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*Research Article*

# Fixed Points of Multivalued Contractive Mappings in Partial Metric Spaces

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The aim of this paper is to present fixed point results of multivalued mappings in the framework of partial metric spaces. Some examples are presented to support the results proved herein. Our results generalize and extend various results in the existing literature. As an application of our main result, the existence and uniqueness of bounded solution of functional equations arising in dynamic programming are established.

## 1. Introduction

In 1922, Banach proved his celebrated contraction principle [1]. As it is well known, there have been published remarkable research articles about fixed points theory for different classes of contractive mappings, on some spaces such as quasi-metric spaces [2], cone metric spaces [3], convex metric spaces [4], partially ordered metric spaces [5–10], $G$-metric spaces [11–15], partial metric spaces [16, 17], quasi-partial metric spaces [18], fuzzy metric spaces [19], and Menger spaces [20]. Also, studies either on approximate fixed point or on qualitative aspects of numerical procedures for approximating fixed points are available in literature; please, see [4, 21–23].

The concept of a partial metric space is introduced by Matthews [24], as a part of the study of denotational semantics of dataflow networks. He gave a modified version of the Banach contraction principle, more suitable in this context (see also [25, 26]). In fact, (complete) partial metric spaces constitute a suitable framework to model several distinguished examples of the theory of computation and also to model metric spaces via domain theory (see [24, 27–31]).

It was shown that, in some cases, the results of fixed point in partial metric spaces can be obtained directly from their induced metric counterparts [32–34]. However, some conclusions important for the application of partial metrics in information sciences cannot be obtained in this way. For example, if $x$ is a fixed point of map $f$, then, by using the method from [32], we cannot conclude that $p(fx, fx) = 0 = p(x, x)$. For further details, we refer the reader to [35, 36].

Recently, Aydi et al. [37] introduced the concept of a partial Hausdorff metric. They initiated study of fixed point theory for multivalued mappings on partial metric space using the partial Hausdorff metric and proved an analogue of the well-known Nadler fixed point theorem.

In this paper, we obtain several fixed point results of multivalued mappings in partial metric spaces. Our results extend, unify, and generalize the comparable results in [38–41].

## 2. Preliminaries

In the sequel the letters $\mathbb{R}$, $\mathbb{R}^+$, and $\mathbb{N}^*$ will denote the set of all real numbers, the set of all nonnegative real numbers, and the set of all positive integer numbers, respectively.

Consistent with [24, 42], the following definitions and results will be needed in the sequel.

*Definition 1.* Let $X$ be a nonempty set. A function $p : X \times X \to \mathbb{R}^+$ is said to be a partial metric on $X$ if, for any $x, y, z \in X$, the following conditions hold:

($p_1$) $p(x, x) = p(y, y) = p(x, y)$ if and only if $x = y$;

($p_2$) $p(x, x) \leq p(x, y)$;

($p_3$) $p(x, y) = p(y, x)$;

($p_4$) $p(x, z) \leq p(x, y) + p(y, z) - p(y, y)$.

The pair $(X, p)$ is called a *partial metric space*.

If $p(x, y) = 0$, then ($p_1$) and ($p_2$) imply that $x = y$. But the converse does not hold in general.

A trivial example of a partial metric space is the pair $(\mathbb{R}^+, p)$, where

$$p : \mathbb{R}^+ \times \mathbb{R}^+ \longrightarrow \mathbb{R}^+, \qquad p(x, y) = \max\{x, y\}. \quad (1)$$

*Example 2* (see [24]). If $X = \{[a, b] : a, b \in \mathbb{R}, a \leq b\}$, then

$$p([a, b], [c, d]) = \max\{b, d\} - \min\{a, c\} \quad (2)$$

defines a partial metric $p$ on $X$.

For more examples of partial metric spaces, we refer to [17, 29, 31, 43–45].

Each partial metric $p$ on $X$ generates a $T_0$ topology $\tau_p$ on $X$, whose base is the family of open $p$-balls $\{B_p(x, \varepsilon) : x \in X, \varepsilon > 0\}$, where $B_p(x, \varepsilon) = \{y \in X : p(x, y) < p(x, x) + \varepsilon\}$, for all $x \in X$ and $\varepsilon > 0$.

Observe (see [24, p. 187]) that a sequence $\{x_n\}$ in a partial metric space $(X, p)$ converges to a point $x \in X$, with respect to $\tau_p$, if and only if $p(x, x) = \lim_{n\to\infty} p(x, x_n)$.

If $p$ is a partial metric on $X$, then the function

$$\begin{gathered} p^S : X \times X \longrightarrow \mathbb{R}^+, \\ p^S(x, y) = 2p(x, y) - p(x, x) - p(y, y), \end{gathered} \quad (3)$$

defines a metric on $X$.

Furthermore, a sequence $\{x_n\}$ converges in $(X, p^S)$ to a point $x \in X$ if and only if

$$\lim_{n,m\to\infty} p(x_n, x_m) = \lim_{n\to\infty} p(x_n, x) = p(x, x). \quad (4)$$

*Definition 3* (see [24]). Let $(X, p)$ be a partial metric space.

(a) A sequence $\{x_n\}$ in $X$ is said to be a *Cauchy sequence* if $\lim_{n,m\to\infty} p(x_n, x_m)$ exists and is finite.

(b) $(X, p)$ is said to be *complete* if every Cauchy sequence $\{x_n\}$ in $X$ converges with respect to $\tau_p$ to a point $x \in X$ such that $\lim_{n\to\infty} p(x, x_n) = p(x, x)$. In this case, we say that the partial metric $p$ is complete.

