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Research Article

Signless Laplacian Spectral Conditions for Hamiltonicity of Graphs

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We establish some signless Laplacian spectral radius conditions for a graph to be Hamiltonian or traceable or Hamilton-connected.

1. Introduction

Let a graph, $G = (V, E)$ be a simple graph of order n with vertex set $V = \{v_1, v_2, \ldots, v_n\}$ and edge set E. Denote by $e(G) := |E|$ the number of edges of the graph G. Write by K_n a complete graph of order n, O_n an empty graph of order n (without edges), and $K_{n,m}$ a complete bipartite graph with two parts having n, m vertices, respectively. The graph G is said to be Hamiltonian, if it has a Hamiltonian cycle which is a cycle of order n contained in G. The graph G is said to be traceable if it has a Hamiltonian path which is a path of order n contained in G. The problem of deciding whether a graph is Hamiltonian is Hamiltonian problem, which is one of the most difficult classical problems in graph theory. Indeed, it is NP-complete problem.

The adjacency matrix of G is defined to be a matrix $A(G) = [a_{ij}]_{n\times n}$, where $a_{ij} = 1$ if v_i is adjacent to v_j and $a_{ij} = 0$ otherwise. The largest eigenvalue of $A(G)$ is called to be the spectral radius of G, which is denoted by $\mu(G)$. The degree matrix of G is written by $D(G) = \mathrm{diag}(d_G(v_1), d_G(v_2), \ldots, d_G(v_n))$, where $d_G(v_i)$ $(i = 1, 2, \ldots, n)$ denotes the degree of the vertex v_i in the graph G. The signless Laplacian matrix of G is defined by $Q(G) = D(G) + A(G)$. The largest eigenvalue of $Q(G)$ is called to be the signless Laplacian spectral radius of G, which is denoted by $q(G)$.

Recently, using spectral graph theory to study the Hamiltonian problem has received a lot of attention. Some spectral conditions for a graph to be Hamiltonian or traceable have been given in [1–6]. In this paper, we still study the Hamiltonicity of a graph. Firstly, we present a signless Laplacian spectral radius condition for a bipartite graph to be Hamiltonian in Section 2. Secondly, we give some signless Laplacian spectral radius conditions for a graph to be traceable or Hamilton-connected in Section 3 and Section 4, respectively.

2. Signless Laplacian Spectral Radius in Hamiltonian Bipartite Graphs

The definition of the closure of a balanced bipartite graph can be found in [7, 8]. For a positive integer k, the *k-closure of* a balanced bipartite graph $G_{BPT} := (X, Y; E)$, where $|X| = |Y|$, written by $\mathscr{C}_k(G_{BPT})$, is a graph obtained from G_{BPT} by successively joining pairs of nonadjacent vertices $x \in X$ and $y \in Y$, whose degree sum is at least k, until no such pairs remain. By the definition of the $\mathscr{C}_k(G_{BPT})$, we have that $d_{\mathscr{C}_k(G_{BPT})}(x) + d_{\mathscr{C}_k(G_{BPT})}(y) \le k-1$ for any pair of nonadjacent vertices $x \in X$ and $y \in Y$ of $\mathscr{C}_k(G_{BPT})$.

Lemma 1 (see [9]). *Let $G_{BPT} = (X, Y; E)$ be a connected balanced bipartite graph, where $|X| = |Y| = r \ge 2$. Then, G_{BPT} is Hamiltonian if and only if $\mathscr{C}_{r+1}(G_{BPT})$ is Hamiltonian.*

For a graph G, write $Z(G) := \sum_{uv\in E(G)}(d_G(u) + d_G(v)) = \sum_{u\in V(G)} d_G^2(u)$, and let $\Delta(G)$ be maximum degree of G. A regular graph is a graph for which every vertex in the graph has the same degree. A semi-regular graph is a bipartite graph for which every vertex in the same partite set has the same degree.

Lemma 2 (see [2]). *Let G be a graph with at least one edge. Then,*

$$q(G) \geq \frac{Z(G)}{e(G)}, \tag{1}$$

if and only if G is regular or semi-regular.

