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Research Article

Convergence Theorems for Hierarchical Fixed Point Problems and Variational Inequalities

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This paper deals with a modified iterative projection method for approximating a solution of hierarchical fixed point problems for nearly nonexpansive mappings. Some strong convergence theorems for the proposed method are presented under certain approximate assumptions of mappings and parameters. As a special case, this projection method solves some quadratic minimization problem. It should be noted that the proposed method can be regarded as a generalized version of Wang and Xu (2013), Ceng et al. (2011), Sahu et al. (2012), and many other authors.

1. Introduction

Throughout this paper, C is a nonempty closed convex subset of a real Hilbert space H; $\langle \cdot, \cdot \rangle$ denotes the associated inner product; $\|\cdot\|$ stands for the corresponding norm. Let I be the identity mapping on C and P_C the metric projection of H onto C. To begin with, let us recall the following concepts which are commonly used in the context of convex and nonlinear analysis. For all $x, y \in C$, a mapping $F : C \to H$ is said to be monotone if $\langle Fx - Fy, x - y \rangle \geq 0$, η-strongly monotone if there exists a positive real number η such that $\langle Fx - Fy, x - y \rangle \geq \eta \|x - y\|^2$, and L-Lipschitzian if there exists a positive real number L such that $\|Fx - Fy\| \leq L\|x - y\|$. Let us also recall that a mapping $T : C \to C$ is said to be contraction if there exists a constant $k \in [0, 1)$ such that $\|Tx - Ty\| \leq k\|x - y\|$, nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$, and asymptotically nonexpansive if for each $n \geq 1$ there exists a positive constant $k_n \geq 1$ with $\lim_{n\to\infty} k_n = 1$ such that $\|T^n x - T^n y\| \leq k_n \|x - y\|$, for all $x, y \in C$.

As a generalization of asymptotically nonexpansive mappings, Sahu [1] introduced the class of nearly Lipschitzian mappings. Let us fix a sequence $\{a_n\}$ in $[0, \infty)$ with $a_n \to 0$. A mapping $T : C \to C$ is called nearly Lipschitzian with respect to $\{a_n\}$ if for each $n \geq 1$ there exists a constant $k_n \geq 0$ such that

$$\|T^n x - T^n y\| \leq k_n (\|x - y\| + a_n), \quad \forall x, y \in C. \tag{1}$$

The infimum constant k_n for which (1) holds will be denoted by $\eta(T^n)$ and called nearly Lipschitz constant. Notice that

$$\eta(T^n) = \sup \left\{ \frac{\|T^n x - T^n y\|}{\|x - y\| + a_n} : x, y \in C, x \neq y \right\}. \tag{2}$$

A nearly Lipschitzian mapping T with sequence $\{a_n, \eta(T^n)\}$ is said to be nearly nonexpansive if $\eta(T^n) \leq 1$ for $n \geq 1$. In this paper, we study a nearly nonexpansive mapping which is studied by some authors (see [1–4]). This type of mappings (not necessarily continuous) is important with regard to generalization of the asymptotically nonexpansive mappings. Particularly, the class of this type of mappings is an intermediate class between the class of asymptotically nonexpansive mappings and that of mappings of asymptotically nonexpansive type (please see [1] for nearly nonexpansive mappings examples).

Now, we focus on the hierarchical fixed point problem for a nearly nonexpansive mapping T with respect to a nonexpansive mapping S. This problem is to find a point $x^* \in \operatorname{Fix}(T)$ satisfying

$$\langle (I-S)x^*, x^* - x\rangle \le 0, \quad \forall x \in \operatorname{Fix}(T), \tag{3}$$

where $\operatorname{Fix}(T)$ is the set of fixed points of T; that is, $\operatorname{Fix}(T) = \{x \in C : Tx = x\}$. It is easy to see that the problem (3) is equivalent to the problem of finding a point $x^* \in C$ that satisfies $x^* = P_{\operatorname{Fix}(T)} S x^*$.

Let $N_{\operatorname{Fix}(T)}$ be the normal cone to $\operatorname{Fix}(T)$ defined by

$$N_{\operatorname{Fix}(T)}x = \begin{cases} \{u \in H : \langle y - x, u\rangle \le 0, \\ \quad \forall y \in \operatorname{Fix}(T)\}, & \text{if } x \in \operatorname{Fix}(T) \\ \emptyset, & \text{if } x \notin \operatorname{Fix}(T). \end{cases} \tag{4}$$

Then, the hierarchical fixed point problem is equivalent to the variational inclusion problem which is to find a point $x^* \in C$ such that

$$0 \in (I-S)x^* + N_{\operatorname{Fix}(T)}x^*. \tag{5}$$

The existence problem of hierarchical fixed points for a nonlinear mapping and approximation problem has been studied by several authors (see [5–12]). In 2006, Marino and Xu [13] introduced the following viscosity iterative method:

$$x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n A) T x_n, \quad \forall n \ge 0, \tag{6}$$

where f is a contraction, T is a nonexpansive mapping, and A is a strongly positive bounded linear operator on H; that is, $\langle Ax, x\rangle \ge \gamma\|x\|$, $\forall x \in H$ for some $\gamma > 0$. Under the appropriate conditions, they proved that the sequence $\{x_n\}$ defined by (6) converges strongly to the unique solution of the variational inequality

$$\langle (\gamma f - A)x^*, x - x^*\rangle \le 0, \quad \forall x \in C, \tag{7}$$

which is the optimality condition for the minimization problem

$$\min_{x\in C} \frac{1}{2}\langle Ax, x\rangle - h(x), \tag{8}$$

where h is a potential function for γf; that is, $h'(x) = \gamma f(x)$ for all $x \in H$.

