstrate the applications of the abstract results.

Example 9. Let u be a solution of the following problem:

$$\begin{gathered}(u^p)_t = \nabla\cdot(u^n\nabla u) + u^q \quad \text{in } D\times(0,T),\\ \frac{\partial u}{\partial n} + \gamma u = 0 \quad \text{on } \partial D\times(0,T),\\ u(x,0) = u_0(x) > 0 \quad \text{in } \overline{D},\end{gathered} \tag{75}$$

where $D \subset \mathbb{R}^N$ $(N \geq 2)$ is a bounded domain with smooth boundary ∂D, $p > 0, -\infty < n < +\infty, -\infty < q < +\infty$. Here,

$$\begin{gathered}g(u) = u^p, \quad a(u,t) = u^n,\\ b(x) = 1, \quad h(t) = 1, \quad f(u) = u^q.\end{gathered} \tag{76}$$

Assume that

$$\alpha = \min_{\overline{D}}\left\{1 + \frac{\nabla\cdot(u_0^n\nabla u_0)}{u_0^q}\right\} > 0 \tag{77}$$

and one of the following three assumptions holds.

(i) In the case $0 < \alpha < 1, p-q < 0 \leq n+1 \leq p$ or $p-q \leq n+1 < 0$.

(ii) In the case $\alpha = 1, p-q < 0 \leq n+1$ or $p-q \leq n+1 < 0$.

(iii) In the case $\alpha > 1, p-q < 0 < p \leq n+1$.

It follows from Theorems 1–3 that u blows up in a finite time T, and

$$\begin{gathered}T \leq P^{-1}\left(\frac{1}{\alpha}\int_{M_0}^{+\infty}\frac{g'(s)}{f(s)}ds\right) = \frac{p}{\alpha(q-p)M_0^{q-p}},\\ u(x,t) \leq H^{-1}\left(\alpha\int_0^T h(t)\,dt\right) = \left(\frac{p}{\alpha(q-p)(T-t)}\right)^{1/(q-p)}.\end{gathered} \tag{78}$$

Assume that

$$\beta = \max_{\overline{D}}\left\{1 + \frac{\nabla\cdot(u_0^n\nabla u_0)}{u_0^q}\right\} > 0 \tag{79}$$

and one of the following three assumptions holds.

(i) In the case $0 < \beta < 1, 0 \leq p-q \leq p \leq n+1$ or $p < p-q \leq n+1$.

(ii) In the case $\beta = 1, 0 \leq p-q \leq n+1$.

(iii) In the case $\beta > 1, 0 \leq p-q \leq n+1 \leq p$.

By Theorems 5–7, u must be a global solution and

$$\begin{aligned}u(x,t) &\leq F^{-1}\left(\beta\int_0^t h(t)\,dt + F(u_0(x))\right)\\ &= \begin{cases}\left[\dfrac{\beta(p-q)}{p}t + (u_0(x))^{p-q}\right]^{1/(p-q)}, & p-q > 0,\\ u_0(x)e^{\alpha t/p}, & p-q = 0.\end{cases}\end{aligned} \tag{80}$$

Example 10. Let u be a solution of the following problem:

$$\begin{gathered}(ue^u)_t = \nabla\cdot\left(e^{u+t}\left(2-\sum_{i=1}^3 x_i^2\right)\nabla u\right) + u(1+u)^2e^{u+t}\\ \text{in } D\times(0,T),\\ \frac{\partial u}{\partial n} + 2u = 0 \quad \text{on } \partial D\times(0,T),\\ u(x,0) = 2-\sum_{i=1}^3 x_i^2 \quad \text{in } \overline{D},\end{gathered} \tag{81}$$

where $D = \{x = (x_1,x_2,x_3) \mid \sum_{i=1}^3 x_i^2 < 1\}$ is the unit ball of $\mathbb{R}^3$. Now, we have

$$\begin{gathered}g(u) = ue^u, \quad a(u,t) = e^{u+t}, \quad b(x) = 2-\sum_{i=1}^3 x_i^2,\\ h(t) = e^t, \quad f(u) = u(1+u)^2e^u,\\ u_0(x) = 2-\sum_{i=1}^3 x_i^2, \quad \gamma = 2.\end{gathered} \tag{82}$$

