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Research Article

Lie Groupoids and Generalized Contact Manifolds

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We investigate relationships between Lie groupoids and generalized almost contact manifolds. We first relate the notions of integrable Jacobi pairs and contact groupoids on generalized contact manifolds, and then we show that there is a one to one correspondence between linear operators and multiplicative forms satisfying Hitchin pair. Finally, we find equivalent conditions among the integrability conditions of generalized almost contact manifolds, the condition of compatibility of source, and target maps of contact groupoids with contact form and generalized contact maps.

1. Introduction

A groupoid is a small category in which all morphisms are invertible. More precisely, a groupoid (G, G_0) consists of two sets, G and G_0, called arrows and objects, respectively, with maps $s, t : G \to G_0$ called source and target. It is equipped with a composition $m : G_2 \to G$ defined on the subset $G_2 = \{(g, h) \in G \times G \mid s(g) = t(h)\}$, (see Figure 1), an inclusion map of objects $e : G_0 \to G$ and an inverse map $i : G \to G$. For a groupoid, the following properties are satisfied: $s(gh) = s(h)$, $t(gh) = t(g)$, $s(g^{-1}) = t(g)$, $t(g^{-1}) = s(g)$, and $g(hf) = (gh)f$ whenever both sides are defined, $g^{-1}g = 1_{s(g)}$, $gg^{-1} = 1_{t(g)}$. Here, we have used gh, 1_x and g^{-1} instead of $m(g, h)$, $e(x)$ and $i(g)$. Generally, a groupoid (G, G_0) is denoted by the set of arrows G. A topological groupoid is a groupoid G whose set of arrows and set of objects are both topological spaces whose structure maps s, t, e, i, m are all continuous and s, t are open maps.

A Lie groupoid is a groupoid G whose set of arrows and set of objects are both manifolds whose structure maps s, t, e, i, m are all smooth maps and s, t are submersions. The latter condition ensures that s and t-fibres are manifolds. One can see from the above definition that the space G_2 of composable arrows is a submanifold of $G \times G$. We note that the notion of Lie groupoids was introduced by Ehresmann [1]. Relations among Lie groupoids, Lie algebroids, and other algebraic structures have been investigated by many authors [2–6].

On the other hand, Lie algebroids were first introduced by Pradines [7] as infinitesimal objects associated with the Lie groupoids. More precisely, a Lie algebroid structure on a real vector bundle A on a manifold M is defined by a vector bundle map $\rho_A : A \to TM$, the anchor of A, and an $\mathbb{R}$-Lie algebra bracket on $\Gamma(A)$, $[,]_A$ satisfying the Leibnitz rule

$$[\alpha, f\beta]_A = f[\alpha, \beta]_A + L_{\rho_A(\alpha)}(f)\,\beta \tag{1}$$

for all $\alpha, \beta \in \Gamma(A)$, $f \in C^{\infty}(M)$, where $L_{\rho_A(\alpha)}$ is the Lie derivative with respect to the vector field $\rho_A(\alpha)$, where $\Gamma(A)$ denotes the set of sections in A.

In [8], Hitchin introduced the notion of generalized complex manifolds by unifying and extending the usual notions of complex and symplectic manifolds. Later, such manifolds have been studied widely by Gualtieri. He also introduced the notion of generalized Kähler manifold [9]. On the other hand, the concept of generalized almost contact structure on odd-dimensional manifolds has been studied in [10–12].

Recently, Crainic [13] showed that there is a close relationship between the equations of a generalized complex manifold and a Lie groupoid. More precisely, he obtained that the complicated equations of such manifolds turn into simple structures for Lie groupoids.

In this paper, we investigate relationships between the complicated equations of generalized contact structures and Lie groupoids. We showed that the equations of such manifolds are useful to obtain equivalent results on a contact

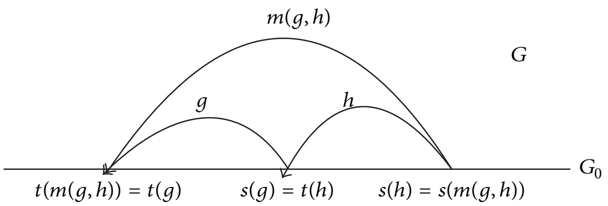

Figure 1: Two arrows g and $h \in G$, with the target of h, $t(h) \in G_0$, equal to the source of g, $s(g) \in G_0$, and the composed arrow $m(g,h)$ [5].

groupoid. The paper is organized as follows. In Section 2, we gather main definitions and results used in the other sections. In Section 3, we first state necessary and sufficient conditions for generalized almost contact structure to be integrable, and then we obtain a relation between integrable Jacobi pairs and contact groupoids defined on generalized manifolds. Moreover, we observe that there is a close relationship between $(1,1)$-tensors satisfying certain conditions in terms of tensor fields defined on generalized manifolds and multiplicative forms. Finally, we find one to one correspondence among generalized contact map, source and target maps, and the conditions of a generalized contact structure to be integrable.