In 2011, Ceng et al. [14] generalized the iterative method (6) of Marino and Xu [13] by taking a Lipschitzian mapping V and Lipschitzian and strongly monotone operator F instead of the mappings f and A, respectively. They gave the following iterative method:

$$x_{n+1} = P_C\left[\alpha_n \rho V x_n + (I - \alpha_n \mu F) T x_n\right], \quad \forall n \ge 0, \tag{9}$$

where P_C is a metric projection and T is a nonexpansive mapping, and they also proved that the sequence $\{x_n\}$ generated by (9) converges strongly to the unique solution of the variational inequality

$$\langle (\rho V - \mu F)x^*, x - x^*\rangle \le 0, \quad \forall x \in \operatorname{Fix}(T). \tag{10}$$

Recently, motivated by the iteration method (9) of Ceng et al. [14], Wang and Xu [15] studied the following iterative method for a hierarchical fixed point problem:

$$\begin{aligned} y_n &= \beta_n S x_n + (1-\beta_n) x_n, \\ x_{n+1} &= P_C\left[\alpha_n \rho V x_n + (I - \alpha_n \mu F) T y_n\right], \\ &\qquad \forall n \ge 0, \end{aligned} \tag{11}$$

where $S, T : C \to C$ are nonexpansive mappings, $V : C \to H$ is a γ-Lipschitzian mapping, and $F : C \to H$ is a L-Lipschitzian and η-strongly monotone operator. They proved that under some suitable assumptions on the sequences $\{\alpha_n\}$ and $\{\beta_n\}$, the sequence $\{x_n\}$ generated by (11) converges strongly to the hierarchical fixed point of T with respect to the mapping S which is the unique solution of the variational inequality (10). With this study, Wang and Xu extend and improve the many recent results of other authors.

Let $\{T_n\}$ be a sequence of mappings from C into H and fix a sequence $\{a_n\}$ in $[0, \infty)$ with $a_n \to 0$. Then, $\{T_n\}$ is called a sequence of nearly nonexpansive mappings [16] with respect to a sequence $\{a_n\}$ if

$$\|T_n x - T_n y\| \le \|x - y\| + a_n, \quad \forall x, y \in C,\ n \ge 1. \tag{12}$$

It is obvious that the sequence of nearly nonexpansive mappings is a wider class of sequence of nonexpansive mappings. For the sequence of nearly nonexpansive mappings defined by (12), Sahu et al. [3] introduced a new iteration method to solve the hierarchical fixed point problem and variational inequality problem.

Remark 1. Let $\{T_n\}$ be a sequence of nearly nonexpansive mappings. Then,

(1) for each $n \ge 1$, T_n is not a nearly nonexpansive mapping,

(2) if T is a mapping on C defined by $Tx = \lim_{n\to\infty} T_n x$ for all $x \in C$, then it is clear that T is a nonexpansive mapping.

Recently, in 2012, Sahu et al. [4] introduced the following iterative method for the sequence of nearly nonexpansive mappings $\{T_n\}$ defined by (12):

$$x_{n+1} = P_C\left[\alpha_n \rho V x_n + (1 - \alpha_n \mu F) T_n x_n\right], \quad \forall n \ge 1. \tag{13}$$

They proved that the sequence $\{x_n\}$ generated by (13) converges strongly to the unique solution of the variational inequality (10).

Remark 2. Since the mapping T_n is not a nearly nonexpansive mapping for each $n \ge 1$, if one takes $T_n := T$ for all $n \ge 1$ such that T is a nearly nonexpansive mapping, then the iteration (13) is not well defined. Hence, the main result of Sahu et al. [4] is no longer valid for a nearly nonexpansive mapping.

In this paper, motivated and inspired by the work of Wang and Xu [15], we introduce a modified iterative projection method to find a hierarchical fixed point of a nearly nonexpansive mapping with respect to a nonexpansive mapping. We show that our iterative method converges strongly to the unique solution of the variational inequality (10). As a special case, the presented projection method solves some quadratic minimization problem. Also, our method improves and generalizes corresponding results of Yao et al. [5], Marino and Xu [13], Ceng et al. [14], Wang and Xu [15], Moudafi [17], Xu [18], Tian [19], and Suzuki [20].

2. Preliminaries

This section contains some lemmas and definitions which will be used in the proof of our main result in the following section. We write $x_n \rightharpoonup x$ to indicate that the sequence $\{x_n\}$ converges weakly to x and $x_n \to x$ for the strong convergence. A mapping $P_C : H \to C$ is called a metric projection if there exists a unique nearest point in C denoted by $P_C x$ such that

$$\left\|x - P_C x\right\| = \inf_{y\in C}\left\|x - y\right\|, \quad \forall x \in H. \tag{14}$$

It is easy to see that P_C is a nonexpansive mapping and it satisfies the following inequality:

$$\left\langle x - P_C x, y - P_C x\right\rangle \le 0, \quad \forall x \in H,\ y \in C. \tag{15}$$

Now, we give the definitions of a demicontinuous mapping, asymptotic radius, and asymptotic center.