Let M be a Hermitian matrix of order n, and let $\lambda_1(M) \geq \lambda_2(M) \geq \cdots \geq \lambda_n(M)$ be the eigenvalues of M.

Lemma 3 (see [10]). *Let B and C be Hermitian matrices of order n, $1 \leq i, j \leq n$. Then,*

$$\lambda_i(B) + \lambda_j(C) \geq \lambda_{i+j-1}(B+C), \tag{2}$$

if $i + j \leq n + 1$.

Lemma 4. *Let G be a graph. Then,*

$$q(G) \leq \mu(G) + \Delta(G). \tag{3}$$

Proof. Because $Q(G) = A(G) + D(G)$, by Lemma 3,

$$\lambda_1(A(G)) + \lambda_1(D(G)) \geq \lambda_1(Q(G)). \tag{4}$$

We notice that $\lambda_1(A(G)) = \mu(G)$, $\lambda_1(D(G)) = \Delta(G)$, and $\lambda_1(Q(G)) = q(G)$. So, the result follows. □

Let $G_{BPT} = (X, Y; E)$ be a bipartite graph, the quasi-complement of G_{BPT} is denoted by $G^*_{BPT} := (X, Y; E')$, where $E' = \{xy : x \in X, y \in Y, xy \notin E\}$.

Theorem 5. *Let $G_{BPT} = (X, Y; E)$ be a connected balanced bipartite graph, where $|X| = |Y| = r \geq 2$. If*

$$q(G^*_{BPT}) < r, \tag{5}$$

then G_{BPT} is Hamiltonian.

Proof. Suppose that G_{BPT} is not Hamiltonian. Then, $H_{BPT} := \mathscr{C}_{r+1}(G_{BPT})$ is not Hamiltonian too by Lemma 1, and therefore, H_{BPT} is not $K_{r,r}$. Thus, there exists a vertex $x \in X$ and a vertex $y \in Y$ such that $xy \notin E(H_{BPT})$. We find that $d_{H_{BPT}}(x) + d_{H_{BPT}}(y) \leq r$ for any pair of nonadjacent vertices $x \in X$ and $y \in Y$ in H_{BPT}. So,

$$d_{H^*_{BPT}}(x) + d_{H^*_{BPT}}(y) = r - d_{H_{BPT}}(x) + r - d_{H_{BPT}}(y) \geq r, \tag{6}$$

for any pair of adjacent vertices $x \in X$, $y \in Y$ in H^*_{BPT}. Hence,

$$Z(H^*_{BPT}) = \sum_{xy \in E(H^*_{BPT})} \left(d_{H^*_{BPT}}(x) + d_{H^*_{BPT}}(y)\right) \geq re(H^*_{BPT}). \tag{7}$$

By Lemma 2, we have that

$$q(H^*_{BPT}) \geq \frac{Z(H^*_{BPT})}{e(H^*_{BPT})} \geq r. \tag{8}$$

As H^*_{BPT} is a subgraph of G^*_{BPT}, by Perron-Frobenius theorem,

$$q(G^*_{BPT}) \geq q(H^*_{BPT}). \tag{9}$$

Thus, by (5), (8), and (9), we have that

$$r > q(G^*_{BPT}) \geq q(H^*_{BPT}) \geq \frac{Z(H^*_{BPT})}{e(H^*_{BPT})} \geq r, \tag{10}$$

a contradiction. □

Li [4] has given a sufficient condition for a bipartite graph to be Hamiltonian as follows.

Theorem 6 (see [4]). *Let $G_{BPT} = (X, Y; E)$ be a connected balanced bipartite graph, where $|X| = |Y| = r \geq 2$. If*

$$\mu(G^*_{BPT}) \leq \sqrt{\frac{r-2}{2}}, \tag{11}$$

then G_{BPT} is Hamiltonian.