**Lemma 4** (see [24, 42]). *Let $(X, p)$ be a partial metric space. Then,*

(i) *a sequence $\{x_n\}$ in $X$ is a Cauchy sequence in $(X, p)$ if and only if it is a Cauchy sequence in metric space $(X, p^S)$;*

(ii) *a partial metric space $(X, p)$ is complete if and only if the metric space $(X, p^S)$ is complete.*

Let $(X, p)$ be a partial metric space. Let $P(X)$ and $P_{cl}(X)$ ($P_{CB}(X)$) be the family of all nonempty and nonempty and closed (nonempty, closed, and bounded) subsets of the partial metric space $(X, p)$. Note that here closedness is considered in $(X, \tau_p)$ ($\tau_p$ is the topology induced by $p$) while boundedness is given as follows: $A$ is a bounded subset in $(X, p)$ if there exist $x_0 \in X$ and $M \geq 0$ such that, for all $a \in A$, we have $a \in B_p(x_0, M)$; that is, $p(x_0, a) < p(a, a) + M$.

For $A, B \in P_{CB}(X)$ and $x \in X$, [37] defines

$$\begin{gathered} p(x, A) = \inf\{p(x, a), a \in A\}, \\ \delta_p(A, B) = \sup\{p(a, B) : a \in A\}, \\ \delta_p(B, A) = \sup\{p(b, A) : b \in B\}. \end{gathered} \quad (5)$$

It is easy to check that $p(x, A) = 0 \Rightarrow p^S(x, A) = 0$, where $p^S(x, A) = \inf\{p^S(x, a), a \in A\}$.

*Remark 5* (see [42]). Let $(X, p)$ be a partial metric space and let $A$ be any nonempty set in $(X, p)$. Then

$$a \in \overline{A} \quad \text{iff } p(a, A) = p(a, a), \quad (6)$$

where $\overline{A}$ denotes the closure of $A$ with respect to the partial metric $p$. Note that $A$ is closed in $(X, p)$ if and only if $A = \overline{A}$.

Let $X$ be any nonempty set and let $T : X \to P(X)$ be a given mapping. For any fixed $x_0 \in X$, a sequence $\{x_n\}$ in $X$ such that $x_{n+1} \in T(x_n)$ is called a $T$-orbital sequence about $x_0$. Collection of all such sequences will be denoted by $O(T, x_0)$. Further a point $z \in X$ is called a fixed point of $T$ if and only if $z \in T(z)$ [46]. The set of all fixed points of multivalued mapping $T$ is denoted by $F(T)$.

We have the following partial metric space version of Definition 1.8 in [47].

*Definition 6.* Let $X$ be any nonempty set, $x_0, z \in X$, and let $T : X \to P(X)$. A mapping $f : X \to \mathbb{R}$ is said to be $T$-*orbitally lower semicontinuous* at $z$ with respect to $x_0$ if $\{x_n\} \in O(T, x_0)$ and $x_n$ converges to $z$ implying that $f(z) \leq \liminf_{n\to\infty} f(x_n)$.

## 3. Fixed Points of Multivalued Mapping

In this section, we obtain several fixed point results of multivalued mappings satisfying more general contractive conditions than those of Cho et al. [47], Ćirić [48], and Feng and Liu [38] in the frame work of partial metric spaces.

**Theorem 7.** *Let $X$ be a partial metric space and let $T : X \to P_{cl}(X)$ be a multivalued mapping. Suppose that there exist* *functions* $\phi : [0,\infty) \to [0,1)$ *and* $\psi : [0,\infty) \to [c,1]$ *such that*

$$\phi(t) < \psi(t) \quad \forall t \in [0,\infty),$$
$$\limsup_{t\to r^+} \frac{\phi(t)}{\psi(t)} < 1 \quad \forall r \in [0,\infty), \tag{7}$$

*where* $c \in (0,1)$. *If, for any* $x \in X$, *there exists* $y \in T(x)$ *satisfying*

$$\psi(p(x,y))\, p(x,y) \le p(x,T(x)), \tag{8}$$

$$p(y,T(y)) \le \phi(p(x,y))\, p(x,y), \tag{9}$$

*then, for each* $x_0 \in X$, *there exists* $\{x_n\}$ *in* $O(T,x_0)$ *such that* $\{x_n\}$ *is Cauchy sequence. Further, if* $\{x_n\}$ *converges to* $z$ *and the function* $f(x) = p(x,T(x))$ *is* $T$*-orbitally lower semicontinuous at* $z$ *with respect to* $x_0$, *then* $z$ *is a fixed point of* $T$. *If* $T(z)=\{z\}$, *then* $p(z,z)=0$.

*Proof.* Let $x_0$ be a given point in $X$. Since $T(x_0) \in P_{cl}(X)$, we can choose $x_1 \in T(x_0)$ such that

$$\psi(p(x_0,x_1))\, p(x_0,x_1) \le p(x_0,T(x_0)),$$
$$p(x_1,T(x_1)) \le \phi(p(x_0,x_1))\, p(x_0,x_1). \tag{10}$$

Then, we have

$$p(x_1,T(x_1)) \le \frac{\phi(p(x_0,x_1))}{\psi(p(x_0,x_1))} p(x_0,T(x_0)). \tag{11}$$

