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Research Article

Implicit Numerical Solutions for Solving Stochastic Differential Equations with Jumps

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To realize the applications of stochastic differential equations with jumps, much attention has recently been paid to the construction of efficient numerical solutions of the equations. Considering the fact that the use of the explicit methods often results in instability and inaccurate approximations in solving stochastic differential equations, we propose two implicit methods, the θ-Taylor method and the balanced θ-Taylor method, for numerically solving the stochastic differential equation with jumps and prove that the numerical solutions are convergent with strong order 1.0. For a linear scalar test equation, the mean-square stability regions of the methods are derived. Finally, numerical examples are given to evaluate the performance of the methods.

1. Introduction

Stochastic differential equations (SDEs) have been one of the most important mathematical tools for dealing with many problems in a variety of practical areas. However, SDEs are in general so complex that the analytical solutions can rarely be obtained. Thus, it is a common way to numerically solve SDEs. Since the explicit numerical methods often result in instability and inaccurate approximations to the solutions unless the step-size is very small, it is often necessary to use some implicit methods in numerically solving SDEs.

Generally speaking, there are two kinds of implicit numerical methods. One is the semi-implicit methods in which the drift components are computed implicitly while the diffusion components are computed explicitly. Higham [1, 2] studied the stochastic θ-method for SDEs and SDEs with jumps (SDEJs). When $\theta = 1$, the stochastic θ-method is the backward Euler method. The backward Euler method is discussed in [3–5] and the references therein. Hu and Gan [6] proposed a class of drift-implicit one-step methods for neutral stochastic delay differential equations with jump diffusion. Higham and Kloeden [3, 7] constructed the split-step backward Euler method and the compensated split-step backward Euler method for SDEJs. Ding et al. [8] introduced the split-step θ-method which is more general than the split-step backward Euler method. Wang and Gan [9] studied split-step one-leg θ methods for SDEs. Buckwar and Sickenberger [10] compared the mean-square stability properties of the θ-Maruyama and θ-Milstein methods for SDEs.

The other is the fully implicit methods in which both the drift components and the diffusion components are computed implicitly. Since implicit stochastic terms in the implicit methods lead to infinite absolute moments of the numerical solution, extensive research has been done to address this issue [11–26]. For example, Milstein et al. [11] proposed the balanced implicit method for the numerical solutions of SDEs. Burrage and Tian [12] suggested three implicit Taylor methods: the implicit Euler-Taylor method with strong order 0.5, the implicit Milstein-Taylor method with strong order 1.0, and the implicit Taylor method with strong order 1.5. Kahl and Schurz [16] introduced the balanced Milstein method for ordinary SDEs. Wang and Liu [20, 21] proposed the semi-implicit Milstein method and the split-step backward balanced Milstein method for stiff stochastic systems. Furthermore, Haghighi and Hosseini [23] developed a class of general split-step balanced numerical methods for SDEs.

Let $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\in[t_0,T]}, \mathbb{P})$ be a complete probability space with the filtration $\{\mathscr{F}_t\}_{t\in[t_0,T]}$ satisfying the usual conditions that $\mathscr{F}_t$ is right-continuous and $\mathscr{F}_0$ contains all $\mathbb{P}$-null sets. In this paper, we consider the stochastic differential equations with jumps of the form

$$\begin{gathered} \mathrm{d}x(t) = f(x(t))\,\mathrm{d}t + g(x(t))\,\mathrm{d}W(t) + h(x(t))\,\mathrm{d}N(t), \\ t \in [t_0, T]; \\ x(t_0) = x_0, \end{gathered} \tag{1}$$

where $W(t)$ is $\mathscr{F}_t$-adapted Wiener process and $N(t)$ is a scalar poisson process with intensity λ and is independent of $W(t)$. Hu and Gan [22, 25] proposed the balanced method for SDEJs (1) and stochastic pantograph equations with jumps, respectively, and proved that the numerical solution converges to the analytical solution with rate 1/2. The asymptotic stability of the balanced method for SDEJs (1) was obtained in [26]. To obtain higher order numerical schemes and improve the accuracy of the numerical solutions, we propose two kinds of implicit Taylor methods and prove that the numerical solutions converge to the true solutions of SDEJs (1) with rate 1.0.