In order to determine the constant α, we assume

$$w := \sum_{i=1}^3 x_i^2. \tag{83}$$

Then, $0 \le w \le 1$ and

$$\begin{aligned}\alpha &= \min_{\overline{D}}\left\{1+\frac{\nabla\cdot\left(a\left(u_0,0\right)b\left(x\right)\nabla u_0\right)}{h\left(0\right)f\left(u_0\right)}\right\}\\ &= \min_{\overline{D}}\left\{1+\frac{-12+18\sum_{i=1}^{3}x_i^2-4\left(\sum_{i=1}^{3}x_i^2\right)^2}{\left(2-\sum_{i=1}^{3}x_i^2\right)\left(3-\sum_{i=1}^{3}x_i^2\right)^2}\right\}\\ &= \min_{0\le w\le 1}\left\{1+\frac{-12+18w-4w^2}{\left(2-w\right)\left(3-w\right)^2}\right\}=\frac{1}{3}.\end{aligned}\tag{84}$$

It is easy to check that (6)-(9) hold. By Theorem 1, u must blow up in a finite time T, and

$$\begin{aligned}T &\le P^{-1}\left(\frac{1}{\alpha}\int_{M_0}^{+\infty}\frac{g'\left(s\right)}{f\left(s\right)}ds\right)=\ln\left(\frac{1}{3}\ln\frac{3}{2}+1\right),\\ u\left(x,t\right) &\le H^{-1}\left(\alpha\int_t^T h\left(t\right)dt\right)=\frac{1}{e^{(1/3)(e^T-e^t)}-1}.\end{aligned}\tag{85}$$

Example 11. Let u be a solution of the following problem:

$$\begin{gathered}\left(ue^u\right)_t=\nabla\cdot\left(e^{u-t}\left(2-\sum_{i=1}^{3}x_i^2\right)\nabla u\right)+\left(1+u\right)e^{u-t}\\ \text{in } D\times\left(0,T\right),\\ \frac{\partial u}{\partial n}+2u=0 \quad \text{on } \partial D\times\left(0,T\right),\\ u\left(x,0\right)=2-\sum_{i=1}^{3}x_i^2 \quad \text{in } \overline{D},\end{gathered}\tag{86}$$

where $D=\{x=(x_1,x_2,x_3)\mid \sum_{i=1}^{3}x_i^2<1\}$ is the unit ball of $\mathbb{R}^3$. Now,

$$\begin{gathered}g\left(u\right)=ue^u,\qquad a\left(u,t\right)=e^{u-t},\qquad b\left(x\right)=2-\sum_{i=1}^{3}x_i^2,\\ h\left(t\right)=e^{-t},\qquad f\left(u\right)=\left(1+u\right)e^u,\\ u_0\left(x\right)=2-\sum_{i=1}^{3}x_i^2,\qquad \gamma=2.\end{gathered}\tag{87}$$

By setting

$$w:=\sum_{i=1}^{3}x_i^2,\tag{88}$$

we have $0\le w\le 1$ and

$$\begin{aligned}\beta &= \max_{\overline{D}}\left\{1+\frac{\nabla\cdot\left(a\left(u_0,0\right)b\left(x\right)\nabla u_0\right)}{h\left(0\right)f\left(u_0\right)}\right\}\\ &= \max_{\overline{D}}\left\{1+\frac{-12+18\sum_{i=1}^{3}x_i^2-4\left(\sum_{i=1}^{3}x_i^2\right)^2}{3-\sum_{i=1}^{3}x_i^2}\right\}\\ &= \max_{0\le w\le 1}\left\{1+\frac{-12+18w-4w^2}{\left(3-w\right)}\right\}=2.\end{aligned}\tag{89}$$

It is easy to check that (68)-(69) and (50)-(51) hold. By Theorem 7, u must be a global solution, and

$$\begin{aligned}u\left(x,t\right) &\le F^{-1}\left(\beta\int_0^t h\left(t\right)dt+F\left(u_0\left(x\right)\right)\right)\\ &= 2\left(2-e^{-t}\right)-\sum_{i=1}^{3}x_i^2.\end{aligned}\tag{90}$$

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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References

[1] H. Amann, "Quasilinear parabolic systems under nonlinear boundary conditions," *Archive for Rational Mechanics and Analysis*, vol. 92, no. 2, pp. 153–192, 1986.

[2] R. P. Sperb, *Maximum Principles and Their Applications*, vol. 157 of *Mathematics in Science and Engineering*, Academic Press, New York, NY, USA, 1981.

[3] H. A. Levine, "The role of critical exponents in blowup theorems," *SIAM Review*, vol. 32, no. 2, pp. 262–288, 1990.

[4] A. A. Samarskii, V. A. Galaktionov, S. P. Kurdyumov, and A. P. Mikhailov, *Blow-Up in Problems for Quasilinear Parabolic Equations*, Nauka, Moscow, Russia, 1987 (Russian), English translation: Walter de Gruyter, Berlin, Germany, 1995.