Remark 7. We now compare Theorems 5 and 6. If $\mu(G^*_{BPT}) \leq \sqrt{(r-2)/2}$ and $\Delta(G^*_{BPT}) < r - \sqrt{(r-2)/2}$, we have that $q(G^*_{BPT}) < r$ by Lemma 4. Hence Theorem 5 improves Theorem 6 when $\Delta(G^*_{BPT}) < r - \sqrt{(r-2)/2}$. For example, let G_{BPT} be a regular connected balanced bipartite graph with degree $(r+1)/2$, where r is odd and $|X| = |Y| = r \geq 6$. Then, its quasi-complement G^*_{BPT} is a regular graph with degrees $(r-1)/2$, $\mu(G^*_{BPT}) = (r-1)/2$, and $q(G^*_{BPT}) = r - 1$. G_{BPT} satisfies the condition of Theorems 5, and hence, it is Hamiltonian. But it does not satisfy the condition of Theorem 6.

3. Signless Laplacian Spectral Radius in Traceable Graphs

Write $K_{n-1} + v$ for K_{n-1} together with an isolated vertex. Let $G = (V(G), E(G))$ and $H = (V(H), E(H))$ be two disjoint graphs. The disjoint union of G and H, denoted by $G \cup H$, is the graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$. If $G_1 \cong \cdots \cong G_k$, we write kG_1 for $G_1 \cup \cdots \cup G_k$. The join of G and H, denoted by $G \vee H$, is the graph obtained from $G \cup H$ by adding edges joining every vertex of G to every vertex of H.

Lemma 8 (see [3]). *Let G be a connected graph of order $n \geq 4$. If*

$$e(G) \geq \frac{(n-2)(n-3)}{2} + 2, \tag{12}$$

then G is traceable unless $G \cong K_1 \vee (K_{n-3} \cup 2K_1)$, $K_2 \vee (3K_1 \cup K_2)$, or $K_4 \vee (6K_1)$.

Let G be a graph containing a vertex v. Denote $m_G(v) = m(v) = (1/d_G(v)) \sum_{u \in N_G(v)} d_G(u)$ if $d_G(v) > 0$ and $m_G(v) = 0$ otherwise, where $N_G(v)$ or simply $N(v)$ denotes the neighborhood of v in G.

Lemma 9 (see [11]). *Let G be a graph of order n. Then,*

$$\max \{d_G(v) + m_G(v) : v \in V(G)\} \leq \frac{2e(G)}{n-1} + n - 2, \quad (13)$$

with equality if and only if $G \supseteq K_{1,n-1}$ or $G = K_{n-1} + v$.

Lemma 10 (see [12]). *Let G be a connected graph. Then*

$$q(G) \leq \max \{d_G(v) + m_G(v) : v \in V(G)\}, \quad (14)$$

with equality if and only if G is a regular graph or a semi-regular graph.

In fact, if G is disconnected, there exists a component H of G such that

$$\begin{aligned} q(G) = q(H) &\leq \max \{d_H(v) + m_H(v) : v \in V(H)\} \\ &\leq \max \{d_G(v) + m_G(v) : v \in V(G)\}. \end{aligned} \quad (15)$$

So the inequality (14) also holds when G is a disconnected graph. By Lemmas 9 and 10, we have the following result; also see [13].

Corollary 11. *Let G be a graph of order n. Then,*

$$q(G) \leq \frac{2e(G)}{n-1} + n - 2. \quad (16)$$

If G is connected, then the equality in (16) *holds if and only if $G = K_{1,n-1}$ or $G = K_n$. Otherwise, the equality in* (16) *holds if and only if $G = K_{n-1} + v$.*

Given a graph G of order n, a vector $X \in \mathbb{R}^n$ is called a function defined on G, if there is a 1-1 map φ from $V(G)$ to the entries of X, simply written as $X_u = \varphi(u)$ for each $u \in V(G)$, X_u is also called the value of u given by X. If X is an eigenvector of $Q(G)$ corresponding to the eigenvalue q, then X is defined naturally on G; that is, X_u is the entry of X corresponding to the vertex u. One can find that

$$[q - d_G(v)] X_v = \sum_{u \in N_G(v)} X_u, \quad \text{for each } v \in V(G), \quad (17)$$

where $N_G(v)$ denotes the neighborhood of v in G. The equation (17) is called (q, X)-eigenequation of G.