Let C be a nonempty subset of a Banach space X and $T : C \to C$ a mapping. T is called demicontinuous if $x_n \to x$ in X implies $Tx_n \rightharpoonup Tx$ in C.

Let C be a nonempty closed convex subset of a uniformly convex Banach space X, $\{x_n\}$ a bounded sequence in X, and $r : C \to [0,\infty)$ a functional defined by

$$r(x) = \limsup_{n\to\infty}\left\|x_n - x\right\|, \quad x \in C. \tag{16}$$

The infimum of $r(\cdot)$ over C is called asymptotic radius of $\{x_n\}$ with respect to C and is denoted by $r(C,\{x_n\})$. A point $\tilde{x} \in C$ is said to be an asymptotic center of the sequence $\{x_n\}$ with respect to C if

$$r(\tilde{x}) = \min\{r(x) : x \in C\}. \tag{17}$$

The set of all asymptotic centers is denoted by $A(C,\{x_n\})$. Related with these definitions, we will use the following in our main results.

Theorem 3 (see [21]). *Let C be a nonempty closed convex subset of a uniformly convex Banach space X satisfying the Opial condition. If $\{x_n\}$ is a sequence in C such that $x_n \rightharpoonup \tilde{x}$, then $\tilde{x}$ is the asymptotic center of $\{x_n\}$ in C.*

Lemma 4 (see [1]). *Let C be a nonempty closed convex subset of a uniformly convex Banach space X and $T : C \to C$ a demicontinuous nearly Lipschitzian mapping with sequence $\{a_n, \eta(T^n)\}$ such that $\lim_{n\to\infty}\eta(T^n) \le 1$. If $\{x_n\}$ is a bounded sequence in C such that*

$$\lim_{m\to\infty}\left(\lim_{n\to\infty}\left\|x_n - T^m x_n\right\|\right) = 0, \qquad A(C,\{x_n\}) = \{\tilde{x}\}, \tag{18}$$

then $\tilde{x}$ is a fixed point of T.

Lemma 5 (see [14]). *Let $V : C \to H$ be a γ-Lipschitzian mapping and let $F : C \to H$ be a L-Lipschitzian and η-strongly monotone operator; then, for $0 \le \rho\gamma < \mu\eta$,*

$$\begin{aligned}&\left\langle(\mu F - \rho V)x - (\mu F - \rho V)y, x - y\right\rangle\\ &\quad\ge(\mu\eta - \rho\gamma)\left\|x - y\right\|^2, \quad \forall x, y \in C.\end{aligned} \tag{19}$$

That is to say, the operator $\mu F - \rho V$ is $\mu\eta - \rho\gamma$-strongly monotone.

Lemma 6 (see [22]). *Let C be a nonempty subset of a real Hilbert space H. Suppose that $\lambda \in (0,1)$ and $\mu > 0$. Let $F : C \to H$ be a L-Lipschitzian and η-strongly monotone operator. Define the mapping $G : C \to H$ by*

$$Gx = x - \lambda\mu Fx, \quad \forall x \in C. \tag{20}$$

Then, G is a contraction that provided $\mu < 2\eta/L^2$. More precisely, for $\mu \in (0, 2\eta/L^2)$,

$$\left\|Gx - Gy\right\| \le (1 - \lambda\nu)\left\|x - y\right\|, \quad \forall x, y \in C, \tag{21}$$

where $\nu = 1 - \sqrt{1 - \mu(2\eta - \mu L^2)}$.

Lemma 7 (see [10]). *Assume that $\{x_n\}$ is a sequence of nonnegative real numbers such that*

$$x_{n+1} \le (1 - \alpha_n)x_n + \alpha_n\beta_n, \quad \forall n \ge 1, \tag{22}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences of real numbers which satisfy the following conditions:

(i) *$\{\alpha_n\} \subset [0,1]$, and $\sum_{n=1}^{\infty}\alpha_n = \infty$;*
(ii) *either $\limsup_{n\to\infty}\beta_n \le 0$ or $\sum_{n=1}^{\infty}\alpha_n\beta_n < \infty$.*

Then $\lim_{n\to\infty}x_n = 0$.

3. Main Result

Theorem 8. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let S be a nonexpansive mapping and T a demicontinuous nearly nonexpansive mapping on C with respect to the sequence $\{a_n\}$ such that $\mathrm{Fix}(T) \ne \emptyset$. Let $V : C \to H$ be a γ-Lipschitzian mapping and $F : C \to H$ a L-Lipschitzian and η-strongly monotone operator such that the coefficients satisfy $0 < \mu < 2\eta/L^2$, $0 \le \rho\gamma < \nu$, where $\nu = 1 - \sqrt{1 - \mu(2\eta - \mu L^2)}$. For an arbitrarily initial value $x_1 \in C$, consider the sequence $\{x_n\}$ in C generated by*

$$\begin{aligned}y_n &= \beta_n S x_n + (1 - \beta_n)x_n,\\ x_{n+1} &= P_C\left[\alpha_n\rho V x_n + (I - \alpha_n\mu F)T^n y_n\right], \\ &\qquad \forall n \ge 1,\end{aligned} \tag{23}$$

where $\{\alpha_n\}$ *and* $\{\beta_n\}$ *are sequences in* $[0,1]$ *satisfying the following conditions:*

(i) $\lim_{n\to\infty}\alpha_n = 0$, *and* $\sum_{n=1}^{\infty}\alpha_n = \infty$;

(ii) $\lim_{n\to\infty}(a_n/\alpha_n) = 0$, $\lim_{n\to\infty}(\beta_n/\alpha_n) = 0$, $\lim_{n\to\infty}(|\alpha_n - \alpha_{n-1}|/\alpha_n) = 0$, *and* $\lim_{n\to\infty}(|\beta_n - \beta_{n-1}|/\alpha_n) = 0$;

(iii) $\lim_{n\to\infty}\|T^n x - T^{n-1}x\| = 0$, *and* $\lim_{n\to\infty}(\|T^n x - T^{n-1}x\|/\alpha_n) = 0$, $\forall x \in C$.

