Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2014, Article ID 175982, 4 pages
http://dx.doi.org/10.1155/2014/175982

Research Article

Q-Symmetry and Conditional *Q*-Symmetries for Boussinesq Equation

Hassan A. Zedan[1,2] and Seham Sh. Tantawy[3]

[1] *Mathematics Department, Faculty of Science, King Abdulaziz University, P. O. Box 80203, Jeddah 21589, Saudi Arabia*
[2] *Mathematics Department, Faculty of Science, Kafr El-Sheikh University, Kafr El-Sheikh, Egypt*
[3] *Mathematics Department, Faculty of Education, Ain Shams University, Cairo 11566, Egypt*

Correspondence should be addressed to Hassan A. Zedan; hassanzedan2003@yahoo.com

Received 30 August 2013; Revised 26 November 2013; Accepted 27 November 2013; Published 16 January 2014

Academic Editor: Chaudry M. Khalique

We study in this paper the Q-symmetry and conditional Q-symmetries of Boussinesq equation. The solutions which we obtain, in this case, are in the form of convergent power series with easily computable components.

1. Introduction

The Boussinesq equation, which belongs to the KdV family of equations and describes motions of long waves in shallow water under gravity propagating in both directions, is given by

$$u_{tt} + u_x^2 + uu_{xx} + u_{xxx} = 0, \tag{1}$$

where $u(x,t)$ is a sufficiently often differentiable function.

A great deal of research work has been invested in recent years for the study of the Boussinesq equation. Many effective methods for obtaining exact solutions of Boussinesq equation have been proposed, such as variational iteration method [1], Travelling wave solutions [2], potential method [3], scattering method [4], the (G'/G) expansion method [5], optimal and symmetry reductions [6], and projective Riccati equations method [7].

The aim of this paper is to calculate and list the Q-symmetry and conditional Q-symmetries of Boussinesq equation. We can say today that many mathematicians, mechanicians and physicists, such as Euler, D'Alembert, Poincare, Volterra, Whittaker, Bateman, implicitly used conditional symmetries for the construction of exact solutions of the linear wave equation.

Nontrivial conditional symmetries of a PDE (partial differential equation) allow us to obtain in explicit form such solutions which cannot be found by using the symmetries of the whole set of solutions of the given PDE [8]. Moreover, conditional symmetries make the class of PDEs reduce to a system of ODEs (ordinary differential equations). As a rule, the reduced equations one obtains from conditional symmetries and from Q-symmetry are significantly simpler than those found by reduction using symmetries of the full set of solutions. This allows us to construct exact solutions of the reduced equations.

2. Conditional Q-Symmetries

The classical symmetry properties can be extended if one studies (1) together with the invariant surface of the symmetry generator as an overdetermined system of partial differential equations [9]. That is, one studies the Lie symmetry properties of the system

$$u_{tt} + u_x^2 + uu_{xx} + u_{xxx} = 0, \tag{2}$$

$$\eta(x,t,u) - \xi_1(x,t,u)\, u_x - \xi_2(x,t,u)\, u_t = 0, \tag{3}$$

where (3) is the invariant surfaces corresponding to the Lie symmetry group generator

$$Z = \xi_1(x,t,u)\frac{\partial}{\partial x} + \xi_2(x,t,u)\frac{\partial}{\partial t} + \eta(x,t,u)\frac{\partial}{\partial u}. \tag{4}$$

The invariance condition leading to conditional Q-symmetries for (2) is given by

$$Z^{(3)} F|_{\{F^{(j)}=0, Q^{(k)}=0\}} = 0, \tag{5}$$

where

$$\begin{aligned} F_1 &= u_{tt} + u_x^2 + u u_{xx} + u_{xxx}, \\ Q &= \eta(x,t,u) - \xi_1(x,t,u) u_x - \xi_2(x,t,u) u_t. \end{aligned} \tag{6}$$

