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Research Article

Optimal Pole Assignment of Linear Systems by the Sylvester Matrix Equations

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The problem of state feedback optimal pole assignment is to design a feedback gain such that the closed-loop system has desired eigenvalues and such that certain quadratic performance index is minimized. Optimal pole assignment controller can guarantee both good dynamic response and well robustness properties of the closed-loop system. With the help of a class of linear matrix equations, necessary and sufficient conditions for the existence of a solution to the optimal pole assignment problem are proposed in this paper. By properly choosing the free parameters in the parametric solutions to this class of linear matrix equations, complete solutions to the optimal pole assignment problem can be obtained. A numerical example is used to illustrate the effectiveness of the proposed approach.

1. Introduction and Problem Formulation

The linear quadratic optimal control problem for linear systems is to design a linear state feedback controller such that a quadratic performance index function is minimized. It is well known in the literature (see, e.g., [1]) that the linear quadratic optimal controller possesses very good robustness properties; for example, it has at least a 6 dB gain margin and a 60° phase margin for single input linear system [2]. On the other hand, as a nonoptimal design approach, the pole assignment approach can arbitrarily assign the poles of the closed-loop system to any place so that a very satisfactory transient performance of the closed-loop system can be achieved. However, when the weighting matrices in the performance index function are fixed in the linear quadratic optimal control problem, the controller is unique and, thus, the pole locations of the resulting optimal closed-loop system are uniquely determined. As a result, the relationship between the weighting matrices and the locations of the poles of the resulting optimal closed-loop system is not clear. Hence, to take advantages of both the linear quadratic optimal control approach and the pole assignment approach, the so-called problem of designing a feedback gain which shifts the poles of a given linear system to some prescribed positions and simultaneously minimizes a quadratic cost function has been widely studied in the past several decades, especially in the case of multi-input system (see [3–8] and the references therein).

To make the problem clear, let us consider the following linear system:

$$\dot{x} = Ax + Bu, \tag{1}$$

where $A \in \mathbf{R}^{n\times n}$ and $B \in \mathbf{R}^{n\times m}$ are the system matrix and input matrix, respectively. Let the quadratic performance index function be given by

$$J = \frac{1}{2}\int_0^\infty \left[x^T(t)\,Qx(t) + u^T(t)\,Ru(t)\right]dt, \tag{2}$$

where R is a constant symmetric positive definite matrix and Q is a constant symmetric positive (or semipositive) definite matrix. The well-known linear quadratic optimal control is to design u such that J in (2) is minimized. The solution to this problem can be summarized as follows.

Theorem 1. *Assume that $[A, B]$ is controllable, $R > 0$, $Q \geq 0$ are given, and $[A\ \ Q]$ is observable. Then the solution to the* *infinite-time linear quadratic optimal control problem is given by*

$$u^* = Kx = -R^{-1}B^T P x, \tag{3}$$

where $P \in \mathbf{R}^{n\times n}$ *is unique positive definite solution to the following algebraic Riccati equation:*

$$A^T P + PA - PBR^{-1}B^T P + Q = 0. \tag{4}$$

Moreover, the resulting optimal closed-loop system

$$\dot{x} = \left(A - BR^{-1}B^T P\right)x, \quad x(0) = x_0, \tag{5}$$

is such that the optimal performance value is given by

$$J = \frac{1}{2}x_0^T P x_0. \tag{6}$$

By the following theorem, we know that the closed-loop system (5) is asymptotically stable.

Theorem 2 (see [1]). *Assume that the conditions of Theorem 1 are satisfied. Then, the closed-loop system* (5) *consisting of the linear system* (1) *and the linear state feedback* (3) *is asymptotically stable; namely, all the eigenvalues of the closed-loop system matrix* $A - BR^{-1}B^T P$ *have negative real parts.*

The feedback gain matrix $R^{-1}B^T P$ of the quadratic optimal controller can ensure that the closed-loop system has very good robustness properties. Moreover, for single-input linear systems, it has been proven that the closed-loop system has a phase margin of 60 degrees and a magnitude margin of $[1/2, \infty)$ at least. For general multiple-input systems, we can easily prove the following result.