Then, the sequence $\{x_n\}$ *converges strongly to* $x^* \in \mathrm{Fix}(T)$*, where* x^* *is the unique solution of the variational inequality* (10).

Proof. Since the mapping $\mu F - \rho V$ is a strongly monotone operator from Lemma 5, it is known that the variational inequality (10) has a unique solution. Let us denote this solution by $x^* \in \mathrm{Fix}(T)$. Now, we divide our proof into five steps.

Step 1. First we show that the sequence $\{x_n\}$ generated by (23) is bounded. From condition (ii), without loss of generality, we may suppose that $\beta_n \le \alpha_n$, for all $n \ge 1$. Hence, we get $\lim_{n\to\infty}\beta_n = 0$. Let $p \in \mathrm{Fix}(T)$ and $t_n = \alpha_n \rho V x_n + (I - \alpha_n \mu F)T^n y_n$. Then we have

$$\begin{aligned}\|y_n - p\| &= \|\beta_n S x_n + (1-\beta_n)x_n - p\| \\ &\le (1-\beta_n)\|x_n - p\| + \beta_n\|Sx_n - p\| \\ &\le (1-\beta_n)\|x_n - p\| + \beta_n\|Sx_n - Sp\| \\ &\quad + \beta_n\|Sp - p\| \\ &\le \|x_n - p\| + \beta_n\|Sp - p\|,\end{aligned} \tag{24}$$

and, by using the definition of nearly nonexpansive mapping and Lemma 6, we obtain

$$\begin{aligned}\|x_{n+1} - p\| &= \|P_C t_n - P_C p\| \\ &\le \|t_n - p\| \\ &= \|\alpha_n \rho V x_n + (I - \alpha_n\mu F)T^n y_n - p\| \\ &\le \alpha_n\|\rho V x_n - \mu F p\| \\ &\quad + \|(I - \alpha_n\mu F)T^n y_n - (I - \alpha_n\mu F)T^n p\| \\ &\le \alpha_n\rho\gamma\|x_n - p\| + \alpha_n\|\rho V p - \mu F p\| \\ &\quad + (1 - \alpha_n\nu)(\|y_n - p\| + a_n).\end{aligned} \tag{25}$$

From (24) and (25), we get

$$\begin{aligned}&\|x_{n+1} - p\| \\ &\quad \le \alpha_n\rho\gamma\|x_n - p\| + \alpha_n\|\rho V p - \mu F p\| \\ &\qquad + (1-\alpha_n\nu)(\|x_n - p\| + \beta_n\|Sp - p\| + a_n) \\ &\quad \le (1 - \alpha_n(\nu - \rho\gamma))\|x_n - p\| \\ &\qquad + \alpha_n(\nu - \rho\gamma) \\ &\qquad \times\left[\frac{1}{(\nu-\rho\gamma)}\left(\|\rho V p - \mu F p\| + \|Sp - p\| + \frac{a_n}{\alpha_n}\right)\right].\end{aligned} \tag{26}$$

From condition (ii), this sequence is bounded, and so we can write

$$\begin{aligned}\|x_{n+1} - p\| \le\ &(1 - \alpha_n(\nu - \rho\gamma))\|x_n - p\| \\ &+ \alpha_n(\nu - \rho\gamma)M,\end{aligned} \tag{27}$$

where

$$\frac{1}{(\nu-\rho\gamma)}\left(\|\rho V p - \mu F p\| + \|Sp - p\| + \frac{a_n}{\alpha_n}\right) \le M, \quad \forall n \ge 1. \tag{28}$$

Therefore, by induction, we get

$$\|x_{n+1} - p\| \le \max\{\|x_1 - p\|, M\}. \tag{29}$$

Hence, we obtain that $\{x_n\}$ is bounded. So, the sequences $\{y_n\}$, $\{Tx_n\}$, $\{Sx_n\}$, $\{Vx_n\}$, and $\{FTy_n\}$ are bounded too.

Step 2. Secondly, we show that $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$. By using the iteration (23), we have

$$\begin{aligned}\|y_n - y_{n-1}\| &= \|\beta_n S x_n + (1-\beta_n)x_n - \beta_{n-1}Sx_{n-1} \\ &\qquad - (1-\beta_{n-1})x_{n-1}\| \\ &\le \beta_n\|Sx_n - Sx_{n-1}\| + (1-\beta_n)\|x_n - x_{n-1}\| \\ &\quad + |\beta_n - \beta_{n-1}|(\|Sx_{n-1}\| + \|x_{n-1}\|) \\ &\le \|x_n - x_{n-1}\| + |\beta_n - \beta_{n-1}|M_1,\end{aligned} \tag{30}$$

where M_1 is a constant such that $\sup_{n\ge1}\{\|Sx_n\| + \|x_n\|\} \le M_1$, and