We define $\mu : [0,\infty) \to [0,\infty)$ by $\mu(t) = \phi(t)/\psi(t)$ for all $t \in [0,\infty)$. Then by definition of $\phi$ and $\psi$, it follows that $\mu(t) < 1$ for all $t \in [0,\infty)$ and $\limsup_{t\to r^+}\mu(t) < 1$ for all $r \in [0,\infty)$. From (11), we have

$$p(x_1,T(x_1)) \le \mu(p(x_0,x_1))\, p(x_0,T(x_0)). \tag{12}$$

Continuing this way, we can obtain a sequence $\{x_n\}$ in $X$ such that $x_{n+1} \in T(x_n)$ which satisfies

$$\psi(p(x_n,x_{n+1}))\, p(x_n,x_{n+1}) \le p(x_n,T(x_n)), \tag{13}$$

$$p(x_{n+1},T(x_{n+1})) \le \phi(p(x_n,x_{n+1}))\, p(x_n,x_{n+1}). \tag{14}$$

Now by (13) and (14), we have

$$p(x_{n+1},T(x_{n+1})) \le \mu(p(x_n,x_{n+1}))\, p(x_n,T(x_n)). \tag{15}$$

As $\mu(t) < 1$, we have

$$p(x_{n+1},T(x_{n+1})) < p(x_n,T(x_n)), \tag{16}$$

for all $n \ge 0$. Thus, $\{p(x_n,T(x_n))\}$ is (strictly) decreasing sequence of positive real numbers. Consequently, there exists $\alpha \ge 0$ such that $\{p(x_n,T(x_n))\}$ converges to $\alpha$. Since $0 < c \le \psi(t)$ for all $t \in [0,\infty)$, it follows from (13) that $cp(x_n,x_{n+1}) \le \psi(p(x_n,x_{n+1}))p(x_n,x_{n+1}) \le p(x_n,T(x_n))$, and hence we have

$$0 \le p(x_n,x_{n+1}) \le \frac{1}{c} p(x_n,T(x_n)). \tag{17}$$

On taking upper limit as $n \to \infty$ on both sides of (17), we have

$$\alpha \le \limsup_{n\to\infty} \mu(p(x_n,x_{n+1}))\,\alpha, \tag{18}$$

which implies that $\alpha = 0$; that is, $\lim_{n\to\infty} p(x_n,T(x_n)) = 0$.

Now we show that $\{x_n\}$ is a Cauchy sequence. Put $\gamma = \limsup_{p(x_n,x_{n+1})\to 0^+} \mu(p(x_n,x_{n+1}))$. We can choose a real number $k \in [\gamma,1)$ such that there exists a positive integer $n_1$ such that $\mu(p(x_n,x_{n+1})) \le k$ for all $n \ge n_1$. Thus, from (15), we have $p(x_n,T(x_n)) \le kp(x_{n-1},T(x_{n-1}))$ for all $n \ge n_1$. So for all $m,n \in \mathbb{N}$ with $m > n \ge n_1+1$,

$$p(x_m,T(x_m)) \le k^{m-n} p(x_{n-1},T(x_{n-1})). \tag{19}$$

Also, from (17) and (19), we have

$$p(x_m,x_{m+1}) \le \frac{k^{m-n}}{c} p(x_{n-1},T(x_{n-1})) \tag{20}$$

for all $m > n > n_1 + 1$. Now

$$\begin{aligned} p^S&(x_m,x_{m+1}) \\ &= 2p(x_m,x_{m+1}) - p(x_m,x_m) - p(x_{m+1},x_{m+1}) \\ &\le 2p(x_m,x_{m+1}) + p(x_m,x_m) + p(x_{m+1},x_{m+1}) \\ &\le 4p(x_m,x_{m+1}) \le \frac{4}{c} k^{m-n} p(x_{n-1},T(x_{n-1})). \end{aligned} \tag{21}$$

Thus

$$\begin{aligned} p^S&(x_n,x_m) \\ &\le p^S(x_n,x_{n+1}) + p^S(x_{n+1},x_{n+2}) + \cdots + p^S(x_{m-1},x_m) \\ &\le \frac{4}{c}\left[k + k^2 + \cdots k^{m-n-1}\right] p(x_{n-1},T(x_{n-1})) \\ &\le \frac{4}{c}\left(\frac{1}{1-k}\right) p(x_{n-1},T(x_{n-1})), \end{aligned} \tag{22}$$

for all $m > n \ge n_1+1$. Using $\lim_{n\to\infty} p(x_{n-1},T(x_{n-1})) = 0$, we get that $\{x_n\}$ is a Cauchy sequence in the metric space $(O(T,x_0),p^S)$. By Lemma 4, $\{x_n\}$ is a Cauchy sequence in $(O(T,x_0),p)$.

Next, we assume that there exists an element $z$ in $O(T,x_0)$ such that

$$p(z,z) = \lim_{n\to\infty} p(x_n,z) = \lim_{n,m\to\infty} p(x_n,x_m) \tag{23}$$

and the function $p(x,T(x))$ is $T$-orbitally lower semicontinuous at $z$ with respect to $x_0$. Then it follows that

$$\begin{aligned} 0 &\le p(z,T(z)) \\ &\le \liminf_{n\to\infty} p(x_n,T(x_n)) \\ &\le \lim_{n\to\infty} p(x_n,T(x_n)) = 0. \end{aligned} \tag{24}$$

Thus $p(z,T(z)) = 0$. Since $T(z)$ is closed, $z \in T(z)$.