Theorem 3. *The quadratic optimal controller* (3) *has at least the magnitude margin* $[1/2, \infty)$*; namely, by multiplying an arbitrary constant* λ $(\lambda \geq 1/2)$ *by the quadratic optimal control state feedback gain* $K = -R^{-1}B^T P$*, the resulting closed-loop system remains asymptotically stable.*

Proof. We only need to prove that, for any $\lambda \geq 1/2$, the closed-loop system

$$\dot{x} = (A + \lambda BK)x = \left(A - \lambda BR^{-1}B^T P\right)x \tag{7}$$

remains asymptotically stable. Choose a Lyapunov function as $V = x^T P x$. The time-derivative of V along the trajectory of closed-loop system (7) can be evaluated as

$$\begin{aligned}\dot{V} &= \dot{x}^T P x + x^T P \dot{x} \\ &= x^T\left(A - \lambda BR^{-1}B^T P\right)^T P x + x^T P\left(A - \lambda BR^{-1}B^T P\right)x \\ &= x^T\left(A^T P + PA - 2\lambda PBR^{-1}B^T P\right)x \\ &= x^T\left(-Q + PBR^{-1}B^T P - 2\lambda PBR^{-1}B^T P\right)x \\ &= x^T\left(-Q + (1-2\lambda)PBR^{-1}B^T P\right)x,\end{aligned} \tag{8}$$

where Q is positive definite, and, thus, $-Q < 0$. Notice that R is positive definite; then, $PBR^{-1}B^T P \geq 0$. Moreover, as $\lambda \geq 1/2$, then $(1-2\lambda)PBR^{-1}B^T P \leq 0$. Therefore, $\dot{V} < 0$. By the Lyapunov stability theorem, the closed-loop system (7) is asymptotically stable. The proof is complete. □

From the above theorem, we know that the quadratic optimal controller has a good robustness property; namely, after multiplying a factor $\gamma \in [1/2, \infty)$ by the feedback gain, the resulting closed-loop system remains asymptotically stable. Because of the common existence of parameters drift and delay effect in engineering systems (see, e.g., [9–13]), the feedback controllers designed by theoretical method are often subject to parameter perturbations after a period of running time. But if the perturbation satisfies certain conditions, the linear quadratic optimal controller can still guarantee the stability of the closed-loop system. So the linear quadratic optimal controller is very efficient in engineering applications. However, it is easy to prove by some counter-examples that controllers designed by ordinary pole placement technique have such good robustness properties. On the other hand, the transient performance of a finite dimensional linear system is completely determined by the locations of the eigenvalues of the closed-loop system. However, the poles of closed-loop system resulting from the quadratic optimal controller are still not clear; that is, if the weighting matrices Q and R are prescribed, the feedback gain matrix is uniquely determined, while the locations of the poles of the closed-loop systems can not be determined by specifying Q and R in advance.

Therefore, if we can combine the linear quadratic optimal control approach and the pole placement technique together to design the feedback gain, then such gain can not only place the poles of the closed-loop systems to the desired position, but also minimize certain quadratic performance index functions. Then, such a feedback gain matrix has the advantages of controller designed by both the pole placement approach and the linear quadratic optimal approach. This approach by combining the pole placement and the linear quadratic optimal control is named as optimal pole assignment. To make this problem clear, we state it as follows.

Problem 4. Let $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ be a prescribed set, where $\mathrm{Re}\{\lambda_i\} < 0$ and $\lambda \in \Lambda$ implies $\overline{\lambda} \in \Lambda$. Find a state feedback controller $u = Kx$ such that

$$\lambda(A + BK) = \Lambda \tag{9}$$

and the quadratic performance index (2) is minimized for some $Q > 0$ and $R > 0$.

This optimal pole assignment problem has been studied in the literature for many years (see [14–19] and the references therein). The existing solutions are basically to use the idea of inverse optimal control method, which should be solved in the frequency domain [15, 18, 19]. The alternative methods in the time domain are generally based on optimization, for example, the approach given in [16]. However, this method cannot guarantee the exact position of the poles of the closed-loop systems.

In this paper, we present a new method for the optimal pole assignment problem. The main idea is to use the relationship between the algebraic Riccati equation and the corresponding Hamiltonian matrix. Then, by choosing appropriate free parameters in the parametric solutions of a class of linear matrix equations, the resulting feedback gain matrix can simultaneously minimize some quadratic performance index and assign the poles of the closed-loop systems to the desired locations. The advantages of the proposed approach are summarized as follows.

(i) When the pole locations are prescribed, all the possible weighting matrixes Q in the quadratic performance index functions can be solved in theory. Without loss of generality, we have assumed that R is known as in all the other methods.