$$\begin{aligned}&\|x_{n+1} - x_n\| \\ &\quad \le \|P_C t_n - P_C t_{n-1}\| \\ &\quad \le \|\alpha_n\rho V x_n + (I - \alpha_n\mu F)T^n y_n - \alpha_{n-1}\rho V x_{n-1} \\ &\qquad - (I - \alpha_{n-1}\mu F)T^{n-1}y_{n-1}\| \\ &\quad \le \|\alpha_n\rho V(x_n - x_{n-1}) + (\alpha_n - \alpha_{n-1})\rho V x_{n-1} \\ &\qquad + (I - \alpha_n\mu F)T^n y_n - (I - \alpha_n\mu F)T^n y_{n-1} \\ &\qquad + T^n y_{n-1} - T^{n-1}y_{n-1} + \alpha_{n-1}\mu F T^{n-1}y_{n-1} \\ &\qquad - \alpha_n\mu F T^n y_{n-1}\|\end{aligned}$$

$$\begin{aligned}
&\leq \alpha_n\rho\gamma \|x_n - x_{n-1}\| + \rho|\alpha_n - \alpha_{n-1}| \|Vx_{n-1}\| \\
&\quad + (1-\alpha_n\nu)\|T^n y_n - T^n y_{n-1}\| \\
&\quad + \|T^n y_{n-1} - T^{n-1} y_{n-1}\| \\
&\quad + \mu\|\alpha_{n-1}FT^{n-1}y_{n-1} - \alpha_n FT^n y_{n-1}\| \\
&\leq \alpha_n\rho\gamma \|x_n - x_{n-1}\| + \rho|\alpha_n - \alpha_{n-1}| \|Vx_{n-1}\| \\
&\quad + (1-\alpha_n\nu)[\|y_n - y_{n-1}\| + a_n] \\
&\quad + \|T^n y_{n-1} - T^{n-1} y_{n-1}\| \\
&\quad + \mu\|\alpha_{n-1}(FT^{n-1}y_{n-1} - FT^n y_{n-1}) \\
&\qquad - (\alpha_n - \alpha_{n-1})FT^n y_{n-1}\|.
\end{aligned} \tag{31}$$

So, from (30) and (31), we get

$$\begin{aligned}
&\|x_{n+1} - x_n\| \\
&\quad\leq \alpha_n\rho\gamma\|x_n - x_{n-1}\| + \rho|\alpha_n - \alpha_{n-1}|\|Vx_{n-1}\| \\
&\qquad + (1-\alpha_n\nu)\|x_n - x_{n-1}\| \\
&\qquad + (1-\alpha_n\nu)|\beta_n - \beta_{n-1}| M_1 \\
&\qquad + (1-\alpha_n\nu)a_n + \|T^n y_{n-1} - T^{n-1}y_{n-1}\| \\
&\qquad + \mu\alpha_{n-1}L\|T^n y_{n-1} - T^{n-1}y_{n-1}\| \\
&\qquad + |\alpha_n - \alpha_{n-1}|\|FT^n y_{n-1}\| \\
&\quad\leq (1-\alpha_n(\nu-\rho\gamma))\|x_n - x_{n-1}\| \\
&\qquad + |\alpha_n - \alpha_{n-1}|(\rho\|Vx_{n-1}\| + \|FT^n y_{n-1}\|) \\
&\qquad + (1+\mu\alpha_{n-1}L)\|T^n y_{n-1} - T^{n-1}y_{n-1}\| \\
&\qquad + |\beta_n - \beta_{n-1}| M_1 + a_n \\
&\quad\leq (1-\alpha_n(\nu-\rho\gamma))\|x_n - x_{n-1}\| + \alpha_n(\nu-\rho\gamma)\delta_n,
\end{aligned} \tag{32}$$

where

$$\begin{aligned}
\delta_n &= \frac{1}{(\nu-\rho\gamma)} \\
&\quad\times\left[(1+\mu\alpha_{n-1}L)\frac{\|T^n y_{n-1} - T^{n-1}y_{n-1}\|}{\alpha_n}\right. \\
&\qquad\left. + \left(\left|\frac{\alpha_n - \alpha_{n-1}}{\alpha_n}\right| + \left|\frac{\beta_n - \beta_{n-1}}{\alpha_n}\right|\right) M_2 + \frac{a_n}{\alpha_n}\right], \\
&\quad \sup_{n\geq 1}\{\rho\|Vx_{n-1}\| + \|FT_n y_{n-1}\|, M_1\} \leq M_2.
\end{aligned} \tag{33}$$

By using the conditions (ii) and (iii), since $\lim\ \sup_{n\to\infty}\delta_n \leq 0$, it follows from Lemma 7 that

$$\|x_{n+1} - x_n\| \longrightarrow 0, \quad \text{as } n \longrightarrow \infty. \tag{34}$$