[5] C. Bandle and H. Brunner, "Blowup in diffusion equations: a survey," *Journal of Computational and Applied Mathematics*, vol. 97, no. 1-2, pp. 3–22, 1998.

[6] K. Deng and H. A. Levine, "The role of critical exponents in blow-up theorems: the sequel," *Journal of Mathematical Analysis and Applications*, vol. 243, no. 1, pp. 85–126, 2000.

[7] V. A. Galaktionov and J. L. Vázquez, "The problem of blow-up in nonlinear parabolic equations," *Discrete and Continuous Dynamical Systems A*, vol. 8, no. 2, pp. 399–433, 2002.

[8] P. Quittner and P. Souplet, *Superlinear Parabolic Problems: Blow-Up, Global Existence and Steady States*, Birkhäuser Advanced Texts, Birkhäuser, Basel, Switzerland, 2007.

[9] B. Hu and H.-M. Yin, "The profile near blowup time for solution of the heat equation with a nonlinear boundary condition," *Transactions of the American Mathematical Society*, vol. 346, no. 1, pp. 117–135, 1994.

[10] Y. Qi, M. Wang, and Z. Wang, "Existence and non-existence of global solutions of diffusion systems with nonlinear boundary conditions," *Proceedings of the Royal Society of Edinburgh A. Mathematics*, vol. 134, no. 6, pp. 1199–1217, 2004.

[11] C. Enache, "Blow-up, global existence and exponential decay estimates for a class of quasilinear parabolic problems," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 69, no. 9, pp. 2864–2874, 2008.

[12] H. Zhang, “Blow-up solutions and global solutions for nonlinear parabolic problems,” *Nonlinear Analysis: Theory, Methods & Applications*, vol. 69, no. 12, pp. 4567–4574, 2008.

[13] J. Ding, X. Gao, and S. Li, “Global existence and blow-up problems for reaction diffusion model with multiple nonlinearities,” *Journal of Mathematical Analysis and Applications*, vol. 343, no. 1, pp. 159–169, 2008.

[14] J. Ding and S. Li, “Blow-up and global solutions for nonlinear reaction-diffusion equations with Neumann boundary conditions,” *Nonlinear Analysis: Theory, Methods & Applications*, vol. 68, no. 3, pp. 507–514, 2008.

[15] C. Enache, “Blow-up phenomena for a class of quasilinear parabolic problems under Robin boundary condition,” *Applied Mathematics Letters*, vol. 24, no. 3, pp. 288–292, 2011.

[16] L. Zhang, “Blow-up of solutions for a class of nonlinear parabolic equations,” *Zeitschrift für Analysis und ihre Anwendungen*, vol. 25, no. 4, pp. 479–486, 2006.

[17] J. Ding, “Global and blow-up solutions for nonlinear parabolic equations with Robin boundary conditions,” *Computers & Mathematics with Applications*, vol. 65, no. 11, pp. 1808–1822, 2013.

[18] A. Friedman and B. McLeod, “Blow-up of positive solutions of semilinear heat equations,” *Indiana University Mathematics Journal*, vol. 34, no. 2, pp. 425–447, 1985.

[19] L. E. Payne and P. W. Schaefer, “Blow-up in parabolic problems under Robin boundary conditions,” *Applicable Analysis*, vol. 87, no. 6, pp. 699–707, 2008.

[20] Y. Li, Y. Liu, and C. Lin, “Blow-up phenomena for some nonlinear parabolic problems under mixed boundary conditions,” *Nonlinear Analysis: Real World Applications*, vol. 11, no. 5, pp. 3815–3823, 2010.

[21] J.-F. Rault, “The Fujita phenomenon in exterior domains under the Robin boundary conditions,” *Comptes Rendus de l’Académie des Sciences. Série I. Mathematics*, vol. 349, no. 19-20, pp. 1059–1061, 2011.

[22] Y. Li, Y. Liu, and S. Xiao, “Blow-up phenomena for some nonlinear parabolic problems under Robin boundary conditions,” *Mathematical and Computer Modelling*, vol. 54, no. 11-12, pp. 3065–3069, 2011.

[23] Y. Liu, S. Luo, and Y. Ye, “Blow-up phenomena for a parabolic problem with a gradient nonlinearity under nonlinear boundary conditions,” *Computers & Mathematics with Applications*, vol. 65, no. 8, pp. 1194–1199, 2013.

[24] J. Ding, “Global and blow-up solutions for nonlinear parabolic problems with a gradient term under Robin boundary conditions,” *Boundary Value Problems*, vol. 2013, article 237, pp. 1–12, 2013.

[25] M. H. Protter and H. F. Weinberger, *Maximum Principles in Differential Equations*, Prentice-Hall, Englewood Cliffs, NJ, USA, 1967.