Here $Z^{(3)}$ denotes the second prolongation of Z, namely,

$$\begin{aligned} Z^{(3)} = Z &+ \gamma_1 \frac{\partial}{\partial u_x} + \gamma_2 \frac{\partial}{\partial u_t} + \gamma_{11} \frac{\partial}{\partial u_{xx}} \\ &+ \gamma_{22} \frac{\partial}{\partial u_{tt}} + \gamma_{111} \frac{\partial}{\partial u_{xxx}}, \end{aligned} \tag{7}$$

where

$$\begin{aligned} \gamma_1 &= D_x(\eta_1) - u_x D_x(\xi_1) - u_t D_x(\xi_2), \\ \gamma_2 &= D_t(\eta_1) - u_x D_t(\xi_1) - u_t D_t(\xi_2), \\ \gamma_{11} &= D_x(\gamma_1) - u_{xx} D_x(\xi_1) - u_{xt} D_x(\xi_2), \\ \gamma_{22} &= D_t(\gamma_2) - u_{xt} D_t(\xi_1) - u_{tt} D_t(\xi_2), \\ \gamma_{111} &= D_x(\gamma_{11}) - u_{xxx} D_x(\xi_1) - u_{xxt} D_x(\xi_2). \end{aligned} \tag{8}$$

A generator Z which satisfies condition (5) is called a conditional Q-symmetry generator, where by the invariant surface (3). The $F^{(j)}$ and $Q^{(k)}$ denote the jth and kth prolongations, respectively. D_x and D_t denote the total derivative with respect to x and with respect to t, respectively.

We now derive the general determining equations for the conditional Q-symmetry generators for (2). We set $\xi_1 = \xi_1(x,t,u)$, $\xi_2 = \xi_2(x,t,u)$, and $\eta = \eta(x,t,u)$. The invariance condition (5) leads to the following expression:

$$\gamma_{22} + u\gamma_{11} + \eta u_{xx} + 2u_x\gamma_1 + \gamma_{111} = 0. \tag{9}$$

This leads to

$$\begin{aligned} & u_{xx}\eta + u_{tt}\eta_u + \eta_u u_{xxx} - 2u_{xt}\xi_t \\ &\quad - u_x u_{tt}\xi_u - 2u_t u_{xt}\xi_u - 3u_{xx}^2\xi_u \\ &\quad - 4u_x u_{xxx}\xi_u - 3u_{xxx}\xi_x \\ &\quad + 2u_x u_{xx}\left(u_x\eta_u + \eta_x - u_x^2\xi_u - u_x\xi_x\right) \\ &\quad + \eta_{tt} + 2u_t\eta_{tu} + u_t^2\eta_{uu} \\ &\quad + 3u_x u_{xx}\eta_{uu} + 3u_{xx}\eta_{xu} - u_x\xi_{tt} \\ &\quad - 2u_t u_x\xi_{tu} - u_t^2 u_x\xi_{uu} - 6u_x^2 u_{xx}\xi_{uu} \\ &\quad - 9u_x u_{xx}\xi_{xu} - 3u_{xx}\xi_{xx} \\ &\quad + u\left(u_{xx}\eta_u - 3u_x u_{xx}\xi_u - 2u_{xx}\xi_x + u_x^2\eta_{uu}\right. \\ &\qquad \left. + 2u_x\eta_{xu} + \eta_{xx} - u_x^3\xi_{uu} - 2u_x^2\xi_{xu} - u_x\xi_{xx}\right) \\ &\quad + u_x^3\eta_{uuu} + 3u_x^2\eta_{xuu} + 3u_x\eta_{xxu} + \eta_{xxx} - u_x^4\xi_{uuu} \\ &\quad - 3u_x^3\xi_{xuu} - 3u_x^2\xi_{xxu} - u_x\xi_{xxx} = 0. \end{aligned} \tag{10}$$

In particular, from $Q = 0$ follows

$$\xi_2(x,t,u) u_t = \eta(x,t,u) - \xi_1(x,t,u) u_x. \tag{11}$$

The determining equations for the conditional Q-symmetry generator Z are now obtained by equating to zero the coefficients of the independent coordinates. By solving this system of linear partial differential equations for the infinitesimal $\xi_1(x,t,u)$, $\xi_2(x,t,u)$, and $\eta(x,t,u)$, we obtain

$$\begin{aligned} \eta(x,t,u) &= -\frac{2k_3 u}{3}, \\ \xi_1(x,t,u) &= k_1 + \frac{2k_3 x}{3}, \\ \xi_2(x,t,u) &= k_2 + k_3 t, \end{aligned} \tag{12}$$

where k_1, k_2, and k_3 are arbitrary constants.