Step 3. Next, we show that $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$. For $n \geq m \geq 1$, we get

$$\begin{aligned}
&\|T^n y_n - T^m x_n\| \\
&\quad\leq \|T^n y_n - T^{n-1}y_n\| + \|T^{n-1}y_n - T^{n-2}y_n\| \\
&\qquad + \cdots + \|T^m y_n - T^m x_n\| \\
&\quad\leq \|T^n y_n - T^{n-1}y_n\| + \|T^{n-1}y_n - T^{n-2}y_n\| \\
&\qquad + \cdots + \|y_n - x_n\| + a_m,
\end{aligned} \tag{35}$$

and so

$$\begin{aligned}
&\|x_{n+1} - T^m x_n\| \\
&\quad= \|P_C t_n - P_C T^m x_n\| \\
&\quad\leq \|\alpha_n\rho Vx_n + (I - \alpha_n\mu F)T^n y_n - T^m x_n\| \\
&\quad\leq \alpha_n\|\rho Vx_n - \mu FT^n y_n\| + \|T^n y_n - T^m x_n\| \\
&\quad\leq \alpha_n\|\rho Vx_n - \mu FT^n y_n\| + \|T^n y_n - T^{n-1}y_n\| \\
&\qquad + \|T^{n-1}y_n - T^{n-2}y_n\| + \cdots + \|y_n - x_n\| + a_m.
\end{aligned} \tag{36}$$

Hence, we obtain from (35) and (36) that

$$\begin{aligned}
&\|x_n - T^m x_n\| \\
&\quad\leq \|x_n - x_{n+1}\| + \|x_{n+1} - T^m x_n\| \\
&\quad\leq \|x_n - x_{n+1}\| + \alpha_n\|\rho Vx_n - \mu FT^n y_n\| \\
&\qquad + \|T^n y_n - T^{n-1}y_n\| + \|T^{n-1}y_n - T^{n-2}y_n\| \\
&\qquad + \cdots + \|y_n - x_n\| + a_m \\
&\quad\leq \|x_n - x_{n+1}\| + \alpha_n\|\rho Vx_n - \mu FT^n y_n\| \\
&\qquad + \|T^n y_n - T^{n-1}y_n\| + \|T^{n-1}y_n - T^{n-2}y_n\| \\
&\qquad + \cdots + \beta_n\|Sx_n - x_n\| + a_m.
\end{aligned} \tag{37}$$

Since $\|\rho Vx_n - \mu FT^n y_n\|$ and $\|Sx_n - x_n\|$ are bounded, it follows from (34), (37), condition (i), and condition (iii) that

$$\lim_{m\to\infty}\left(\lim_{n\to\infty}\|x_n - T^m x_n\|\right) = 0. \tag{38}$$

Combining (38) and condition (iii), we have

$$\begin{aligned}
\|x_n - Tx_n\| \leq{}& \|x_n - T^m x_n\| \\
&+ \|T^m x_n - Tx_n\| \longrightarrow 0, \quad \text{as } n, m \longrightarrow \infty.
\end{aligned} \tag{39}$$

Step 4. Now, we show that $\limsup_{n\to\infty}\langle(\rho V - \mu F)x^*, x_n - x^*\rangle \leq 0$, where x^* is the unique solution of the variational inequality (10). Since the sequence $\{x_n\}$ is bounded, it has a weak convergent subsequence $\{x_{n_k}\}$ such that

$$\begin{aligned}
&\limsup_{n\to\infty}\langle(\rho V - \mu F)x^*, x_n - x^*\rangle \\
&\quad= \limsup_{k\to\infty}\langle(\rho V - \mu F)x^*, x_{n_k} - x^*\rangle.
\end{aligned} \tag{40}$$

Let $x_{n_k} \rightharpoonup \tilde{x}$, as $k \to \infty$. Then, Opial's condition guarantees that the weakly subsequential limit of $\{x_n\}$ is unique. Hence, this implies that $x_n \rightharpoonup \tilde{x}$, as $n \to \infty$. So, it follows from (38), Theorem 3, and Lemma 4 that $\tilde{x} \in \mathrm{Fix}(T)$. Therefore

$$\limsup_{n\to\infty} \langle (\rho V - \mu F) x^*, x_n - x^* \rangle = \langle (\rho V - \mu F) x^*, \tilde{x} - x^* \rangle \le 0. \tag{41}$$

Step 5. Finally, we show that the sequence $\{x_n\}$ converges strongly to x^*. By using inequality (15), we have

$$\begin{aligned}
&\|x_{n+1} - x^*\|^2 \\
&= \langle P_C t_n - x^*, x_{n+1} - x^* \rangle \\
&= \langle P_C t_n - t_n, x_{n+1} - x^* \rangle + \langle t_n - x^*, x_{n+1} - x^* \rangle \\
&\le \langle \alpha_n \rho V x_n + (I - \alpha_n \mu F) T^n y_n - x^*, x_{n+1} - x^* \rangle \\
&= \langle \alpha_n (\rho V x_n - \mu F x^*) + (I - \alpha_n \mu F) T^n y_n \\
&\qquad - (I - \alpha_n \mu F) T^n x^*, x_{n+1} - x^* \rangle \\
&= \alpha_n \rho \langle V x_n - V x^*, x_{n+1} - x^* \rangle \\
&\quad + \alpha_n \langle \rho V x^* - \mu F x^*, x_{n+1} - x^* \rangle \\
&\quad + \langle (I - \alpha_n \mu F) T^n y_n - (I - \alpha_n \mu F) T^n x^*, x_{n+1} - x^* \rangle \\
&\le \alpha_n \rho \gamma \|x_n - x^*\| \|x_{n+1} - x^*\| \\
&\quad + \alpha_n \langle \rho V x^* - \mu F x^*, x_{n+1} - x^* \rangle \\
&\quad + (1 - \alpha_n \nu) (\|y_n - x^*\| + a_n) \|x_{n+1} - x^*\|.
\end{aligned} \tag{42}$$