(ii) For a given Q, the corresponding positive definite solution P to the algebraic Riccati equation can be obtained.

(iii) Different from some other numerical methods, the proposed method can even give analytical solution P to the algebraic Riccati equation in some cases. Just remember that it is well known that the analytical solutions of the algebraic Riccati equations are generally not available.

2. Algebraic Riccati Equation and Hamiltonian Matrix

Consider the following algebraic Riccati equation:

$$A^T P + PA - PBR^{-1}B^T P + Q = 0, \tag{10}$$

where $A \in \mathbf{R}^{n\times n}$ and $B \in \mathbf{R}^{n\times m}$ are given system parameter matrix, $R \in \mathbf{R}^{m\times m}$ is positive definite, and $Q \in \mathbf{R}^{n\times n}$ is semipositive definite. Define a $2n \times 2n$ matrix as follows:

$$H = \begin{bmatrix} A & -BR^{-1}B^T \\ -Q & -A^T \end{bmatrix}, \tag{11}$$

which is called the Hamiltonian matrix of the algebraic Riccati equation (10).

Assume that λ_i, $i = 1, 2, \ldots, n$, are eigenvalues of the Hamiltonian matrix H and v_i are the corresponding eigenvectors. Denote the corresponding Jordan canonical form of H by J and define the $2n \times n$ matrix X as follows:

$$X = \begin{bmatrix} v_1 & v_2 & \ldots & v_n \end{bmatrix}. \tag{12}$$

Then, according to the above definitions, we get

$$HX = XJ. \tag{13}$$

Let matrixes X_1 and X_2 with dimensions $n \times n$ satisfy the following partition:

$$X = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}; \tag{14}$$

then, for the solution to the algebraic Riccati equation (10), we have the following conclusion whose proof can be found in [20].

Theorem 5. *Let P be the unique positive definite solution to the algebraic Riccati equation* (10). *Then, P can be expressed as $P = X_2 X_1^{-1}$.*

For the Hamiltonian matrix (11), we recall the following conclusion.

Lemma 6 (see [20]). *Eigenvalues of Hamiltonian matrix H are symmetric with respect to the origin; namely, if λ is the eigenvalue of H, then $-\lambda$, λ^*, and $-\lambda^*$ are all eigenvalues of H.*

3. Main Results

Our main result of this paper is stated as follows.

Theorem 7. *Assume that (A, B) is controllable. Then, Problem 4 is solvable if and only if there exist two matrices $X_1 \in \mathbf{R}^{n\times n}$ and $X_2 \in \mathbf{R}^{n\times n}$ such that the following inequality is satisfied:*

$$\begin{gathered} F^T X_2^T X_1 + X_1^T X_2 F + X_2^T BR^{-1}B^T X_2 < 0, \\ X_1^T X_2 = X_2^T X_1 > 0, \end{gathered} \tag{15}$$

where F is any matrix such that $\lambda(F) = \Lambda$ and (X_1, X_2) satisfies the following linear matrix equation:

$$AX_1 - X_1 F = BR^{-1}B^T X_2. \tag{16}$$

Moreover, if (X_1, X_2) is a feasible solution to the above two conditions, then the weighting matrices in the quadratic performance index (2) *can be chosen as*

$$\begin{gathered} Q = -A^T X_2 X_1^{-1} - X_2 F X_1^{-1}, \\ P = X_2 X_1^{-1}. \end{gathered} \tag{17}$$

Proof. Let

$$H = \begin{bmatrix} A & -BR^{-1}B^T \\ -Q & -A^T \end{bmatrix}, \tag{18}$$

whose set of arbitrary n eigenvalues is denoted as Λ_1 and the corresponding Jordan canonical form is denoted as J. Then, there must exist an eigenvector matrix X such that the following formula holds:

$$HX = XF. \tag{19}$$

Let X be partitioned according to (14). Then, the above formula can be rewritten as

$$\begin{bmatrix} A & -BR^{-1}B^T \\ -Q & -A^T \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} F. \tag{20}$$