2. Preliminaries

In this section, we give basic facts of Jacobi geometry, Lie groupoids and Lie algebroids. We first recall notions of contact manifold and contact groupoid from [5]. A contact manifold is a smooth (odd-dimensional) manifold M with 1-form $\eta \in \Omega^1(M)$ such that $\eta \wedge (d\eta)^n \neq 0$. η is called the contact form of M. Let G be a Lie groupoid on M and η a form on Lie groupoid G; then η is called r-multiplicative if

$$m^*\eta = pr_2^*\left(e^{-r}\right) pr_1^*\eta + pr_2^*\eta, \tag{2}$$

where $pr_i : G \times G \to G, i = 1, 2$, are the canonical projections and $r : G \to \mathbb{R}$, $r(gh) = r(g) + r(h)$ is a function [14]. A contact groupoid over a manifold M is a Lie groupoid G over M together with a contact form η on G such that η is r-multiplicative. We recall that multiplicative of a 2-form ω is defined by

$$m^*\omega = pr_1^*\omega + pr_2^*\omega. \tag{3}$$

We now recall the notion of Jacobi manifolds. A Jacobi manifold is a smooth manifold M equipped with a bivector field π and a vector field E such that

$$[\pi, \pi] = -2E \wedge \pi, \quad [E, \pi] = 0, \tag{4}$$

where $[,]$ denotes the Schouten bracket. In this case, (π, E) defines a bracket on $C^\infty(M, \mathbb{R})$, which is called Jacobi bracket and is given, for all $f, g \in C^\infty(M, \mathbb{R})$, by

$$\{f, g\} = \pi\left(df, dg\right) + fE\left(g\right) - gE\left(f\right). \tag{5}$$

The Jacobi bracket endows $C^\infty(M, \mathbb{R})$ with a local Lie algebra structure in the sense of Kirillov [15].

We now give a relation between Lie algebroid and Lie groupoid; more details can be found in [16]. Given a Lie groupoid G on M, the associated Lie algebroid $A = \mathrm{Lie}(G)$ has fibres $A_x = \mathrm{Ker}(ds)_x = T_x(G(-, x))$, for any $x \in M$. Any $\alpha \in \Gamma(A)$ extends to a unique right-invariant vector field on G, which will be denoted by the same letter α. The usual Lie bracket on vector fields induces the bracket on $\Gamma(A)$, and the anchor is given by $\rho = dt : A \to TM$.

Given a Lie algebroid A, an integration of A is a Lie groupoid G together with an isomorphism $A \cong \mathrm{Lie}(G)$. If such a G exists, then it is said that A is integrable. In contrast with the case of Lie algebras, not every Lie algebroid admits an integration. However, if a Lie algebroid is integrable, then there exists a canonical source-simply connected integration G, and any other source-simply connected integration is smoothly isomorphic to G. *From now on, we assume that all Lie groupoids are source-simply connected.*

We now recall the notion of IM form (infinitesimal multiplicative form) on a Lie algebroid [17], which will be useful when we deal with relations between Lie groupoids and Lie algebroids. An IM form on a Lie algebroid A is a bundle map

$$u : A \longrightarrow T^*M \tag{6}$$

satisfying the following properties:

(i) $\langle u(\alpha), \rho(\beta)\rangle = -\langle u(\beta), \rho(\alpha)\rangle$

(ii) $u([\alpha, \beta]) = \mathscr{L}_\alpha(u(\beta)) - \mathscr{L}_\beta(u(\alpha)) + d\langle u(\alpha), \rho(\beta)\rangle$

for $\alpha, \beta \in \Gamma(A)$, where $\rho = \rho_A$ and $\langle,\rangle$ denotes the usual pairing between a vector space and its dual.

If A is a Lie algebroid of a Lie groupoid G, then a closed multiplicative 2-form ω on G induces an IM form u_ω of A by

$$\left\langle u_\omega\left(\alpha\right), X\right\rangle = \omega\left(\alpha, X\right). \tag{7}$$

For the relationship between IM form and closed 2-form, we have the following.

Theorem 1 (see [17]). *If A is an integrable Lie algebroid and if G is its integration, then $\omega \mapsto u_\omega$ is a one to one correspondence between closed multiplicative 2-forms on G and IM forms of A.*

Finally, in this section, we give brief information on the notion of generalized geometry; details can be found in [9]. A central idea in generalized geometry is that $TM \oplus T^*M$ should be thought of as a generalized tangent bundle to manifold M. If X and ξ denote a vector field and a dual vector field on M, respectively, then we write (X, ξ) (or $X + \xi$) as a typical element of $TM \oplus T^*M$. The Courant bracket of two sections $(X, \xi), (Y, \eta)$ of $TM \oplus T^*M = \mathscr{T}\mathscr{M}$ is defined by

$$[\![\left(X, \xi\right), \left(Y, \eta\right)]\!] = [X, Y] + \mathscr{L}_X\eta - \mathscr{L}_Y\xi - \frac{1}{2}d\left(i_X\eta - i_Y\xi\right), \tag{8}$$

where d, $\mathscr{L}_X$, and i_X denote exterior derivative, Lie derivative, and interior derivative with respect to X, respectively. The Courant bracket is antisymmetric, but it does not satisfy the Jacobi identity. Here, we use the notations $\beta(\pi^\sharp\alpha) = \pi(\alpha, \beta)$

and $\omega_\sharp(X)(Y) = \omega(X,Y)$, which are defined as $\pi^\sharp : T^*M \to TM$, $\omega_\sharp : TM \to T^*M$ for any 1-forms α and β, 2-form ω and bivector field π, and vector fields X and Y. Also we denote by $[,]_\pi$ the bracket on the space of 1-forms on M defined by

$$[\alpha,\beta]_\pi = \mathscr{L}_{\pi^\sharp\alpha}\beta - \mathscr{L}_{\pi^\sharp\beta}\alpha - d\pi(\alpha,\beta). \tag{9}$$

3. Lie Groupoids and Generalized Contact Structures

In this section, we first give a characterization for generalized contact structures to be integrable; then we obtain certain relationships between generalized contact manifolds and contact groupoids. We recall a generalized almost contact pair and then a generalized almost contact structure.