Theorem 12. *Let G be a connected graph of order $n \geq 4$. If*

$$q(G) \geq \frac{2(n-2)^2 + 4}{n-1}, \quad (18)$$

then G is traceable.

Proof. By Corollary 11 and (18), we have

$$e(G) \geq \frac{(n-1)q(G) - (n-1)(n-2)}{2} \geq \frac{(n-2)(n-3)}{2} + 2. \quad (19)$$

Suppose that G is non-traceable. Then, by Lemma 8 and (19), $G \cong K_1 \vee (K_{n-3} \cup 2K_1)$, $K_2 \vee (3K_1 \cup K_2)$, or $K_4 \vee (6K_1)$.

If $G \cong K_1 \vee (K_{n-3} \cup 2K_1)$, let $X = (X_1, X_2, \ldots, X_n)^T$ be the eigenvector of $Q(G)$ corresponding to eigenvalue $q(G)$. By (18), we know that $q(G) \neq 1, n-4$. Thus, by (17), all vertices of degree 1 have the same values given by X, say X_1; all vertices of degree $n-3$ have the same values by X, say X_2. Denote by X_3 the value of the vertex of degree $n-1$ given by X. Also, by (17), we have

$$\begin{aligned} (q(G) - 1) X_1 &= X_3, \\ (q(G) - (n-3)) X_2 &= (n-4) X_2 + X_3, \\ (q(G) - (n-1)) X_3 &= 2X_1 + (n-3) X_2. \end{aligned} \quad (20)$$

Transform (20) into a matrix equation $(B - q(G)\mathbf{I})X' = 0$, where $X' = (X_1, X_2, X_3)^T$ and

$$B = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2n-7 & 1 \\ 2 & n-3 & n-1 \end{bmatrix}. \quad (21)$$

Thus, $q(G)$ is the largest root of the following equation:

$$q^3 + (-3n+7) q^2 + (2n^2 - 7n) q - 2n^2 + 14n - 24 = 0. \quad (22)$$

Let $f(x) = x^3 + (-3n+7)x^2 + (2n^2 - 7n)x - 2n^2 + 14n - 24$; then $f'(x) = 3x^2 + 2(-3n+7)x + 2n^2 - 7n$. Let $f'(x) = 0$; we have two values x_1 and x_2, such that $f'(x_1) = f'(x_2) = 0$, where

$$\begin{aligned} x_1 &= \frac{3n - 7 - \sqrt{3n^2 - 21n + 49}}{3}, \\ x_2 &= \frac{3n - 7 + \sqrt{3n^2 - 21n + 49}}{3}. \end{aligned} \quad (23)$$

Hence, $f(x)$ is strictly increasing with respect to x for $x > x_2$.

Because $f(2(n-3)) = 2n^2 - 17n + 33 > 0$ and $(2(n-2)^2 + 4)/(n-1) > 2(n-3) > x_2$, we have that $f((2(n-2)^2 + 4)/(n-1)) > 0$, which implies that $q(G) < (2(n-2)^2 + 4)/(n-1)$.

If $G \cong K_2 \vee (3K_1 \cup K_2)$, let $X = (X_1, X_2, \ldots, X_7)^T$ be the eigenvector of $Q(G)$ corresponding to eigenvalue $q(G)$. By (18), we know that $q(G) \neq 2, 5$. Thus, by (17), three vertices of degree 2 have the same values given by X, say X_1; two vertices of degree 3 have the same values, say X_2; two vertices of degree 6 have the same values, say X_3. Also, by (17), we have

$$\begin{aligned} (q(G) - 2) X_1 &= 2X_3, \\ (q(G) - 3) X_2 &= X_2 + 2X_3, \\ (q(G) - 6) X_3 &= 3X_1 + 2X_2 + X_3. \end{aligned} \quad (24)$$