Based on the relationship of the Hamiltonian matrix and the associated algebraic Riccati equation, we know that the algebraic Riccati equation

$$A^T P + PA - PBR^{-1}B^T P = -Q \tag{21}$$

has a positive definite solution if and only if X_1 is invertible, and, in this case, the solution can be expressed as

$$P = X_2 X_1^{-1}. \tag{22}$$

Let us prove that the following formula holds:

$$\sigma\left(A - BR^{-1}B^T P\right) = \sigma(F). \tag{23}$$

By expanding (20), we obtain two equations as follows:

$$\begin{gathered} AX_1 - BR^{-1}B^T X_2 = X_1 F, \\ -QX_1 - A^T X_2 = X_2 F. \end{gathered} \tag{24}$$

As X_1 is invertible, multiplying both sides of the first equation of (24) by X_1^{-1} on the right gives

$$AX_1 X_1^{-1} - BR^{-1}B^T X_2 X_1^{-1} = X_1 F X_1^{-1}. \tag{25}$$

By using (22), the above equation can be simplified as

$$A - BR^{-1}B^T P = X_1 F X_1^{-1}. \tag{26}$$

Therefore, we have $\sigma(A - BR^{-1}B^T P) = \sigma(F)$; namely, (23) is satisfied.

For the second equation of (24), we multiply both sides by the matrix X_1^T on the right hand side to give

$$X_1^T Q X_1 = -X_1^T A^T X_2 - X_1^T X_2 F. \tag{27}$$

For the first equation of (24), we obtain

$$AX_1 - X_1 F = BR^{-1}B^T X_2. \tag{28}$$

Substituting (28) into (27) and simplifying give

$$X_1^T Q X_1 = \left(-X_1 F - BR^{-1}B^T X_2\right)^T X_2 - X_1^T X_2 F. \tag{29}$$

Since X_1 is nonsingular, we know that Q is positive definite if and only if $X_1^T Q X_1$ is positive definite; namely,

$$F^T X_2^T X_1 + X_1^T X_2 F + X_2^T BR^{-1}B^T X_2 < 0, \tag{30}$$

which is just the first condition in (15). On the other hand, P is requested to be a symmetric positive-definite, which, based on (22), is equivalent to

$$P = X_2 X_1^{-1} = X_1^{-T} X_2^T > 0. \tag{31}$$

Multiplying the left hand side and the right hand side of the above equation by, respectively, X_1^T and X_1 gives

$$X_1^T X_2 = X_2^T X_1 > 0, \tag{32}$$

which is just the second condition in (15). Moreover, the first equation of (24) is exactly the one in (16). Since the above process is reversible, the proof is complete. □

From the above theorem, we know that, in order to solve the optimal pole assignment problem, the key step is to solve the linear matrix equation in the form of (16). If we set

$$Y = R^{-1}B^T X_2, \qquad X = X_1, \tag{33}$$

then the equation in (16) can be expressed as

$$AX - XF = BY, \tag{34}$$

which is generally referred to as the generalized Sylvester matrix equation and has many advanced applications in control theory [21], for example, constrained control (see, e.g., [22, 23]) and stabilization of time-delay systems (see, e.g., [24, 25]). Regarding the solutions to the linear matrix equation (34), we recall the following results from [26].

Theorem 8 (see [26]). *Let (A, B) be controllable and let $(N(s), D(s)) \in \mathbf{R}^{n\times m} \times \mathbf{R}^{m\times m}$ be a pair of right coprime polynomial matrices such that*

$$(sI_n - A)^{-1} B = N(s) D(s)^{-1}. \tag{35}$$

Let $(N(s), D(s))$ be expanded as

$$N(s) = \sum_{i=0}^{p} N_i s^i, \qquad D(s) = \sum_{i=0}^{p} D_i s^i, \tag{36}$$

where $N_i \in \mathbf{R}^{n\times m}$, $D_i \in \mathbf{R}^{m\times m}$, $i = 1, 2, \ldots, p$, are constant matrices. Then, the complete solutions to the linear matrix equation (34) can be expressed as

$$\begin{gathered} X = \sum_{i=0}^{p} N_i Z F^i, \\ Y = \sum_{i=0}^{p} D_i Z F^i, \end{gathered} \tag{37}$$

where $Z \in \mathbf{R}^{m\times m}$ is an arbitrary parameter matrix.

For (33), we let $X_1 = X$. Furthermore, in order to obtain X_2, we need to solve the following linear matrix equation:

$$Y = R^{-1}B^T X_2, \tag{38}$$

where $R \in \mathbf{R}^{m\times m}$ and $B \in \mathbf{R}^{m\times m}$. As $(R^{-1}B^T)^{-1}$ does not exist in general, even when R is nonsingular, the solution to (38) is not unique. In order to obtain all solutions, we introduce the following lemma.