Definition 2 (see [12]). A generalized almost contact pair $(\mathscr{I}, F+\eta)$ on a smooth odd-dimensional manifold M consists of a bundle endomorphism $\mathscr{I}$ of $TM \oplus T^*M$ and a section $F+\eta$ of $\mathscr{T}\mathscr{M}$ such that

$$\begin{gathered}\mathscr{I} + \mathscr{I}^* = 0; \qquad \eta(F) = 1; \\ \mathscr{I}(F) = 0; \qquad \mathscr{I}(\eta) = 0; \qquad \mathscr{I}^2 = -Id + F \odot \eta,\end{gathered} \tag{10}$$

where $F\odot\eta(X+\alpha) := \eta(X)F + \alpha(F)\eta$, for any $X+\alpha \in \Gamma(\mathscr{T}\mathscr{M})$. Since $\mathscr{I}$ has a matrix form,

$$\mathscr{I} = \begin{bmatrix} \varphi & \pi^\sharp \\ \theta^\sharp & -\varphi^* \end{bmatrix}, \tag{11}$$

where φ is a $(1,1)$-tensor, π is a bivector field, θ is a 2-form, and $\varphi^* : T^*M \to T^*M$ is dual of φ, one sees that a generalized almost contact pair is equivalent to a quintuplet $(F,\eta,\pi,\theta,\varphi)$, where F is a vector field, η a 1-form.

Definition 3 (see [12]). A generalized almost contact structure on M is an equivalent class of such pairs $(\mathscr{I}, F+\eta)$.

We now present two examples of generalized almost contact manifolds.

Example 4 (see [11]). An $(2n+1)$-dimensional smooth manifold M has an almost contact structure (φ, F, η) if it admits a tensor field φ of type $(1,1)$, a vector field F, and a 1-form η satisfying the following compatibility conditions:

$$\begin{gathered}\varphi(F) = 0, \qquad \eta\circ\varphi = 0, \\ \eta(F) = 1, \qquad \varphi^2 = -id + \eta\otimes F.\end{gathered} \tag{12}$$

Associated with any almost contact structure, we have an almost generalized contact structure by setting

$$\mathscr{I} = \begin{bmatrix} \varphi & 0 \\ 0 & -\varphi^* \end{bmatrix}. \tag{13}$$

Example 5 (see [11]). On the three-dimensional Heisenberg group H_3, we choose a basis $\{X_1, X_2, X_3\}$ and let $\{\alpha_1,\alpha_2,\alpha_3\}$ be a dual frame. For $t = rc + irs$, where $c = \cos v$ and $s = \sin v$ for some real number v, we define

$$\begin{aligned}\varphi_t &= \frac{2rc}{1-r^2}\left(X_2\otimes\alpha^2 + X_3\otimes\alpha^3\right), \\ \sigma_t &= \frac{r^2 - 2rs + 1}{1-r^2}\alpha^2\wedge\alpha^3, \\ \pi_t &= \frac{r^2+2rs+1}{1-r^2}X_2\wedge X_3, \\ \eta = \alpha^1, &\qquad F = X_1 - bX_2 + aX_3,\end{aligned} \tag{14}$$

for any real numbers a, b. We also define

$$\mathscr{I} = \begin{bmatrix} \varphi_t & \pi_t^\sharp \\ \sigma_{t\flat} & -\varphi_t^* \end{bmatrix}. \tag{15}$$

Then, $\mathscr{I}_t = (F, \eta, \phi_t, \pi_t\sigma_t)$ is a family of generalized almost contact structures.

Given a generalized almost contact pair $(\mathscr{I}, F+\eta)$, we define

$$\begin{aligned}E^{(1,0)} &= \{e - i\mathscr{I}(e) \mid e \in \ker\eta \oplus \ker F\}, \\ E^{(0,1)} &= \{e + i\mathscr{I}(e) \mid e \in \ker\eta \oplus \ker F\}.\end{aligned} \tag{16}$$

The endomorphism $\mathscr{I}$ is linearly extended to the complexified bundle $\mathscr{T}\mathscr{M}\otimes\mathbb{C}$. It has three eigenvalues, namely, $\lambda = 0$, $\lambda = i = \sqrt{-1}$, and $\lambda = -i$. The corresponding eigenbundles are $L_F \oplus L_\eta$, $E^{(1,0)}$ and $E^{(0,1)}$, where L_F and L_η are the complex vector bundles of rank 1 generated with F and η, respectively. Define

$$\begin{gathered}L := L_F \oplus E^{(1,0)}, \qquad \overline{L} := L_F \oplus E^{(0,1)}, \\ L^* := L_\eta \oplus E^{(0,1)}, \qquad \overline{L}^* := L_\eta \oplus E^{(1,0)}.\end{gathered} \tag{17}$$

Definition 6 (see [11]). Consider a generalized almost contact pair and let L be its associated maximal isotropic subbundle. One says that the generalized almost contact pair is integrable if the space $\Gamma(L)$ of sections of L is closed under the Courant bracket; that is, $[\![\Gamma(L),\Gamma(L)]\!] \subset \Gamma(L)$. In this case, the generalized almost contact pair is simply called a generalized contact pair. A generalized contact structure is an equivalence class of generalized contact pairs.