Now, if $T(z) = \{z\}$, then from (9) we have

$$0 \leq p(z,z) = p(z,T(z)) \leq \phi(p(z,z))\,p(z,z), \quad (25)$$

where $\phi(p(z,z)) < 1$. Hence $p(z,z) = 0$. □

*Example 8.* Let $X = \{1,2,3\}$ and let $p : X \times X \to \mathbb{R}^+$ be the partial metric defined by

$$\begin{gathered} p(1,1) = \frac{1}{4}, \qquad p(2,2) = p(3,3) = 0, \\ p(1,2) = p(2,1) = \frac{11}{24}, \\ p(1,3) = p(3,1) = \frac{1}{2}, \\ p(2,3) = p(3,2) = \frac{1}{3}. \end{gathered} \quad (26)$$

Define the mapping $T : X \to P_{cl}(X)$ by

$$Tx = \begin{cases} \{2\} & \text{if } x \in \{2,3\}, \\ \{2,3\} & \text{if } x = 1. \end{cases} \quad (27)$$

Note that $Tx$ is closed and bounded for all $x \in X$ in the partial metric space $(X,p)$. Define $\phi : [0,\infty) \to [0,1)$ and $\psi : [0,\infty) \to [c,1]$, where $c \in (0,1)$ as

$$\phi(t) = \begin{cases} \dfrac{t}{8} & \text{if } t \in [0,4), \\ \dfrac{1}{2} & \text{if } t \geq 4, \end{cases}$$
$$\psi(t) = \begin{cases} \dfrac{1}{6} & \text{if } t = 0, \\ \dfrac{t}{4} & \text{if } t \in (0,4), \\ 1 & \text{if } t \geq 4. \end{cases} \quad (28)$$

Clearly, $\phi(t) < \psi(t)$ for all $t \in [0,\infty)$ and $\limsup_{t \to r^+}(\phi(t)/\psi(t)) < 1$ for all $r \in [0,\infty)$. We will show that for all $x, y \in X$, (8) and (9) are satisfied. For this, we consider the following cases.

(i) If $x = 1$, then there exists $y = 2 \in T(1)$ such that

$$\begin{aligned} \psi(p(x,y))\,p(x,y) &= \psi(p(1,2))\,p(1,2) = \frac{121}{2304} \\ &< \frac{11}{24} = p(1,\{2,3\}) = p(x,T(x)), \\ p(y,T(y)) &= p(2,\{2\}) = 0 \\ &< \frac{121}{4608} = \phi(p(1,2))\,p(1,2) \\ &= \phi(p(x,y))\,p(x,y). \end{aligned} \quad (29)$$

(ii) When $x = 2$, then there exists $y = 2 \in T(2)$ such that

$$\begin{aligned} \psi(p(x,y))\,p(x,y) &= \psi(p(2,2))\,p(2,2) = 0 \\ &= p(2,\{2\}) = p(x,T(x)), \\ p(y,T(y)) &= p(2,\{2\}) = 0 \\ &= \phi(p(2,2))\,p(2,2) \\ &= \phi(p(x,y))\,p(x,y). \end{aligned} \quad (30)$$

(iii) For $x = 3$, there exists $y = 2 \in T(1)$ such that

$$\begin{aligned} \psi(p(x,y))\,p(x,y) &= \psi(p(3,2))\,p(3,2) = \frac{1}{36} \\ &< \frac{1}{3} = p(3,\{2\}) = p(x,T(x)), \\ p(y,T(y)) &= p(2,\{2\}) = 0 \\ &< \frac{1}{72} = \phi(p(3,2))\,p(3,2) \\ &= \phi(p(x,y))\,p(x,y). \end{aligned} \quad (31)$$

Thus, all the conditions of Theorem 7 are satisfied. Moreover, $T(2) = \{2\}$ and $p(2,2) = 0$.

The next example shows that one cannot derive the conclusion of Theorem 7 using metric induced by a partial metric.

*Example 9.* Let $X = \mathbb{R}^+$ be the partial metric space with $p(x,y) = \max\{x,y\}$. Define the mapping $T : X \to P_{cl}(X)$ by

$$Tx = \begin{cases} \left[0, \dfrac{7x}{8}\right] & \text{if } x \in [0,1], \\ \left\{\dfrac{1}{2}\right\} & \text{if } x > 1. \end{cases} \quad (32)$$

Note that $Tx$ is closed and bounded for all $x \in X$ in the partial metric space $(X,p)$. Define $\phi : [0,\infty) \to [0,1)$ and $\psi : [0,\infty) \to [c,1]$ by

$$\phi(t) = \begin{cases} \dfrac{t}{8} & \text{if } t \in [0,1), \\ \dfrac{1}{2} & \text{if } t \geq 1, \end{cases}$$
$$\psi(t) = \begin{cases} \dfrac{1}{6} & \text{if } t = 0, \\ \dfrac{t}{4} & \text{if } t \in (0,1), \\ 1 & \text{if } t \geq 1, \end{cases} \quad (33)$$

where $c \in (0,1)$. Clearly, $\phi(t) < \psi(t)$ for all $t \in [0,\infty)$ and $\limsup_{t \to r^+}(\phi(t)/\psi(t)) < 1$ for all $r \in [0,\infty)$. We will show that, for all $x, y \in X$, (8) and (9) are satisfied. For this, we consider the following cases.

When $x = 0$, then, for $y = 0 \in T(x)$, (8) and (9) are satisfied.

For $x \in (0, 1]$, take $y = x/2 \in T(x)$ such that

$$\begin{aligned} \psi(p(x,y))\,p(x,y) &= \psi\left(p\left(x,\frac{x}{2}\right)\right)p\left(x,\frac{x}{2}\right) = \frac{x^2}{4} \\ &< x = p\left(x,\left[0,\frac{7x}{8}\right]\right) = p(x,T(x)), \\ p(y,T(y)) &= p\left(\frac{x}{2},\left[0,\frac{7x}{8}\right]\right) = \frac{x}{2} \\ &< \frac{x^2}{8} = \phi\left(p\left(x,\frac{x}{2}\right)\right)p\left(x,\frac{x}{2}\right) \\ &= \phi(p(x,y))\,p(x,y). \end{aligned} \tag{34}$$

In case $x > 1$, taking $y = 1/2 \in T(x)$, we have

$$\begin{aligned} \psi(p(x,y))\,p(x,y) &= \psi\left(p\left(x,\frac{1}{2}\right)\right)p\left(x,\frac{1}{2}\right) \\ &= x = p\left(x,\left[0,\frac{7x}{8}\right]\right) = p(x,T(x)), \\ p(y,T(y)) &= p\left(\frac{1}{2},\left[0,\frac{7}{16}\right)\right) = \frac{1}{2} \\ &< \frac{x}{2} = \phi\left(p\left(x,\frac{1}{2}\right)\right)p\left(x,\frac{1}{2}\right) \\ &= \phi(p(x,y))\,p(x,y). \end{aligned} \tag{35}$$

Hence for all $x \in X$, there exists $y \in T(x)$ such that (8) and (9) are satisfied. Thus, all the conditions of Theorem 7 are satisfied. Moreover, $T(0) = \{0\}$ and $p(0, 0) = 0$.