Also, by using inequality (24), we get

$$\begin{aligned}
&\|x_{n+1} - x^*\|^2 \\
&\le \alpha_n \rho \gamma \|x_n - x^*\| \|x_{n+1} - x^*\| \\
&\quad + \alpha_n \langle \rho V x^* - \mu F x^*, x_{n+1} - x^* \rangle \\
&\quad + (1 - \alpha_n \nu) (\|x_n - x^*\| + \beta_n \|S x^* - x^*\| + a_n) \\
&\quad \times \|x_{n+1} - x^*\| \\
&\le (1 - \alpha_n (\nu - \rho\gamma)) \|x_n - x^*\| \|x_{n+1} - x^*\| \\
&\quad + \alpha_n \langle \rho V x^* - \mu F x^*, x_{n+1} - x^* \rangle \\
&\quad + (1 - \alpha_n \nu) \beta_n \|S x^* - x^*\| \|x_{n+1} - x^*\| \\
&\quad + (1 - \alpha_n \nu) a_n \|x_{n+1} - x^*\| \\
&\le \frac{(1 - \alpha_n (\nu - \rho\gamma))}{2} (\|x_n - x^*\|^2 + \|x_{n+1} - x^*\|^2) \\
&\quad + \alpha_n \langle \rho V x^* - \mu F x^*, x_{n+1} - x^* \rangle \\
&\quad + \beta_n \|S x^* - x^*\| \|x_{n+1} - x^*\| + a_n \|x_{n+1} - x^*\|,
\end{aligned} \tag{43}$$

which implies that

$$\begin{aligned}
&\|x_{n+1} - x^*\|^2 \\
&\le \frac{(1 - \alpha_n (\nu - \rho\gamma))}{(1 + \alpha_n (\nu - \rho\gamma))} \|x_n - x^*\|^2 \\
&\quad + \frac{2\alpha_n}{(1 + \alpha_n (\nu - \rho\gamma))} \langle \rho V x^* - \mu F x^*, x_{n+1} - x^* \rangle \\
&\quad + \frac{2\beta_n}{(1 + \alpha_n (\nu - \rho\gamma))} \|S x^* - x^*\| \|x_{n+1} - x^*\| \\
&\quad + \frac{2a_n}{(1 + \alpha_n (\nu - \rho\gamma))} \|x_{n+1} - x^*\| \\
&\le (1 - \alpha_n (\nu - \rho\gamma)) \|x_n - x^*\|^2 + \alpha_n (\nu - \rho\gamma) \theta_n,
\end{aligned} \tag{44}$$

where

$$\begin{aligned}
\theta_n = {} & \frac{2}{(1 + \alpha_n (\nu - \rho\gamma)) (\nu - \rho\gamma)} \\
& \times \Big[\langle \rho V x^* - \mu F x^*, x_{n+1} - x^* \rangle \\
& \quad + \frac{\beta_n}{\alpha_n} M_3 + \frac{a_n}{\alpha_n} \|x_{n+1} - x^*\| \Big], \\
& \sup_{n\ge1} \{\|S x^* - x^*\| \|x_{n+1} - x^*\|\} \le M_3.
\end{aligned} \tag{45}$$

From condition (ii), by using Step 4, we get

$$\limsup_{n\to\infty} \theta_n \le 0. \tag{46}$$

So, it follows from Lemma 7 that the sequence $\{x_n\}$ generated by (23) converges strongly to $x^* \in \mathrm{Fix}(T)$ which is the unique solution of the variational inequality (10). This completes the proof. □

Remark 9. In particular, the point x^* is the minimum norm fixed point of T; namely, x^* is the unique solution of the quadratic minimization problem

$$x^* = \operatorname*{argmin}_{x\in\mathrm{Fix}(T)} \|x\|^2. \tag{47}$$

Indeed, since the point x^* is the unique solution of the variational inequality (10), if we take $V = 0$ and $F = I$, then we get

$$\langle \langle \mu x^*, x^* - x \rangle \le 0, \forall x \in \mathrm{Fix}(T) \rangle. \tag{48}$$

So we have

$$\begin{aligned}\langle x^*, x^* - x\rangle &= \langle x^*, x^*\rangle - \langle x^*, x\rangle \\ &\le 0 \Longrightarrow \|x^*\|^2 \le \|x^*\| \|x\| .\end{aligned} \tag{49}$$

Hence, x^* is the unique solution to the quadratic minimization problem (47).

Since a nearly nonexpansive mapping can be reduced to a nonexpansive mapping by taking the sequence $\{a_n\}$ as a zero sequence, under the appropriate changes on the control sequences arising from Lemma 7, we can derive main results of Wang and Xu [15, Theorem 3.1] and Ceng et al. [14, Theorem 3.1] as the following corollaries.

Corollary 10. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $S, T : C \to C$ be nonexpansive mappings such that $\mathrm{Fix}(T) \neq \emptyset$. Let $V : C \to H$ be a γ-Lip-schitzian mapping and $F : C \to H$ a L-Lipschitzian and η-strongly monotone operator such that these coefficients satisfy $0 < \mu < 2\eta/L^2$, $0 \le \rho\gamma < \nu$, where $\nu = 1 - \sqrt{1 - \mu(2\eta - \mu L^2)}$. For an arbitrarily initial value $x_1 \in C$, consider the sequence $\{x_n\}$ in C generated by (11) where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0, 1]$ satisfying the following conditions:*

(i) $\lim_{n\to\infty}\alpha_n = 0$, *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(ii) $\lim_{n\to\infty}(\beta_n/\alpha_n) = 0$;

(iii) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$, *and* $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$.