In the sequel, we give necessary and sufficient conditions for a generalized almost contact structure to be integrable in terms of the above tensor fields. We note that the following result was stated in [12].

Theorem 7. *A generalized almost contact pair corresponding to the quintuplet $(F,\eta,\pi,\theta,\varphi)$ is integrable if and only if the following relations are satisfied:*

(C1)

$$\begin{aligned}&\textbf{(a)}\ \frac{1}{2}[\pi,\pi] = F\wedge\left(\pi^\sharp\otimes\pi^\sharp\right)d\eta, \\ &\textbf{(b)}\ [F,\pi] = -F\wedge\pi^\sharp\mathscr{L}_F\eta;\end{aligned} \tag{18}$$

(C2)

$$\varphi\pi^{\sharp} = \pi^{\sharp}\varphi^{*}$$
$$\varphi^{*}[\alpha,\beta]_{\pi} = \mathscr{L}_{\pi^{\sharp}\alpha}\varphi^{*}\beta - \mathscr{L}_{\pi^{\sharp}\beta}\varphi^{*}\alpha - d\pi(\varphi^{*}\alpha,\beta); \quad (19)$$

(C3)

$$\varphi^{2} + \pi^{\sharp}\theta_{\sharp} = -Id + F\odot\eta,$$
$$N_{\varphi}(X,Y) + d\eta(\varphi X,\varphi Y)F = \pi^{\sharp}(i_{X\wedge Y}d\theta); \quad (20)$$

(C4)

$$\varphi^{*}\theta_{\sharp} = \theta_{\sharp}\varphi,$$
$$d\theta_{\varphi}(X,Y,Z) = d\theta(\varphi X,Y,Z) + d\theta(X,\varphi Y,Z) + d\theta(X,Y,\varphi Z); \quad (21)$$

(C5)

$$\mathscr{L}_{F}\varphi = 0; \qquad \mathscr{L}_{F}\theta = 0, \quad (22)$$

where

$$[\alpha,\beta]_{\pi} = \mathscr{L}_{\pi^{\sharp}\alpha}\beta - \mathscr{L}_{\pi^{\sharp}\beta}\alpha - d\pi(\alpha,\beta),$$
$$\theta_{\varphi}(X,Y) = \theta(\varphi X,Y). \quad (23)$$

We note that if (11) is a generalized contact structure, then

$$\overline{\mathscr{J}} = \begin{bmatrix} \varphi & -\pi^{\sharp} \\ -\theta_{\sharp} & -\varphi^{*} \end{bmatrix} \quad (24)$$

is also a generalized contact structure. $\overline{\mathscr{J}}$ is called the opposite of $\mathscr{J}$. In this paper, we denote a generalized contact manifold endowed with $\overline{\mathscr{J}}$ by $\overline{M}$.

As an analogue of a Hitchin pair on a generalized complex manifold, a Hitchin pair on a generalized almost contact manifold M is a pair $(d\eta,\varphi)$ consisting of a contact form η and a $(1,1)$-tensor φ with the property that $d\eta$ and φ commute (i.e., $d\eta(X,\varphi Y) = d\eta(\varphi X,Y)$). We note that since a generalized almost contact structure is equivalent to a generalized almost complex structure on $M\times\mathbb{R}$, the bivector field π of the generalized almost contact structure is not nondegenerate in general. But we emphasize that we are putting this condition for restricted case.

Lemma 8. *Let M be a generalized almost contact manifold. If π is a nondegenerate bivector field on TM^{*}-Span$\{\eta\}$, $d\eta$ is the inverse 2-form (defined by $(d\eta)_{\sharp} = (\pi^{\sharp})^{-1}$), and π satisfies* (20)*, then $\theta = -d\eta - \varphi^{*}d\eta + \eta\wedge(i_{F}d\eta)$ if and only if $d\eta(\varphi^{2}X,Y) = \varphi^{*}d\eta(X,Y)$.*

Proof. For $X\in\chi(M)$, we apply $(d\eta)_{\sharp}$ to (20) and using the dual structure φ^{*}, we have

$$(d\eta)_{\sharp}\varphi^{2}(X) = -(d\eta)_{\sharp}(X) - (d\eta)_{\sharp}\left(\pi^{\sharp}\theta_{\sharp}(X)\right) + (d\eta)_{\sharp}(\eta(X)F)$$
$$d\eta\left(\varphi^{2}X,Y\right) = -d\eta(X,Y) - \theta_{\sharp}(X)(Y) + d\eta(\eta(X)F,Y). \quad (25)$$

We obtain

$$\varphi^{*}d\eta(X,Y) = -d\eta(X,Y) - \theta(X,Y) + \eta(X)d\eta(F,Y). \quad (26)$$

Since (26) holds, for all X and Y, we get

$$\theta = -d\eta - \varphi^{*}d\eta + \eta\wedge(i_{F}d\eta). \quad (27)$$

In a similar way, one can get the converse. □

From now on, when we mention a nondegenerate bivector field π, we mean it is nondegenerate on TM^{*}-Span$\{\eta\}$. We note that if $d\eta$ is the inverse 2-form of π, nondegenerate π on TM^{*}-Span$\{\eta\}$ implies that $d\eta$ is also nondegenerate on TM-Span$\{F\}$.

We say that 2-form θ is the twist of Hitchin pair $(d\eta,\varphi)$. Note that in this case φ is neither an almost contact structure nor torsion(N_{φ}) free.