On the other hand, we have $p^S(x, y) = |x - y|$. If we take $x \in (1/2, 1)$, then there does not exist any $y \in T(x)$ such that (8) and (9) are satisfied.

Hence we are justified in formulating the following result.

**Theorem 10.** *Let $X$ be a partial metric space and let $T : X \to P_{cl}(X)$ be a mapping. Suppose that there exist functions $\phi : [0,\infty) \to [0, 1)$, $\psi : [0,\infty) \to [c, 1]$ such that*

$$\begin{gathered} \phi(t) < \psi(t) \quad \forall t \in [0,\infty), \\ \limsup_{t\to r^+}\frac{\phi(t)}{\psi(t)} < 1 \quad \forall r \in [0,\infty), \end{gathered} \tag{36}$$

*where $c \in (0, 1)$. If for any $x \in X$ there exists $y \in T(x)$ satisfying*

$$\begin{gathered} \psi(p(x,T(x)))\,p(x,y) \le p(x,T(x)), \\ p(y,T(y)) \le \phi(p(x,T(x)))\,p(x,y), \end{gathered} \tag{37}$$

*then, for each $x_0 \in X$, there exists $\{x_n\}$ in $O(T, x_0)$ such that $\{x_n\}$ is a Cauchy sequence. Further, if $\{x_n\}$ converges to $z$ and the function $f(x) = p(x, T(x))$ is $T$-orbitally lower semicontinuous at $z$ with respect to $x_0$, then $z \in F(T)$.*

*Proof.* Let $x_0$ be a given point in $X$. As in the proof of Theorem 7, we can obtain a sequence $\{x_n\}$ in $X$ such that $x_{n+1} \in T(x_n)$, which satisfies

$$\psi(p(x_n,T(x_n)))\,p(x_n,x_{n+1}) \le p(x_n,T(x_n)), \tag{38}$$

$$p(x_{n+1},T(x_{n+1})) \le \phi(p(x_n,T(x_n)))\,p(x_n,x_{n+1}). \tag{39}$$

From $\mu(t) = \phi(t)/\psi(t)$ for all $t \in [0,\infty)$, we have

$$p(x_{n+1},T(x_{n+1})) \le \mu(p(x_n,T(x_n)))\,p(x_n,T(x_n)). \tag{40}$$

As $\mu(t) < 1$, so we have

$$p(x_{n+1},T(x_{n+1})) < p(x_n,T(x_n)), \tag{41}$$

for all $n \ge 0$, and it follows that $\{p(x_n, T(x_n))\}$ is (strictly) decreasing sequence of positive real numbers. Consequently, there exists $\beta \ge 0$ such that $\{p(x_n, T(x_n))\}$ converges to $\beta$. On taking upper limit as $n \to \infty$ on both sides of (40), we have

$$\beta \le \limsup_{n\to\infty}\mu(p(x_n,T(x_n)))\,\beta, \tag{42}$$

which implies that $\beta = 0$; that is, $\lim_{n\to\infty}p(x_n, T(x_n)) = 0$.

Now we show that $\{x_n\}$ is a Cauchy sequence. Since $0 < c \le \psi(t)$, for all $t \in [0,\infty)$, it follows from (38) that $cp(x_n, x_{n+1}) \le \psi(p(x_n, T(x_n)))p(x_n, x_{n+1}) \le p(x_n, T(x_n))$, and hence we have

$$0 \le p(x_n,x_{n+1}) \le \frac{1}{c}p(x_n,T(x_n)). \tag{43}$$

Thus the sequence $\{p(x_n, x_{n+1})\}$ is bounded.

Following arguments similar to those in the proof of Theorem 7, we obtain that $\{x_n\}$ is a Cauchy sequence in $(O(T, x_0), p)$ and $z \in T(z)$. □

Now, in the next two results, we consider further generalization of the conditions (8), (9), and (37).

**Theorem 11.** *Let $X$ be a partial metric space and let $T : X \to P_{cl}(X)$ be a multivalued mapping. Suppose that there exist functions $\phi : [0,\infty) \to [0,\infty)$, $\psi : [0,\infty) \to (0,\infty)$ such that $\phi$ is nondecreasing and subadditive and they satisfy*

$$\begin{gathered} \phi(t) < \psi(t) \quad \forall t \in [0,\infty), \\ \limsup_{t\to r^+}\frac{\phi(t)}{\psi(t)} < 1 \quad \forall r \in [0,\infty). \end{gathered} \tag{44}$$

*If for any $x \in X$ there exists $y \in T(x)$ satisfying*

$$\begin{gathered} \psi(p(x,y)) \le p(x,T(x)), \\ p(y,T(y)) \le \phi(p(x,y)), \end{gathered} \tag{45}$$

*then, for each $x_0 \in X$, there exists $\{x_n\}$ in $O(T, x_0)$ such that $\{x_n\}$ is a Cauchy sequence. Further, if $\{x_n\}$ converges to $z$ and the function $f(x) = p(x, T(x))$ is $T$-orbitally lower semicontinuous at $z$ with respect to $x_0$, then $z \in F(T)$.*