Then, the sequence $\{x_n\}$ converges strongly to $x^* \in \mathrm{Fix}(T)$, where x^* is the unique solution of the variational inequality (10). *In particular, the point x^* is the minimum norm fixed point of T; that is, x^* is the unique solution of the quadratic minimization problem* (47).

Proof. In the proof of Theorem 8, for all $n \ge 1$, if we take the mapping T as a nonexpansive mapping, then the desired conclusion is obtained. □

Corollary 11. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T : C \to C$ be a nonexpansive mapping such that $\mathrm{Fix}(T) \neq \emptyset$. Let $V : C \to H$ be a γ-Lipschitzian mapping with $F : C \to H$ being a L-Lipschitzian and η-strongly monotone operator such that these coefficients satisfy $0 < \mu < 2\eta/L^2$, $0 \le \rho\gamma < \nu$, where $\nu = 1 - \sqrt{1 - \mu(2\eta - \mu L^2)}$. For an arbitrarily initial value $x_1 \in C$, con-sider the sequence $\{x_n\}$ in C generated by* (9) *where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0, 1]$ satisfying the following conditions:*

(i) $\lim_{n\to\infty}\alpha_n = 0$, *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(ii) *either* $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ *or* $\lim_{n\to\infty}(\alpha_{n+1}/\alpha_n) = 0$.

Then, the sequence $\{x_n\}$ converges strongly to $x^* \in \mathrm{Fix}(T)$, where x^* is the unique solution of the variational inequality (10).

Proof. In the proof of Theorem 8, let $S = I$ where I is the identity mapping and T a nonexpansive mapping. Then, the proof is clear. □

Let $\{T_n\}$ be a sequence of mappings from C into H and let $\{a_m^n\}$ be a sequence in $[0, \infty)$ with $\lim_{m\to\infty} a_m^n = 0$, for each $n \ge 1$. Then, $\{T_n\}$ is called a sequence of nearly nonexpansive mappings with respect to a sequence $\{a_m^n\}$ if

$$\begin{gathered}\|T_n^m x - T_n^m y\| \le \|x - y\| + a_m^n, \\ \forall x, y \in C, \quad n, m \ge 1.\end{gathered} \tag{50}$$

Considering Remarks 1 and 2, it is too easy to see that the sequence $\{T_n\}$ defined by (50) is different from the sequence defined by (12) of Wong et al. [16]. The iterative scheme defined by (23) can be modified for the sequence of nearly nonexpansive mappings defined by (50). Accordingly, the problem written in the following remark arises.

Remark 12. For a sequence of nearly nonexpansive mappings (50) with a nonempty common fixed points set, it is an open problem whether or not an iteration process generated by this sequence will converge strongly to a common fixed point.

Example 13. Let $H = \mathbb{R}$ and $C = [0, 1]$. Let $S = I$, $Vx = 3x + 1$, $Fx = 4x$, and

$$Tx = \begin{cases} \dfrac{1}{2}, & \text{if } x \in \left[0, \dfrac{1}{2}\right] \\ 0, & \text{if } x \notin \left(\dfrac{1}{2}, 1\right] \end{cases} \tag{51}$$

for all $x \in C$. It is clear that S is nonexpansive, V is γ-Lipschitzian with $\gamma = 3$, F is L-Lipschitzian and η-strongly monotone operator with $L = \eta = 4$, and T is a nearly nonexpansive mapping with respect to the sequence $\{a_n\} = \{1/2n^2\}$. Define sequences $\{\alpha_n\}$ and $\{\beta_n\}$ in $[0, 1]$ by $\alpha_n = 1/(n + 1)$ and $\beta_n = 1/(n^2 + 1)$, and take $\mu = 1/4$, $\rho = 1/5$, and $\nu = 1$. It is easy to see that all the conditions of Theorem 8 are satisfied. So, the sequence $\{x_n\}$ generated by the iterative scheme (23) becomes

$$x_{n+1} = \frac{3x + 1}{5(n + 1)} - \frac{x}{2(n + 1)} + \frac{1}{2}, \quad \forall x \in C, \ n \ge 2, \tag{52}$$

and it converges strongly to $x^* = 1/2$ which is the unique fixed point of T and the unique solution of the variational inequality (10) over $F(T)$. The first ten values of iterative scheme (52) for the different initial values $x_1 = 0.1$, $x_1 = 0.5$, and $x_1 = 0.8$ are as in Table 1.

Table 1: The numerical results are obtained by using FORTRAN90 Programming Language.

x_1	$1.000000E-01$	$5.000000E-01$	$8.000000E-01$
x_2	$5.700000E-01$	$5.833333E-01$	$2.266667E-01$
x_3	$5.642500E-01$	$5.645834E-01$	$5.700000E-01$
x_4	$5.512850E-01$	$5.512916E-01$	$5.514000E-01$
x_5	$5.425214E-01$	$5.425215E-01$	$5.425233E-01$
x_6	$5.363218E-01$	$5.363218E-01$	$5.363218E-01$
x_7	$5.317040E-01$	$5.317040E-01$	$5.317040E-01$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
x_{10}	$5.229571E-01$	$5.229571E-01$	$5.229571E-01$
$\vdots$	$\vdots$	$\vdots$	$\vdots$

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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