Lemma 9. *Let (M,η,φ,F) be an almost contact manifold. $d\eta$ and φ commute if and only if $d\eta + \varphi^{*}d\eta = \eta\wedge(i_{F}d\eta)$.*

Proof. We will only prove the sufficient condition. We have

$$\varphi^{*}d\eta(X,Y) + d\eta(X,Y) - \eta\wedge(i_{F}d\eta)(X,Y) = 0. \quad (28)$$

Since φ^{*} is dual contact structure, we get

$$d\eta(\varphi X,\varphi Y) + d\eta(X,Y) - \eta(X)(i_{F}d\eta)(Y) = 0. \quad (29)$$

Substituting φX by X, and using contact structure property,

$$d\eta(-X + \eta(X)F,\varphi Y) + d\eta(\varphi X,Y) = 0. \quad (30)$$

Hence, we obtain

$$-d\eta(X,\varphi Y) + d\eta(\varphi X,Y) = 0, \quad (31)$$

which shows that $d\eta$ and φ commute. The converse is clear. □

Next, we see that (C1) is satisfied automatically when one chooses $d\eta$ as the 2-form which is the inverse of π defined by $(d\eta)_{\sharp} = (\pi^{\sharp})^{-1}$.

Lemma 10. *Let π be a nondegenerate bivector on a generalized almost contact manifold M, and $d\eta$ the inverse 2-form (defined by $(d\eta)_{\sharp} = (\pi^{\sharp})^{-1}$). Then π satisfies (C1).*

Proof. Since $d\eta$ is a closed form, it is obvious due to [13, Lemma 2.7]. □

Definition 11 (see [14]). The Lie algebroid of the Jacobi manifold (M, π, F) is $T^*M \oplus \mathbb{R}$, with the anchor $\rho : T^*M \oplus \mathbb{R} \to TM$ given by

$$\rho(\omega, \lambda) = (\pi, F)^\sharp (\omega, \lambda) = \pi^\sharp (\omega) + \lambda F, \tag{32}$$

and the bracket

$$\begin{gathered}[(\omega, 0), (\eta, 0)] = ([\omega, \eta]_\pi, 0) - (i_F(\omega \wedge \eta), \pi(\omega, \eta)) \\ [(0,1), (\omega, 0)] = (\mathscr{L}_{(F)}\omega, 0).\end{gathered} \tag{33}$$

The associated groupoid,

$$\Sigma(M) = G(T^*M \oplus \mathbb{R}), \tag{34}$$

is called contact groupoid of the Jacobi manifold M. We say that M is integrable as a Jacobi manifold if the associated algebroid $T^*M \oplus \mathbb{R}$ is integrable (or, equivalently, if $\Sigma(M)$ is smooth).

Thus, we have the following result which shows that there is a close relationship between the condition (C1) and a contact groupoid.

Theorem 12. *Let M be a generalized almost contact manifold and η a contact form. There is a 1-1 correspondence between*

(i) *integrable Jacobi pair (F, π) on M (i.e., (F, π) is satisfying (C1), integrable),*

(ii) *contact groupoids (Σ, η) over M.*

Proof. Since $(\eta, d\eta)$ is a contact pair and (F, π) satisfies (C1), then (F, π) is an integrable Jacobi pair [18]. From Definition 11, one sees that a contact groupoid is obtained from an integrable Jacobi pair.

The converse is clear. □

We now give the conditions for (C2) in terms of $d\eta$ and φ.

Lemma 13. *Let M be a generalized almost contact manifold and $d\eta$ a 2-form. Given a nondegenerate bivector π on TM^*-$\{\eta\}$ (i.e., $\pi^\sharp = ((d\eta)_\sharp)^{-1}$) and a map $\varphi : TM \to TM$, then π and φ satisfy (C2) if and only if $d\eta$ and φ commute.*

We now give a correspondence between generalized contact structures with nondegenerate π and Hitchin pairs $(d\eta, \varphi)$.

Proposition 14. *There is a one to one correspondence between generalized contact structures given by* (11) *with π nondegenerate and Hitchin pairs $(d\eta, \varphi)$ such that $d\eta(X,Y) = d\eta(\varphi X, \varphi Y)$. In this correspondence, π is the inverse of $d\eta$, and θ is the twist of the Hitchin pair $(d\eta, \varphi)$.*

Proof. Since $(d\eta, \varphi)$ is Hitchin pair, $d\eta$ and $(d\eta)_\varphi$ are closed. By using the following equation (see [13]):

$$\begin{aligned}i_{N_\varphi(X,Y)}(d\eta) = {} & i_{\varphi X\wedge Y + X\wedge\varphi Y}\left(d(d\eta)_\varphi\right) - i_{\varphi X\wedge\varphi Y}(d(d\eta)) \\ & - i_{X\wedge Y}(d(\varphi^* d\eta)),\end{aligned} \tag{35}$$

we get

$$i_{N_\varphi(X,Y)}(d\eta) = -i_{X\wedge Y}(d(\varphi^* d\eta)). \tag{36}$$

Since $\theta = -d\eta - \varphi^* d\eta + \eta \wedge (i_F d\eta)$, we derive

$$(d\eta)_\sharp \left(N_\varphi(X,Y)\right) = -i_{X\wedge Y}(d(-\theta + \eta \wedge (i_F d\eta))). \tag{37}$$