Lemma 9 (see [27]). *The matrix A^- is called a generalized inverse matrix of A if it satisfies $AA^-A = A$. If A is a full row rank matrix, then the general solutions to the matrix equation $AX = B$ can be expressed as*

$$X = A^- B + \left(I - A^- A\right) U, \tag{39}$$

where $U \in \mathbf{R}^{n\times n}$ is an arbitrary matrix.

Since we can assume without loss of generality that B is a full column rank, the condition of the above lemma is satisfied. Thus, we get

$$X_2 = \left(R^{-1}B^T\right)^{-} Y + \left(I - \left(R^{-1}B^T\right)^{-} R^{-1}B^T U\right), \quad (40)$$

where $U \in \mathbf{R}^{n\times n}$ is an arbitrary matrix, and

$$\left(R^{-1}B^T\right)^{-} = \left(R^{-1}B^T\right)^T \left(\left(R^{-1}B^T\right)\left(R^{-1}B^T\right)^T\right)^{-1}. \quad (41)$$

4. A Numerical Example

We consider a linear time-invariant system characterized by (1) in which A, B, and the real-valued Jordan canonical form F associated with the desired eigenvalue set $\Lambda = \{-1, -1 \pm i\}$ are given by

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix},$$
$$F = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & -1 & -1 \end{bmatrix}. \quad (42)$$

Solving the right coprime factorizations of $(sI - A)^{-1}B$ satisfying (35) gives $D(s)$ and $N(s)$ as follows:

$$N(s) = \begin{bmatrix} s & 0 \\ 1 & 0 \\ 0 & -1 \end{bmatrix}, \qquad D(s) = \begin{bmatrix} s^2 & 1 \\ 0 & -s \end{bmatrix}. \quad (43)$$

Then, all the solution to the linear matrix equation (16) can be expressed by

$$Y = \begin{bmatrix} y_{21} + z_{11} & y_{22} + 2z_{13} & y_{23} - 2z_{12} \\ y_{21} & y_{22} + y_{23} & -y_{22} + y_{23} \end{bmatrix},$$
$$X = \begin{bmatrix} z_{11} & z_{12} + z_{13} & -z_{12} + z_{13} \\ -z_{11} & -z_{12} & -z_{13} \\ y_{21} & y_{22} & y_{23} \end{bmatrix}, \quad (44)$$

where the parametric matrix Z,

$$Z = \begin{bmatrix} z_{11} & z_{12} & z_{13} \\ y_{21} & y_{22} & y_{23} \end{bmatrix}, \quad (45)$$

can be chosen arbitrarily. Let $X_1 = X$. Thus, based on Lemma 9 and (40), we can obtain

$$X_2 = \begin{bmatrix} y_{21} + z_{11} & y_{22} + 2z_{13} & y_{23} - 2z_{12} \\ u_{21} & u_{22} & u_{23} \\ y_{21} & y_{22} + y_{23} & -y_{22} + y_{23} \end{bmatrix}, \quad (46)$$

where u_{ij} are arbitrary scalars. To ensure that P is symmetric, we solve the following equation:

$$X_1^T X_2 = X_2^T X_1, \quad (47)$$

to get the following three equations:

$$\begin{aligned} y_{21} &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}, \qquad \left(b^2 - 4ac \geq 0\right), \\ y_{22} &= \Big(\left(z_{11}u_{22} - z_{11}z_{13} - y_{23}y_{21} + z_{12}y_{21} \right.\\ &\qquad \left. + z_{12}z_{11} + z_{13}y_{21} - z_{12}u_{21}\right) \\ &\qquad \times \left(z_{11}\right)^{-1} \Big), \\ y_{23} &= \Big(\left(z_{11}^2 z_{13} + z_{12}z_{11}^2 + z_{11}^2 u_{23} + y_{21}z_{11}u_{22} + z_{12}y_{21}^2 \right.\\ &\qquad \left. + z_{13}y_{21}^2 - y_{21}z_{12}u_{21} - z_{13}u_{21}z_{11}\right) \\ &\qquad \times \left(z_{11}^2 + y_{21}^2\right)^{-1} \Big), \end{aligned} \quad (48)$$