Transform (24) into a matrix equation $(B - q(G)\mathbf{I})X' = 0$, where $X' = (X_1, X_2, X_3)^T$ and

$$B = \begin{bmatrix} 2 & 0 & 2 \\ 0 & 4 & 2 \\ 3 & 2 & 7 \end{bmatrix}. \quad (25)$$

Thus, $q(G)$ is the largest root of the following equation:

$$q^3 - 13q^2 + 40q - 24 = 0. \tag{26}$$

Let $g(x) = x^3 - 13x^2 + 40x - 24$; we can easily get that $g(x)$ is strictly increasing with respect to x for $x > 20/3$.

Consider $g(9) = 12 > 0$, which implies that $q(G) < 9$.

If $G \cong K_4 \vee (6K_1)$, we easily calculate $q(G) = 8 + 2\sqrt{10} < 44/3$.

Thus, in either case, we have a contradiction. □

Lu et al. [3] have given a sufficient condition for a graph to be traceable as follows.

Theorem 13 (see [3]). *Let G be a connected graph of order $n \geq 5$. If*

$$\mu(G) \geq \sqrt{(n-3)^2 + 2}, \tag{27}$$

then G is traceable.

Example 14. There are graphs to which Theorem 12 may apply but Theorem 13 may not. Let $G = (K_r \cup K_r) \vee K_1$ of order $n := 2r + 1$, where $r \geq 4$. Surely, the graph G is traceable. By a little computation, $\mu(G)$ is the largest root of the polynomial $f(x) = x[x - (r-1)] - 2r$ and $q(G)$ is the largest root of the polynomial $g(x) = [x - (2r-1)](x - 2r) - 2r$. Hence,

$$\begin{aligned}\mu(G) &= \frac{r + 1 + \sqrt{r^2 + 6r + 1}}{2} < \sqrt{4r^2 - 8r + 6} \\ &= \sqrt{(n-3)^2 + 2},\end{aligned} \tag{28}$$

$$q(G) = 4r - \frac{1}{4} > \frac{(2r-1)^2 + 2}{r} = \frac{2(n-2)^2 + 4}{n-1}.$$

So, we can apply Theorem 12 but not Theorem 13 for G to be traceable.

4. Signless Laplacian Spectral Radius in Hamilton-Connected Graphs

For a graph G of order n, Erdös and Gallai [14] prove that if

$$d_G(u) + d_G(v) \geq n + 1, \tag{29}$$

for any pair of nonadjacent vertices u and v, then G is Hamilton-connected.

The idea for the closure of a graph can be found in [7]. For a positive integer k, the k-closure of a graph $G = (V, E)$, denoted by $\mathscr{C}_k(G)$, is a graph obtained from G by successively joining pairs of nonadjacent vertices $u \in V$ and $v \in V$, whose degree sum is at least k until no such pairs remain. By the definition of the k-closure of G, we have that $d_{\mathscr{C}_k(G)}(u) + d_{\mathscr{C}_k(G)}(v) \leq k - 1$ for any pair of nonadjacent vertices $u \in V$ and $v \in V$ of $\mathscr{C}_k(G)$.

Lemma 15 (see [7]). *Let G be a graph of order n. Then, G is Hamilton-connected if and only if $\mathscr{C}_{n+1}(G)$ is Hamilton-connected.*

Lemma 16. *Let G be a simple graph with degree sequence $(d_G(v_1), d_G(v_2), \ldots, d_G(v_n))$, where $d_G(v_1) \leq d_G(v_2) \leq \cdots \leq d_G(v_n)$ and $n \geq 3$. Suppose that there is no integer $k \leq n/2$ such that $d_G(v_{k-1}) \leq k$ and $d_G(v_{n-k}) \leq n - k$. Then, G is Hamilton-connected.*