Applying $\pi^\sharp$ to (37), then we get

$$\begin{gathered}N_\varphi(X,Y) = -\pi^\sharp (i_{X\wedge Y}(d(-\theta + \eta \wedge (i_F d\eta)))) \\ N_\varphi(X,Y) = \pi^\sharp \left(i_{X\wedge Y}(d\theta) - \pi^\sharp (i_{X\wedge Y} d\eta \wedge (i_F d\eta))\right) \\ N_\varphi(X,Y) = \pi^\sharp (i_{(X\wedge Y)} d\theta) - d\eta(X,Y)F.\end{gathered} \tag{38}$$

By assumption, we have

$$d\eta(X,Y) = d\eta(\varphi X, \varphi Y). \tag{39}$$

Putting this equation into (38), we obtain

$$N_\varphi(X,Y) = \pi^\sharp (i_{(X\wedge Y)} d\theta) - d\eta(\varphi X, \varphi Y)F, \tag{40}$$

which is the second equation of (C3). Now we show that $\varphi^*\theta_\sharp = \theta_\sharp\varphi$. From (26), we obtain

$$\varphi^*\theta_\sharp = \varphi^* \left(-(d\eta)_\sharp - (\varphi^*(d\eta))_\sharp + (\eta \wedge (i_F(d\eta)))_\sharp\right). \tag{41}$$

Hence, we have

$$\varphi^*\theta_\sharp = -(d\eta)_\sharp\varphi - (\varphi^*(d\eta))_\sharp\varphi + (\eta \wedge (i_F(d\eta)))_\sharp\varphi. \tag{42}$$

From definition of twist, we get

$$\varphi^*\theta_\sharp = \theta_\sharp\varphi. \tag{43}$$

This equation is the first equation of (C4). Now, we will obtain

$$\begin{aligned}& d\theta_\varphi(X,Y,Z) \\ & \quad = d\theta(\varphi X, Y, Z) + d\theta(X, \varphi Y, Z) + d\theta(X, Y, \varphi Z),\end{aligned} \tag{44}$$

which is second equation of (C4). Writing the equation as

$$i_{X\wedge Y}\left(d\theta_\varphi\right) = i_{\varphi X\wedge Y + X\wedge\varphi Y}(d\theta) + \varphi^*(i_{X\wedge Y}(d\theta)), \tag{45}$$

and since $\theta = -d\eta - \varphi^* d\eta + \eta \wedge (i_F d\eta)$, then we should find

$$\begin{aligned}& i_{X\wedge Y}\left(d\left(-(\varphi^*(d\eta))_\varphi + (\eta \wedge (i_F(d\eta)))_\varphi\right)\right) \\ & \quad = i_{\varphi X\wedge Y + X\wedge\varphi Y}(d(-\varphi^* d\eta + \eta \wedge (i_F d\eta))) \\ & \qquad + \varphi^*(i_{X\wedge Y}(d(-\varphi^* d\eta + \eta \wedge (i_F d\eta)))).\end{aligned} \tag{46}$$

A straightforward computation shows that

$$\begin{aligned}i_{X\wedge Y}\left(d\left((\varphi^*(d\eta))_\varphi\right)\right) = {} & i_{\varphi X\wedge Y + X\wedge\varphi Y}(d(\varphi^* d\eta)) \\ & + \varphi^*(i_{X\wedge Y}(d(\varphi^* d\eta))).\end{aligned} \tag{47}$$

Using (35), then we get

$$i_{X\wedge Y}\left(d\left(\left(\varphi^{*}\left(d\eta\right)\right)_{\varphi}\right)\right) = i_{\varphi X\wedge Y+X\wedge\varphi Y}\left(d\left(\varphi^{*}d\eta\right)\right) - i_{N_{\varphi}(X,Y)}\left(d\eta\right)_{\varphi}. \tag{48}$$

Since $i_{N(X,Y)}((d\eta)_{\varphi}) = \varphi^{*} i_{N(X,Y)}(d\eta)$, applying $i_{X\wedge Y} d(\varphi^{*}d\eta) = -i_{N_{\varphi}(X,Y)}(d\eta)$ to (48), we have

$$i_{X\wedge Y}\left(d\left(\left(\varphi^{*}\left(d\eta\right)\right)_{\varphi}\right)\right) = i_{\varphi X\wedge Y+X\wedge\varphi Y}\left(d\left(\varphi^{*}d\eta\right)\right) + \varphi^{*}\left(i_{X\wedge Y}\left(d\left(\varphi^{*}d\eta\right)\right)\right). \tag{49}$$

In a similar way, we can obtain (C5).

The converse is clear from Lemmas 10 and 13. □

We note that, similar to 2-forms, given a Lie groupoid G, a $(1,1)$-tensor $J : TG \to TG$ is called multiplicative [13] if for any $(g,h) \in G\times G$ and any $v_g \in T_gG$, $w_h \in T_hG$ such that (v_g, w_h) is tangent to $G\times G$ at (g,h), so is (Jv_g, Jw_h), and

$$(dm)_{g,h}\left(Jv_g, Jw_h\right) = J\left((dm)_{g,h}\left(v_g, w_h\right)\right). \tag{50}$$

Let (M,η) be a contact manifold. Then it is easy to see that there is a one to one correspondence between $(1,1)$-tensors φ commuting with $d\eta$ and 2-forms on M. On the other hand, it is easy to see that (C2) is equivalent to the fact that $\varphi^{*}\circ pr_1$ is an IM form on the Lie algebroid $T^{*}M\oplus\mathbb{R}$ associated Jacobi structure (F,π). Thus, from the above discussion, Lemma 13 and Theorem 1, one can conclude with the following theorem.