The conditional Q-symmetry is given by

$$Z = \left(k_1 + \frac{2k_3 x}{3}\right)\frac{\partial}{\partial x} + (k_2 + k_3 t)\frac{\partial}{\partial t} - \frac{2k_3 u}{3}\frac{\partial}{\partial u}. \tag{13}$$

The general solution of the associated invariant surface condition,

$$\left(k_1 + \frac{2k_3 x}{3}\right)\frac{\partial u}{\partial x} + (k_2 + k_3 t)\frac{\partial u}{\partial t} = -\frac{2k_3 u}{3}, \tag{14}$$

is

$$u(x,t) = \frac{\varphi(z)}{(3k_1 + 2k_3 x)}, \tag{15}$$

where $\varphi(z)$ is arbitrary function of z and

$$z(x,t) = \frac{3k_2 + 2k_3 x}{(k_2 + k_3 t)^{2/3}}. \tag{16}$$

Substituting (15) into (2), we finally obtain the following nonlinear ordinary differential equation for $\varphi(z)$ taking the form

$$\begin{aligned} & -216k_3\varphi(z) + 54\varphi^2(z) + 216k_3 z\varphi'(z) + 5z^4\varphi'(z) \\ &\quad - 72z\varphi(z)\varphi'(z) + 18z^2\varphi'^2(z) - 108k_3 z^2\varphi''(z) \\ &\quad + 2z^5\varphi''(z) + 18z^2\varphi(z)\varphi''(z) + 36k_3 z^3\varphi'''(z) = 0, \end{aligned} \tag{17}$$

where $\varphi'(z) = d\varphi_i/dz$, $\varphi''(z) = d^2\varphi_i/dz^2$, and $\varphi'''(z) = d^3\varphi_i/dz^3$.

Solving an ordinary differential equation (17), we have three cases of solutions for $\varphi(z)$.

Case 1. Consider

$$\varphi(z) = 4k_3, \tag{18}$$

where k_3 is an arbitrary constant.

Case 2. Consider

$$\varphi(z) = -\frac{1}{4}z^3, \qquad k_3 = -3. \tag{19}$$

Case 3. Consider

$$\varphi(z) = 4k_3 - \frac{1}{4}z^3, \qquad k_3 = -\frac{1}{4}. \tag{20}$$

By using (18)–(20) into (15), we have solutions for Boussinesq equation (1) in the following forms.

Family 1. Consider

$$u(x,t) = \frac{4k_3}{(3k_1 + 2k_3 x)}, \tag{21}$$

where k_1 and k_3 are arbitrary constants.

Family 2. Consider

$$u(x,t) = \frac{-(1/4)z^3}{(3k_1 - 6x)}, \tag{22}$$

where $z = (3k_1 - 6x)/(k_2 - 3t)^{2/3}$ and k_1 is an arbitrary constant.

Family 3. Consider

$$u(x,t) = -\frac{1 + (1/4)z^3}{(3k_1 - x/2)}, \tag{23}$$

where $z = (3k_1 - (1/2)x)/(k_2 - (1/4)t)^{2/3}$ and k_1 is an arbitrary constant.

3. *Q*-Symmetry Generators

Before we consider conditional symmetries of (1), let us briefly describe the classical Lie approach and introduce our notation [10]. We are concerned with a partial differential equation of order r with $m+1$ independent variables $(x_0, x_1, \ldots, x_m)$ and one field variable u, that is, an equation of the form

$$F\left(x_0, x_1, \ldots, x_m, u, \frac{\partial u}{\partial x_0}, \ldots, \frac{\partial^r u}{\partial x_{j_1} \cdots \partial x_{j_r}}\right) = 0, \tag{24}$$

where $0 \le j_1 \le j_2 \le \cdots \le j_r \le m$, $j = 0, \ldots, m$. A Lie transformation group that leaves (24) invariant is generated by a Lie symmetry generator Z, defined by

$$\begin{aligned} Z = {} & \sum_{j=0}^{m} \xi_j (x_0, x_1, \ldots, x_m, u, v) \frac{\partial}{\partial x_j} \\ & + \eta (x_0, x_1, \ldots, x_m, u) \frac{\partial}{\partial u}. \end{aligned} \tag{25}$$