*Proof.* Let $x_0$ be a given point in $X$. Since $T(x_0) \in P_{cl}(X)$, we can choose $x_1 \in T(x_0)$ such that

$$\begin{gathered}\psi\left(p\left(x_0, x_1\right)\right) \leq p\left(x_0, T\left(x_0\right)\right), \\ p\left(x_1, T\left(x_1\right)\right) \leq \phi\left(p\left(x_0, x_1\right)\right).\end{gathered} \tag{46}$$

As before by continuing this way, we can obtain a sequence $\{x_n\}$ in $X$ such that $x_{n+1} \in T(x_n)$ which satisfies

$$\begin{gathered}\psi\left(p\left(x_n, x_{n+1}\right)\right) \leq p\left(x_n, T\left(x_n\right)\right), \\ p\left(x_{n+1}, T\left(x_{n+1}\right)\right) \leq \phi\left(p\left(x_n, x_{n+1}\right)\right).\end{gathered} \tag{47}$$

By (47), we have

$$p\left(x_{n+1}, T\left(x_{n+1}\right)\right) \leq \frac{\phi\left(p\left(x_n, x_{n+1}\right)\right)}{\psi\left(p\left(x_n, x_{n+1}\right)\right)} p\left(x_n, T\left(x_n\right)\right), \tag{48}$$

for all $n \geq 0$. We define $\mu : [0,\infty) \to [0,\infty)$ by $\mu(t) = \phi(t)/\psi(t)$ for all $t \in [0,\infty)$. Then by the definitions of $\phi$ and $\psi$, it follows that $\mu(t) < 1$ for all $t \in [0,\infty)$, and $\limsup_{t \to r^+}\mu(t) < 1$ for all $r \in [0,\infty)$. From (48), we have

$$p\left(x_{n+1}, T\left(x_{n+1}\right)\right) \leq \mu\left(p\left(x_n, x_{n+1}\right)\right) p\left(x_n, T\left(x_n\right)\right). \tag{49}$$

As $\mu(t) < 1$, so we have

$$p\left(x_{n+1}, T\left(x_{n+1}\right)\right) < p\left(x_n, T\left(x_n\right)\right), \tag{50}$$

for all $n \geq 0$. Also,

$$\begin{aligned}\phi\left(p\left(x_n, x_{n+1}\right)\right) &< \psi\left(p\left(x_n, x_{n+1}\right)\right) \\ &\leq p\left(x_n, T\left(x_n\right)\right) \leq \phi\left(p\left(x_{n-1}, x_n\right)\right).\end{aligned} \tag{51}$$

By nondecreasing $\phi$, it follows that

$$p\left(x_n, x_{n+1}\right) < p\left(x_{n-1}, x_n\right) \tag{52}$$

for all $n \geq 0$. Thus, $\{p(x_n, T(x_n))\}$ and $\{p(x_n, x_{n+1})\}$ are (strictly) decreasing sequences of positive real numbers. Consequently, there exist $\alpha, \beta \geq 0$ such that $\{p(x_n, T(x_n))\}$ converges to $\alpha$ and $\{p(x_n, x_{n+1})\}$ converges to $\beta$. Now, by taking upper limit as $n \to \infty$ in (49), we have

$$\alpha \leq \limsup_{n \to \infty} \mu\left(p\left(x_n, x_{n+1}\right)\right) \alpha, \tag{53}$$

which implies $\alpha = 0$; that is, $\lim_{n \to \infty} p(x_n, T(x_n)) = 0$.

Now we show that $\{x_n\}$ is a Cauchy sequence. There exists a real number $k \in (0,1)$ such that, for a positive integer $n_1$ with $n \geq n_1$, we have

$$p\left(x_n, T\left(x_n\right)\right) \leq k^{n-n_0} p\left(x_{n_0}, T\left(x_{n_0}\right)\right). \tag{54}$$

Clearly,

$$\begin{aligned}\phi\left(p\left(x_n, x_{n+1}\right)\right) &< \psi\left(p\left(x_n, x_{n+1}\right)\right) \leq p\left(x_n, T\left(x_n\right)\right) \\ &\leq k^{n-n_0} p\left(x_{n_0}, T\left(x_{n_0}\right)\right)\end{aligned} \tag{55}$$

for all $n > n_1$. Now

$$\begin{aligned}&\psi\left(p^S\left(x_m, x_{m+1}\right)\right) \\ &\quad = \psi\left(2p\left(x_m, x_{m+1}\right) - p\left(x_m, x_m\right) - p\left(x_{m+1}, x_{m+1}\right)\right) \\ &\quad \leq \psi\left(2p\left(x_m, x_{m+1}\right) + p\left(x_m, x_m\right) + p\left(x_{m+1}, x_{m+1}\right)\right) \\ &\quad \leq 4\psi\left(p\left(x_m, x_{m+1}\right)\right) \leq 4k^{m-n} p\left(x_{n_0}, T\left(x_{n_0}\right)\right).\end{aligned} \tag{56}$$

Thus

$$\begin{aligned}&\psi\left(p^S\left(x_n, x_m\right)\right) \\ &\quad \leq \psi\left(p^S\left(x_n, x_{n+1}\right) + p^S\left(x_{n+1}, x_{n+2}\right)\right. \\ &\qquad \left. + \cdots + p^S\left(x_{m-1}, x_m\right)\right) \\ &\quad \leq 4\left[k + k^2 + \cdots k^{m-n}\right] p\left(x_{n_0}, T\left(x_{n_0}\right)\right) \\ &\quad \leq 4\left(\frac{k^m}{1-k}\right) p\left(x_{n_0}, T\left(x_{n_0}\right)\right),\end{aligned} \tag{57}$$

for all $m > n \geq n_1 + 1$. This implies that $\{x_n\}$ is a Cauchy sequence in the metric space $(O(T, x_0), p^S)$. By Lemma 4, $\{x_n\}$ is a Cauchy sequence in $(O(T, x_0), p)$. Using arguments as in the proof of Theorem 7, we can show that the limit point of $\{x_n\}$ is a fixed point of $T$. □

In same way, we can prove the following result.