Theorem 15. *Let M be a generalized almost contact manifold. Let (F,π) be an integrable Jacobi structure on M and (Σ,η) a contact groupoid over M. Then there is a natural 1-1 correspondence between*

(i) *$(1,1)$-tensors φ on M satisfying (C2),*

(ii) *multiplicative $(1,1)$-tensors I on Σ with the property that $(I, d\eta)$ is a Hitchin pair.*

We recall the notion of generalized contact map between generalized contact manifolds. This notion is similar to the generalized holomorphic map given in [13].

Let $(M_i, \mathcal{J}_i)$, $i = 1,2$, be two generalized contact manifolds, and let $\varphi_i, \pi_i, \theta_i$ be the components of $\mathcal{J}_i$ in the matrix representation (11). A map $f : M_1 \to M_2$ is called generalized contact if and only if f maps φ_1 into φ_2, F_1 into F_2, and π_1 into π_2, $f^{*}\theta_2 = \theta_1$ and $(df)\circ\varphi_1 = \varphi_2\circ(df)$.

We now state and prove the main result of this paper. This result gives equivalent assertions between the condition (C3), twist θ of $(d\eta, I)$, and contact maps for a contact groupoid over M.

Theorem 16. *Let M be a generalized almost contact manifold and (Σ,η,I) an induced contact groupoid over M with the induced multiplicative $(1,1)$-tensor. Assume that $((F,\pi), I)$ satisfy (C1), (C2) with integrable (F,π). Then, for a θ 2-form on M, the following assertions are equivalent.*

(i) *(C3) is satisfied,*

(ii) *$d\eta + I^{*}d\eta - \eta\wedge(i_F d\eta) = s^{*}\theta - t^{*}\theta$,*

(iii) *$(t,s) : \Sigma \to M\times\overline{M}$ is a generalized contact map; condition of generalized contact map on M is $(dt)\circ\varphi_1 = \varphi_2\circ(dt)$; this condition on $\overline{M}$ is $(ds)\circ\varphi_1 = -\varphi_2\circ(ds)$.*

Proof. (i) ⇔ (ii): define $\phi = \widetilde{\theta} - t^{*}\theta + s^{*}\theta$, such that $\widetilde{\theta} = -d\eta - I^{*}d\eta + \eta\wedge(i_F d\eta)$ is the twist of $(d\eta, I)$ and $A = \ker(ds)|_M$. We know from Theorem 1 that closed multiplicative 2-form ψ on Σ vanishes if and only if IM form $u_{\psi} = 0$; that is, $\psi(X,\alpha) = 0$, such that $X \in TM$, $\alpha \in A$. This case can be applied for forms with higher degree; that is, 3-form ψ vanishes if and only if $\psi(X,Y,\alpha) = 0$.

Since $d\eta$ and $(d\eta)_I$ are closed, from (35) we get $i_{X\wedge Y}(d(I^{*}d\eta)) = -i_{N_I(X,Y)}d\eta$. Putting $\widetilde{\theta} = -d\eta - I^{*}d\eta + \eta\wedge(i_F d\eta)$, we obtain

$$i_{X\wedge Y} d\left(-\widetilde{\theta} + \eta\wedge\left(i_F d\eta\right)\right) = -i_{N_I(X,Y)}d\eta. \tag{51}$$

Since $d\phi = 0 \Leftrightarrow d\phi(X,Y,\alpha) = 0$, we have

$$d\phi\left(X,Y,\alpha\right) = 0 \Longleftrightarrow d\widetilde{\theta}\left(X,Y,\alpha\right) - d\left(t^{*}\theta\right)\left(X,Y,\alpha\right) + d\left(s^{*}\theta\right)\left(X,Y,\alpha\right) = 0. \tag{52}$$

On the other hand, we obtain

$$d\left(t^{*}\theta\right)\left(X,Y,\alpha\right) = d\theta\left(dt\left(X\right), dt\left(Y\right), dt\left(\alpha\right)\right). \tag{53}$$

If we take $dt = \rho$ in (53) for A, we get

$$d\left(t^{*}\theta\right)\left(X,Y,\alpha\right) = d\theta\left(dt\left(X\right), dt\left(Y\right), \rho\left(\alpha\right)\right). \tag{54}$$

On the other hand, from [17], we know that

$$Id_{\Sigma} = m\circ\left(t, Id_{\Sigma}\right). \tag{55}$$

Differentiating (55), we obtain

$$X = dt\left(X\right). \tag{56}$$

Using (56) in (54), we get

$$d\left(t^{*}\theta\right)\left(X,Y,\alpha\right) = d\theta\left(X,Y,\rho\left(\alpha\right)\right). \tag{57}$$

In a similar way, we see that

$$d\left(s^{*}\theta\right)\left(X,Y,\alpha\right) = d\theta\left(ds\left(X\right), ds\left(Y\right), ds\left(\alpha\right)\right). \tag{58}$$

Since $\alpha \in \ker ds$, then $ds(\alpha) = 0$. Hence, $d(s^{*}\theta) = 0$. Thus, we obtain

$$d\widetilde{\theta}\left(X,Y,\alpha\right) = d\theta\left(X,Y,\rho\left(\alpha\right)\right). \tag{59}$$

Using (51) in (59), we derive

$$d\left(\eta\wedge\left(i_F d\eta\right)\right)\left(X,Y,\alpha\right) + d\eta\left(N_I\left(X,Y\right),\alpha\right) = d\theta\left(X,Y,\rho\left(\alpha\right)\right). \tag{60}$$