where a, b, and c are, respectively, given by

$$\begin{aligned} a &= 2z_{13}^2 + z_{12}u_{23} + 2z_{12}^2 - z_{13}u_{22}, \\ b &= -z_{12}z_{11}u_{23} + z_{13}z_{11}u_{22} + z_{11}u_{23}z_{13} - z_{12}^2 u_{21} \\ &\quad - z_{13}^2 u_{21} + z_{11}u_{22}z_{12}, \\ c &= z_{11}^2 z_{13}u_{23} + z_{11}^2 u_{23}^2 + 2z_{11}^2 z_{13}^2 + z_{12}z_{11}^2 u_{22} + z_{11}^2 u_{22}^2 \\ &\quad - 2z_{11}u_{23}z_{13}u_{21} - 2z_{11}^2 z_{13}u_{22} + 2z_{12}z_{11}^2 u_{23} \\ &\quad - z_{12}^2 u_{21}z_{11} + z_{12}^2 u_{21}^2 - 2z_{11}u_{22}z_{12}u_{21} + z_{13}^2 u_{21}^2 \\ &\quad - z_{13}^2 u_{21}z_{11} + 2z_{12}^2 z_{11}^2. \end{aligned} \quad (49)$$

These three equations in (48) show that the actual free parameters that can be arbitrarily chosen are

$$f = \begin{bmatrix} z_{11} & z_{12} & z_{13} & u_{21} & u_{22} & u_{23} \end{bmatrix}. \quad (50)$$

By searching on the interval $[-2, 2]^6$, we can get 18 groups of solutions if the step size is chosen as 1.0, one of which is given by

$$f = \begin{bmatrix} 1 & -1 & -1 & -1 & -1 & 2 \end{bmatrix}. \quad (51)$$

By substituting the above parameters into (X_1, X_2) and using (17), a set of symmetric positive definite solutions to the algebraic Riccati equation can be obtained as follows:

$$P = \begin{bmatrix} \frac{9}{5} & \frac{8}{5} & \frac{3}{5} \\ \frac{8}{5} & \frac{11}{5} & \frac{1}{5} \\ \frac{3}{5} & \frac{1}{5} & \frac{6}{5} \end{bmatrix} > 0,$$
$$Q = \begin{bmatrix} \frac{2}{5} & \frac{4}{5} & -\frac{1}{5} \\ \frac{4}{5} & \frac{13}{5} & -\frac{2}{5} \\ -\frac{1}{5} & -\frac{2}{5} & \frac{3}{5} \end{bmatrix} > 0. \quad (52)$$

Moreover, the corresponding state feedback matrix is given by

$$K = -B^T P = -\begin{bmatrix} \frac{9}{5} & \frac{8}{5} & \frac{3}{5} \\ \frac{3}{5} & \frac{1}{5} & \frac{6}{5} \end{bmatrix}. \tag{53}$$

A large number of numerical calculations show that there are many solutions of P and Q satisfying the conditions of Theorem 7. This extra design freedom can also be optimized to achieve some other control objectives, for example, to minimize the norm of the feedback matrix K. To this end, by searching on the interval $[-4, 4]^6$ with the step size as 0.1, a set of optimal solutions can be obtained as follows:

$$\begin{gathered} Q_{\text{opt}} = \begin{bmatrix} 0.1024 & 0.2285 & -0.2941 \\ 0.2285 & 2.401 & -0.7456 \\ -0.2941 & -0.7456 & 0.8976 \end{bmatrix}, \\ P_{\text{opt}} = \begin{bmatrix} 1.721 & 1.549 & 0.4880 \\ 1.549 & 2.456 & 0.03708 \\ 0.4880 & 0.03708 & 1.279 \end{bmatrix}, \\ K_{\text{opt}} = \begin{bmatrix} 1.721 & 1.549 & 0.4880 \\ 0.4880 & 0.03708 & 1.279 \end{bmatrix}. \end{gathered} \tag{54}$$

5. Conclusions

This paper has considered the problem of designing optimal feedback controllers for linear systems. Based on properties of the algebraic Riccati equation and the corresponding Hamiltonian matrix, the optimal pole assignment problem is transformed into the problem of solving a kind of linear matrix equations with nonlinear inequality constraints. By choosing appropriate free parameters in the parametric solutions to this class of matrix equations, the solutions to the original optimal pole assignment problem can be obtained. Since the resulting optimal feedback control law can guarantee both good transient performance and robustness properties for the closed-loop system, it is expected that the proposed approach will find important applications in the engineering practice.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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