Proof. Let $\overline{H} = \mathscr{C}_{n+1}(G)$ be the $(n+1)$-closure of G. Next, we will prove that $\overline{H}$ is a complete graph; then the result follows according to (29). To the contrary, suppose that $\overline{H}$ is not a complete graph, and let u and v be two nonadjacent vertices in $\overline{H}$ with

$$d_{\overline{H}}(u) \leq d_{\overline{H}}(v) \tag{30}$$

and $d_{\overline{H}}(u) + d_{\overline{H}}(v)$ being as large as possible. By the definition of $\mathscr{C}_{n+1}(G)$, we have

$$d_{\overline{H}}(u) + d_{\overline{H}}(v) \leq n. \tag{31}$$

Denote by S the set of vertices in $V \backslash \{v\}$ which are nonadjacent to v in $\overline{H}$. Denote by T the set of vertices in $V \setminus \{u\}$ which are nonadjacent to u in $\overline{H}$. Then,

$$|S| = n - 1 - d_{\overline{H}}(v), \qquad |T| = n - 1 - d_{\overline{H}}(u). \tag{32}$$

Furthermore, by $d_{\overline{H}}(u) + d_{\overline{H}}(v)$ being as large as possible, each vertex in S has degree at most $d_{\overline{H}}(u)$ and each vertex in $T \cup \{u\}$ has degree at most $d_{\overline{H}}(v)$. Let $k := d_{\overline{H}}(u)$. According to (31) and (32), we have that $|S| = n - 1 - d_{\overline{H}}(v) \geq d_{\overline{H}}(u) - 1 = k - 1$, $|T| + 1 = n - 1 - d_{\overline{H}}(u) + 1 = n - d_{\overline{H}}(u) = n - k$. Then $\overline{H}$ has at least $k - 1$ vertices of degree not exceeding k and at least $n - k$ vertices of degree not exceeding $n - k$. Because G is a spanning subgraph of $\overline{H}$, the same is true for G; that is, $d_G(v_{k-1}) \leq k$ and $d_G(v_{n-k}) \leq n - k$. Because $k \leq n/2$ by (30) and (31), this is contrary to the hypothesis. So we have that the $(n+1)$-closure $\overline{H}$ of G is indeed complete graph and hence that G is Hamilton-connected by (29). □

We write $K_{n-1} + e + e'$ for K_{n-1} together with a vertex joining two vertices of K_{n-1} by edges e, e', respectively.

Lemma 17. *Let G be a connected graph of order $n \geq 6$. If*

$$e(G) \geq \frac{(n-1)(n-2)}{2} + 2, \tag{33}$$

then G is Hamilton-connected unless $G \cong K_{n-1} + e + e'$ or $G \cong O_3 \vee K_3$.

Proof. Suppose that G is not a Hamilton-connected graph with degree sequence $(d_G(v_1), d_G(v_2), \ldots, d_G(v_n))$, where $d_G(v_1) \leq d_G(v_2) \leq \cdots \leq d_G(v_n)$ and $n \geq 6$. By Lemma 16, there is integer $k \le n/2$ such that $d_G(v_{k-1}) \le k$ and $d_G(v_{n-k}) \le n-k$. Since G is connected, $k \ge 2$. Thus,

$$\begin{aligned} e(G) &= \frac{1}{2}\sum_{i=1}^{n} d_G(v_i) \\ &\le \frac{1}{2}\left[(k-1)k + (n-2k+1)(n-k) + k(n-1)\right] \\ &= \frac{1}{2}\left(n^2 - 2nk + 3k^2 - 3k + n\right) \\ &= \frac{8n^2-9}{24} + \frac{3}{2}\left(k - \frac{2n+3}{6}\right)^2. \end{aligned} \tag{34}$$

Because $2 \le k \le n/2$, $(9-2n)/6 \le k-(2n+3)/6 \le (n-3)/6$. Thus, if $n \ge 6$,

$$e(G) \le \frac{8n^2-9}{24} + \frac{3}{2}\left(k - \frac{2n+3}{6}\right)^2 \le \frac{(n-1)(n-2)}{2} + 2. \tag{35}$$