Z_w is the associated vertical form of (25), defined by

$$Z_w = \left(\eta - \sum_{j=0}^{m} \xi_j u_j\right) \frac{\partial}{\partial u}, \tag{26}$$

where $Z_w|_\theta = Z|_\theta$. Here θ is a differential 1-form, called the contact form, which is defined by

$$\theta = du - \sum_{j=0}^{m} u_j dx_j. \tag{27}$$

Equation (24) is called invariant under the prolonged Lie symmetry generators Z_w if

$$L_{\check{Z}_w} F = 0. \tag{28}$$

L denotes the Lie derivative, and $\check{Z}_w$ is found by prolonging the vertical generator Z_w; that is,

$$\check{Z}_w = \sum_{j=0}^{m} D_j (U_1) \frac{\partial}{\partial u_j} + \cdots + \sum_{j_1,\ldots,j_r=0}^{m} D_{j_1,\ldots,j_r} (U_1) \frac{\partial}{\partial u_{j_1,\ldots,j_r}}, \tag{29}$$

where

$$U = \left(\eta - \sum_{j=0}^{m} \xi_j u_j\right), \tag{30}$$

and D_j is the total derivative operator. We give the definition for conditional invariance of (24) as follows.

Definition 1. Equation (24) is called Q-conditionally invariant if

$$L_{\check{Z}_w} F = 0, \tag{31}$$

under the condition

$$Z_w|_\theta = 0. \tag{32}$$

Z_w is called the Q-symmetry generator and $\check{Z}_w$ is called the prolonged vertical Q-symmetry generator. Let us now study (1) by the use of the above definition. From the definition it follows that the Lie derivative (31), for equations

$$F \equiv u_{tt} + u_x^2 + u u_{xx} + u_{xxx} = 0, \tag{33}$$

under the condition

$$Z_w|_\theta = \eta - \xi_1 u_x - \xi_2 u_t = 0, \tag{34}$$

has to be studied. Let us consider the Q-symmetry generator in the form

$$Z = \xi_1 (x,t,u) \frac{\partial}{\partial x} + \xi_2 (x,t,u) \frac{\partial}{\partial t} + \eta (x,t,u) \frac{\partial}{\partial u}. \tag{35}$$

By applying the Lie derivative (31) and condition (32), we get

$$D_{tt}(U) + u D_{xx}(U) + \eta u_{xx} + 2u_x D_x(U) + D_{xxx}(U) = 0, \tag{36}$$

where

$$\begin{aligned} D_x &= \frac{\partial}{\partial x} + u_x \frac{\partial}{\partial u} + u_{xx} \frac{\partial}{\partial u_x} + u_{xt} \frac{\partial}{\partial u_t} + \cdots \\ D_t &= \frac{\partial}{\partial t} + u_t \frac{\partial}{\partial u} + u_{tt} \frac{\partial}{\partial u_t} + u_{xt} \frac{\partial}{\partial u_x} + \cdots . \end{aligned} \tag{37}$$

The determining equations for the Q-symmetry generator Z are now obtained by equating to zero the coefficients of the independent coordinates. By solving this system of linear partial differential equations for the infinitesimal ξ_1, ξ_2, and η, we obtain

$$\begin{aligned} \eta(x,t,u) &= -\frac{2k_3u}{3}, \\ \xi_1(x,t,u) &= k_1 + \frac{2k_3x}{3}, \\ \xi_2(x,t,u) &= k_2 + k_3t. \end{aligned} \tag{38}$$

All of the similarity variables associated with the Lie symmetries (38) can be derived by solving the following characteristic equation:

$$\frac{dx}{\xi_1} = \frac{dt}{\xi_2} = \frac{du}{\eta}. \tag{39}$$

Consequently

$$\frac{dx}{(k_1 + 2k_3x/3)} = \frac{dt}{(k_2 + k_3t)} = \frac{du}{-2k_3u/3}. \tag{40}$$