On the other hand, it is clear that $\phi = 0 \Leftrightarrow \widetilde{\theta} - t^*\theta + s^*\theta = 0$. Thus, we obtain

$$\widetilde{\theta}(X,\alpha) = \theta(X,\rho(\alpha)). \tag{61}$$

Since $\widetilde{\theta} = -d\eta - I^* d\eta + \eta \wedge (i_F d\eta)$, we get

$$\begin{aligned} &-d\eta(X,\alpha) - d\eta(IX, I\alpha) + (\eta \wedge (i_F d\eta))(X,\alpha) \\ &\quad = \theta(X,\rho(\alpha)). \end{aligned} \tag{62}$$

Since Jacobi structure (F,π) is integrable, it defines a Lie algebroid whose anchor map is $\rho = (\pi, F)^\sharp$. Let us use $(\pi,F)^\sharp$ instead of ρ in (60) and (62); then we get

$$\begin{aligned} &d(\eta \wedge (i_F d\eta))(X,Y,\alpha) + d\eta(N_I(X,Y),\alpha) \\ &\quad = d\theta\left(X,Y,\pi^\sharp(\alpha) + fF\right), \end{aligned} \tag{63}$$

$$\begin{aligned} &-d\eta(X,\alpha) - d\eta(IX, I\alpha) + (\eta \wedge (i_F d\eta))(X,\alpha) \\ &\quad = \theta\left(X,\pi^\sharp(\alpha) + fF\right). \end{aligned} \tag{64}$$

Since $d\eta(\alpha, X) = \alpha(X)$, $(d\eta)_I(\alpha, X) = \alpha(\varphi X)$, from (63) we have

$$\begin{aligned} -\alpha(d\eta(X,Y)F) - \alpha\left(N_\varphi(X,Y)\right) &= d\theta\left(X,Y,\pi^\sharp(\alpha) + fF\right) \\ &= i_{X\wedge Y} d\theta\left(\pi^\sharp(\alpha) + fF\right) \\ &= i_{X\wedge Y} d\theta\left(\pi^\sharp(\alpha)\right) + i_{X\wedge Y} d\theta(fF) \\ &= \pi(\alpha, i_{X\wedge Y} d\theta) \\ &= -\alpha\left(\pi^\sharp(i_{X\wedge Y} d\theta)\right); \end{aligned} \tag{65}$$

that is, $\alpha(d\eta(X,Y)F + N_\varphi(X,Y)) = \alpha(\pi^\sharp(i_{X\wedge Y} d\theta))$.

Since the above equation holds for all nondegenerate α, we get

$$d\eta(X,Y)F + N_\varphi(X,Y) = \pi^\sharp(i_{X\wedge Y} d\theta). \tag{66}$$

Then, we arrive at

$$d\eta(\varphi X, \varphi Y)F + N_\varphi(X,Y) = \pi^\sharp(i_{X\wedge Y} d\theta). \tag{67}$$

On the other hand, from (64) we obtain

$$\begin{aligned} \alpha(X) + \alpha\left(\varphi^2 X\right) - \eta(X)\alpha(F) &= \pi(\alpha, i_X\theta) + i_X\theta(fF) \\ &= -\alpha\left(\pi^\sharp \theta_\sharp X\right). \end{aligned} \tag{68}$$

Thus, we get

$$\varphi^2 + \pi^\sharp\theta_\sharp = -Id + F \odot \eta. \tag{69}$$

Then (i)⇔(ii) follows from (67) and (69).

(ii) ⇔ (iii): $d\eta + I^* d\eta - \eta \wedge i_F d\eta = s^*\theta - t^*\theta$ says that (t,s) is compatible with 2-form $d\eta$. Also, it is clear that (t,s) and bivectors are compatible because Σ is a contact groupoid. We will check the compatibility of (t,s) and $(1,1)$-tensors. From compatibility condition of t and s, we get $dt \circ I = \varphi \circ dt$ and $ds \circ I = -\varphi \circ ds$.

For all $\alpha \in A$, $V \in \chi(\Sigma)$, and $f \in \mathbb{R}$, we have

$$d\eta(\alpha, V) = d\eta(\alpha, dtV), \tag{70}$$

which is equivalent to

$$\alpha(V) = \langle u_{(d\eta)}(\alpha), dtV\rangle. \tag{71}$$

Since $u_{(d\eta)} = Id$ and $u_{((d\eta)_I)} = \varphi^* \circ pr_1$ such that $u_{((d\eta)_I)}(\alpha) = \varphi^* \circ pr_1(\alpha, f) = \varphi^*(\alpha)$, we get

$$\begin{aligned} \langle \alpha, \varphi(dt(V))\rangle &= \alpha(\varphi(dt(V))) \\ &= \varphi^*\alpha(dt(V)) \\ &= \left\langle u_{((d\eta)_I)}\alpha, V\right\rangle \\ &= d\eta(\alpha, IV) \\ &= \langle \alpha, dt(IV)\rangle. \end{aligned} \tag{72}$$

Since this equation holds for all $\alpha \in A$, $a(dt) = dt(I)$. Using $s = t \circ i$,

$$\begin{aligned} a(ds(V)) &= ad(t \circ i)V \\ &= \varphi(dt(di(V))) \\ &= -\varphi(dt(V)) \\ &= -ds(IV), \end{aligned} \tag{73}$$

which shows that $a(ds) = -ds(I)$. Thus, proof is completed. □

Conflict of Interests

The author declares that she has no conflict of interests.

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