**Theorem 12.** *Let $X$ be a partial metric space and $T : X \to P_{cl}(X)$ be a multivalued mapping. Suppose that there exist $\phi : [0,\infty) \to [0,\infty)$, $\psi : [0,\infty) \to (0,\infty)$ such that $\phi$ is nondecreasing and subadditive and they satisfy*

$$\begin{gathered}\phi(t) < \psi(t) \quad \forall t \in [0,\infty), \\ \limsup_{t \to r^+} \frac{\phi(t)}{\psi(t)} < 1 \quad \forall r \in [0,\infty).\end{gathered} \tag{58}$$

*If for any $x \in X$ there exists $y \in T(x)$ satisfying*

$$\begin{gathered}\psi\left(p\left(x, y\right)\right) \leq p\left(x, T\left(x\right)\right), \\ p\left(y, T\left(y\right)\right) \leq \phi\left(p\left(x, y\right)\right),\end{gathered} \tag{59}$$

*then, for each $x_0 \in X$, there exists $\{x_n\}$ in $O(T, x_0)$ such that $\{x_n\}$ is a Cauchy sequence. Further, if $\{x_n\}$ converges to $z$ and the function $f(x) = p(x, T(x))$ is $T$-orbitally lower semicontinuous at $z$ with respect to $x_0$, then $z \in F(T)$.*

The following result generalizes and extends Theorem 3.1 in [38] to partial metric spaces.

**Corollary 13.** *Let $(X, p)$ be a partial metric space, and let $T : X \to P_{cl}(X)$ be a multivalued mapping. If there exist constants $b, c \in (0,1)$ with $c < b$ such that, for any $x \in X$, there exists $y \in T(x)$ satisfying*

$$\begin{gathered}bp\left(x, y\right) \leq p\left(x, T\left(x\right)\right), \\ p\left(y, T\left(y\right)\right) \leq cp\left(x, y\right),\end{gathered} \tag{60}$$

*then, for each* $x_0 \in X$, *there exists* $\{x_n\}$ *in* $O(T, x_0)$ *such that* $\{x_n\}$ *is a Cauchy sequence. Further, if* $\{x_n\}$ *converges to* $z$ *and the function* $f(x) = p(x, T(x))$ *is* $T$*-orbitally lower semicontinuous at* $z$ *with respect to* $x_0$, *then* $z \in F(T)$.

The following corollary is an extension of [49] and in view of Corollary 3.2 in [38] is a special case of Theorem 10.

**Corollary 14.** *Let* $(X, p)$ *be a partial metric space, and let* $T : X \to P_{cl}(X)$ *be a multivalued mapping. If there exist constants* $c \in (0, 1)$ *such that, for any* $x \in X$, *there exists* $y \in T(x)$ *satisfying*

$$p(y, T(y)) \leq cp(x, y), \tag{61}$$

*then, for each* $x_0 \in X$, *there exists* $\{x_n\}$ *in* $O(T, x_0)$ *such that* $\{x_n\}$ *is a Cauchy sequence. Further, if* $\{x_n\}$ *converges to* $z$ *and the function* $f(x) = p(x, T(x))$ *is* $T$*-orbitally lower semicontinuous at* $z$ *with respect to* $x_0$, *then* $z \in F(T)$.

**Corollary 15.** *Let* $X$ *be a partial metric space and let* $T : X \to X$ *be a self-mapping. Suppose that, there exist* $\phi : [0,\infty) \to [0,\infty)$, $\psi : [0,\infty) \to (0,\infty)$ *such that* $\phi$ *is nondecreasing and subadditive and they satisfy*

$$\begin{gathered}\phi(t) < \psi(t) \quad \forall t \in [0,\infty), \\ \limsup_{t \to r^+} \frac{\phi(t)}{\psi(t)} < 1 \quad \forall r \in [0,\infty).\end{gathered} \tag{62}$$

*If for any* $x \in X$ *there exists* $T(x) \in X$ *satisfying*

$$\begin{gathered}\psi(p(x, Tx)) \leq p(x, Tx), \\ p(Tx, T^2x) \leq \phi(p(x, Tx)),\end{gathered} \tag{63}$$

*then, for each* $x_0 \in X$, *there exists* $\{x_n\}$ *in* $O(T, x_0)$ *such that* $\{x_n\}$ *is a Cauchy sequence. Further, if* $\{x_n\}$ *converges to* $z$ *and the function* $f(x) = p(x, Tx)$ *is* $T$*-orbitally lower semicontinuous at* $z$ *with respect to* $x_0$, *then* $Tz = z$.

We remark that

(1) if $(X, p)$ is a complete partial metric space in Theorems 7 and 10, $T : X \to P_{cl}(X)$ is a multivalued mapping satisfying all the conditions of Theorems 7 and 10, and the function $f(x) := p(x, T(x))$ is lower semicontinuous on $X$, then there exists $z$ in $X$ such that $z \in F(T)$.

(2) Theorems 7, 10, and 11 extend and generalize Theorems 2.1 and 2.4 in [47], Theorem 3.1 in [38], and Theorems 2.3, 2.4, 2.7 and 2.8 in [41] to partial metric spaces.