We obtain the following similarity variable:

$$z(x,t) = \frac{3k_1 + 2k_3x}{(k_2 + k_3t)^{2/3}}, \tag{41}$$

and the similarity solutions take the form

$$u(x,t) = \frac{F_1(z)}{(k_2 + k_3t)^{2/3}}, \tag{42}$$

where $F_1(z)$ is arbitrary functions of z. Substituting from (42) into (1), we finally obtain nonlinear ordinary differential equation for $F_1(z)$ taking the form

$$\begin{aligned} &36k_3F_1'''(z) + 2z^2F_1''(z) + 18F_1(z)F_1''(z) \\ &\quad + 5F_1(z) + 18F'^2(z) + 9zF_1'(z) = 0, \end{aligned} \tag{43}$$

where $F_i' = d\varphi_i/dz$, $F_i'' = d^2\varphi_i/dz^2$ and $F_i''' = d^3\varphi_i/dz^3$; $(i = 1)$.

Solving a system of an ordinary differential equation (43), we have two cases of solutions for $F_1(z)$.

Case 1. Consider

$$F_1(z) = \frac{4k_3}{z}, \tag{44}$$

where k_3 is an arbitrary constant.

Case 2. Consider

$$F_1(z) = -\frac{1}{4}z^2. \tag{45}$$

Substitut from (44)-(45) into (42) to obtain the solutions for the Boussinesq equation (1) in the following forms.

Family 1. Consider

$$u(x,t) = \frac{4k_3}{z(k_2 + k_3t)^{2/3}}, \tag{46}$$

where k_2 and k_3 are an arbitrary constants.

Family 2. Consider

$$u(x,t) = -\frac{z^2}{4(k_2 + k_3t)^{2/3}}, \tag{47}$$

where $z = (3k_1 + 2k_3x)/(k_2 + k_3t)^{2/3}$ and k_2 and k_3 are arbitrary constants.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

This project was funded by the Deanship of Scientific Research (DSR), King Abdulaziz University, Jeddah, under Grant no. 109/130/1433. The authors, therefore, acknowledge the DSR technical and financial support.

References

[1] E. Salehpour, H. Jafari, and C. M. Khalique, "A modified variational iteration method for solving generalized Boussinesq equation and Lie'nard equation," *International Journal of Physical Sciences*, vol. 6, no. 23, pp. 5406–5411, 2011.

[2] M. Gandarias and M. Santosbruzón, "Travelling wave solutions for a generalized Boussinesq equation by using free software".

[3] E. Y. Abu El Seoud and M. M. Kassem, "Potential method applied to Boussinesq equation," *Applied Mathematics and Computation*, vol. 215, no. 11, pp. 3991–3997, 2010.

[4] S. Xia and J. Yuan, "Existence and scattering of small solutions to a Boussinesq type equation of sixth order," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 73, no. 4, pp. 1015–1027, 2010.

[5] H. Zedan, "New classes of solution for a system of partial differential equations by using G'/G-expansion method," *Nonlinear Science Letters A*, vol. 1, pp. 219–238, 2010.

[6] M. S. Bruzón and M. L. Gandarias, "Symmetries for a family of Boussinesq equations with nonlinear dispersion," *Communications in Nonlinear Science and Numerical Simulation*, vol. 14, no. 8, pp. 3250–3257, 2009.

[7] B.-D. Tian, Y.-H. Qiu, and N. Chen, "Exact solutions for a class of Boussinesq equation," *Applied Mathematical Sciences*, vol. 3, no. 6, pp. 257–265, 2009.

[8] W. I. Fushchych, "Conditional symmetries of the equations of mathematical physics," *Scientic Works*, vol. 5, pp. 9–16, 2003.

[9] R. Näslund, "On conditional Q-symmetries of some quasilinear hyperbolic wave equations," Research Report 11, Luleå University of Technology, Department of Mathematics, 2003.

[10] N. Eular, A. K. Ohler, and W. I. Fushchy, "Q-Symmetry generators and exact solutions for nonlinear heat conduction," *Scientific Work*, vol. 5, pp. 151–164, 2003.