Since $e(G) \ge (n-1)(n-2)/2 + 2$, then all inequalities in the above argument should be equalities. From the last equality in (35), we have $k = 2$ or $k = 3$ and $n = 6$. If $k = 2$, by the equality in (34), G is a graph with $d_G(v_1) = 2, d_G(v_2) = d_G(v_3) = \cdots = d_G(v_{n-2}) = n-2, d_G(v_{n-1}) = d_G(v_n) = n-1$, which implies $G \cong K_{n-1} + e + e'$. If $k = 3$ and $n = 6$, by the equality in (34), G is a graph with $d_G(v_1) = d_G(v_2) = d_G(v_3) = 3, d_G(v_4) = d_G(v_5) = d_G(v_6) = 5$, which implies $G \cong O_3 \vee K_3$. □

Theorem 18. *Let G be a connected graph of order $n \ge 6$. If*

$$q(G) \ge 2(n-2) + \frac{4}{n-1}, \tag{36}$$

then G is Hamilton-connected.

Proof. By Corollary 11 and (36), we have

$$e(G) \ge \frac{q(G)(n-1) - (n-1)(n-2)}{2} \ge \frac{(n-1)(n-2)}{2} + 2. \tag{37}$$

Suppose that G is not Hamilton-connected. Then, by Lemma 17 and (37), $G \cong K_{n-1} + e + e'$ or $G \cong O_3 \vee K_3$.

If $G \cong K_{n-1} + e + e'$. Let $X = (X_1, X_2, \ldots, X_n)^T$ be the eigenvector of $Q(G)$ corresponding to the eigenvalue $q(G)$. By (36), we know that $q(G) \ne n-3$ and $q(G) \ne n-2$. Thus, by (17), all vertices of degree $n-2$ have the same values given by X, say X_1, and all vertices of degree $n-1$ have the same values, say X_2. Denote by X_3 the value of the vertex of degree 2 given by X. Also, by (17), we have

$$\begin{gathered} (q(G) - (n-2))X_1 = (n-4)X_1 + 2X_2, \\ (q(G) - (n-1))X_2 = (n-3)X_1 + X_2 + X_3 \\ (q(G) - 2)X_3 = 2X_2. \end{gathered} \tag{38}$$

Transform (38) into a matrix equation $(B - q(G)\mathbf{I})X' = 0$, where $X' = (X_1, X_2, X_3)^T$ and

$$B = \begin{bmatrix} 2n-6 & 2 & 0 \\ n-3 & n & 1 \\ 0 & 2 & 2 \end{bmatrix}. \tag{39}$$

Thus, $q(G)$ is the largest root of following equation:

$$q^3 + (4-3n)q^2 + \left(2n^2 - 2n - 8\right)q - 4n^2 + 20n - 24 = 0. \tag{40}$$

Let $f(x) = x^3 + (4-3n)x^2 + (2n^2-2n-8)x - 4n^2 + 20n - 24$; then $f'(x) = 3x^2 + 2(4-3n)x + 2n^2 - 2n - 8$. Let $f'(x) = 0$; we have two values x_1 and x_2, such that $f'(x_1) = f'(x_2) = 0$, where

$$\begin{aligned} x_1 &= \frac{3n - 4 - \sqrt{3n^2 - 18n + 40}}{3}, \\ x_2 &= \frac{3n - 4 + \sqrt{3n^2 - 18n + 40}}{3}. \end{aligned} \tag{41}$$

Hence, $f(x)$ is strictly increasing with respect to x for $x > x_2$.

Consider $f(2(n-2) + 4/(n-1)) = 4(n-3)^2(n^2 - 3n + 6)/(n-1)^3 > 0$ and $2(n-2) + 4/(n-1) > x_2$, which implies that $q(G) < 2(n-2) + 4/(n-1)$.

If $G \cong O_3 \vee K_3$. We can calculate that $q(G) = 5 + \sqrt{13} < 8.8 = 2(6-2) + 4/(6-1)$. Thus, in either case, we have a contradiction. □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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