## 4. Application

Let $U$ and $V$ be the Banach spaces with $W \subseteq U$ and $D \subseteq V$. Suppose that

$$\begin{gathered}\tau : W \times D \longrightarrow W, \qquad g, h : W \times D \longrightarrow \mathbb{R}, \\ G : W \times D \times \mathbb{R} \longrightarrow \mathbb{R}.\end{gathered} \tag{64}$$

If we consider $W$ and $D$ as the state and decision spaces, respectively, then the problem of dynamic programming reduces to the problem of solving the functional equation

$$q(x) = \sup_{y \in D} \{h(x, y) + G(x, y, q(\tau(x, y)))\}, \quad \text{for } x \in W. \tag{65}$$

Equation (65) can be reformulated as

$$q(x) = \sup_{y \in D} \{g(x, y) + G(x, y, q(\tau(x, y)))\} - b, \quad \text{for } x \in W, \tag{66}$$

where $b > 0$.

For more on problems of dynamic programming involving such functional equations, we refer the reader to [25, 50–52].

We study the existence and uniqueness of the bounded solution of the functional equation (66) arising in dynamic programming in the setup of partial metric spaces.

Let $B(W)$ denote the set of all bounded real valued functions on $W$. For an arbitrary $h \in B(W)$, define $\|h\| = \sup_{t \in W} |h(t)|$. Then $(B(W), \|\cdot\|)$ is a Banach space endowed with the metric $d$ defined as $d(h, k) = \sup_{t \in W} |ht - kt|$.

Now consider

$$p_B(h, k) = d(h, k) + b = \sup_{t \in W} |h(t) - k(t)| + b, \tag{67}$$

where $h, k \in B(W)$. Then $p_B$ is a partial metric on $B(W)$ (see also [53]).

We need the following two conditions.

($A_1$) $G$ and $g$ are bounded.

($A_2$) For $x \in W$, $h \in B(W)$, and $b > 0$, define

$$Kh(x) = \sup_{y \in D} \{g(x, y) + G(x, y, h(\tau(x, y)))\} - b. \tag{68}$$

Moreover, assume that there exist mappings $\phi : [0,\infty) \to [0,\infty)$ and $\psi : [0,\infty) \to (0,\infty)$ such that $\phi$ is nondecreasing and subadditive and they satisfy

$$\begin{gathered}\phi(t) < \psi(t), \quad \forall t \in [0,\infty), \\ \limsup_{t \to r^+} \frac{\phi(t)}{\psi(t)} < 1, \quad \forall r \in [0,\infty).\end{gathered} \tag{69}$$

Also for any $h \in B(W)$, there exists $K(h) \in B(W)$ such that

$$\begin{gathered}\psi(p_B(h(x), Kh(x))) \leq (h(x) - Kh(x)), \\ \left|Kh(x) - K^2h(x)\right| \leq \phi(p_B(h(x), Kh(x))) - b\end{gathered} \tag{70}$$

hold for all $x \in W$.

**Theorem 16.** *Assume that conditions* ($A_1$) *and* ($A_2$) *are satisfied. If* $K(B(W))$ *is a closed convex subspace of* $B(W)$, *then the functional equation* (66) *has a unique and bounded solution.*

*Proof.* Note that $(B(W), p_B)$ is a complete partial metric space. By $(\mathrm{A}_1)$, $K$ is a self-map of $B(W)$. By (68) in $(\mathrm{A}_2)$ it follows that for any $h \in B(W)$, there exists $K(h) \in B(W)$ such that

$$\begin{gathered}\psi\left(p_B\left(h\left(t\right), Kh\left(t\right)\right)\right) \leq \left(h\left(t\right) - Kh\left(t\right)\right), \\ \left|Kh\left(t\right) - K^2h\left(t\right)\right| \leq \phi\left(p_B\left(h\left(t\right), Kh\left(t\right)\right)\right) - b\end{gathered} \tag{71}$$

hold for all $t \in W$. Now, we have

$$\begin{aligned}&\left(h\left(t\right) - Kh\left(t\right)\right) \\ &\quad = h\left(t\right) - \left[\sup_{y\in D}\left\{g\left(x, y\right) + G\left(x, y, h\left(\tau\left(x, y\right)\right)\right)\right\} - b\right] \\ &\quad = h\left(t\right) - \left[\sup_{y\in D}\left\{g\left(x, y\right) + G\left(x, y, h\left(\tau\left(x, y\right)\right)\right)\right\}\right] + b \\ &\quad \leq \sup_{t\in W}\left|h\left(t\right) - \left[\sup_{y\in D}\left\{g\left(x, y\right) + G\left(x, y, h\left(\tau\left(x, y\right)\right)\right)\right\}\right]\right| + b \\ &\quad = p_B\left(h\left(t\right), Kh\left(t\right)\right).\end{aligned} \tag{72}$$

Also

$$\begin{aligned}p_B\left(K^2h\left(t\right), Kh\left(t\right)\right) &= \sup_{t\in W}\left|K^2h\left(t\right) - Kh\left(t\right)\right| + b \\ &\leq \phi\left(p_B\left(h\left(t\right), Kh\left(t\right)\right)\right).\end{aligned} \tag{73}$$

Note that the above inequalities are true for all $t \in W$, and, for any $h \in B(W)$, there exists $K(h) \in B(W)$ such that

$$\begin{gathered}\psi\left(p_B\left(h, Kh\right)\right) \leq p_B\left(h, Kh\right), \\ p_B\left(K^2h, Kh\right) \leq \phi\left(P_B\left(h, Kh\right)\right).\end{gathered} \tag{74}$$

Therefore by Corollary 15, the map $K$ has a fixed point $h^*$; that is, $h^*(x)$ is a unique and bounded solution of functional equation (66). □

## Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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