The rest of the paper is arranged as follows. In Section 2, we introduce the θ-Taylor methods and the fully implicit balanced θ-Taylor methods for SDEJs (1). The strong convergence properties of these implicit methods are proved in Section 3. The mean-square stability of the numerical solutions is discussed in Section 4. Some numerical experiments are performed in Section 5 to evaluate the performance of the proposed numerical methods.

2. The Numerical Methods

Define a mesh $0 \le t_0 < t_1 < \cdots < t_n < t_{n+1} < \cdots < t_N = T$ on the time interval $[t_0, T]$ with $t_n = n\Delta$ and the step-size $\Delta = T/N$. x_n is the numerical approximation to $x(t_n)$. Based on appropriate stochastic Taylor expansions, Maghsoodi [27] generalized the Milstein scheme to SDEJs and obtained the order 1.0 strong Taylor scheme (Taylor for short) as

$$\begin{aligned} x_{n+1} = x_n &+ f(x_n)\Delta + g(x_n)\Delta W_n \\ &+ \frac{1}{2} g(x_n) g'(x_n)\left[(\Delta W_n)^2 - \Delta\right] \\ &+ \frac{1}{2}(3h(x_n) - h_h(x_n))\Delta N_n \\ &+ (g_h(x_n) - g(x_n))\Delta W_n \Delta N_n \\ &+ \frac{1}{2}(h_h(x_n) - h(x_n))(\Delta N_n)^2 \\ &+ \left(g(x_n)h'(x_n) - g_h(x_n) + g(x_n)\right)\Delta Z_n, \end{aligned} \tag{2}$$

where $g_h(x) = g(x + h(x))$, $h_h(x) = h(x + h(x))$, and $\Delta Z_n = \int_{t_n}^{t_{n+1}} \int_{t_n}^{s} \mathrm{d}W_t \mathrm{d}N_s = \int_{t_n}^{t_{n+1}} (W_s - W_{t_n})\mathrm{d}N_s$.

Note that $\Delta Z_n = \sum_j (W(\tau_j) - W(t_k)) = \sum_{j=N_n+1}^{N_{n+1}} (N_{n+1} - j + 1)(W(\tau_j) - W(\tau_{j-1}))$ [28]. Given a jump time τ_j in $[t_n, t_{n+1})$, $\Delta Z_{1n}(\tau_j) = W(\tau_j) - W(\tau_{j-1}) \sim N(0, \tau_j - \tau_{j-1})$ $(N_n + 1 \le j \le N_{n+1})$. In addition, the random variable $\Delta W_n = W(t_{n+1}) - W(t_n)$ is dependent on $\Delta Z_{1n}(\tau_j)$, and its sample values can be calculated by $\Delta W_n = \sum_{j=N_n+1}^{N_{n+1}} \Delta Z_{1n}(\tau_j) + \Delta Z_{1n}(t_{n+1})$ where $\Delta Z_{1n}(t_{n+1}) = W(t_{n+1}) - W(\tau_{N_{n+1}}) \sim N(0, t_{n+1} - \tau_{N_{n+1}})$.

By changing the explicit deterministic term into implicit term, we have the following θ-Taylor method:

$$\begin{aligned} x_{n+1} = x_n &+ \Delta\left[(1-\theta) f(x_n) + \theta f(x_{n+1})\right] \\ &+ g(x_n)\Delta W_n + \frac{1}{2} g(x_n) g'(x_n)\left[(\Delta W_n)^2 - \Delta\right] \\ &+ \frac{1}{2}(3h(x_n) - h_h(x_n))\Delta N_n \\ &+ (g_h(x_n) - g(x_n))\Delta W_n \Delta N_n \\ &+ \frac{1}{2}(h_h(x_n) - h(x_n))(\Delta N_n)^2 \\ &+ \left(g(x_n)h'(x_n) - g_h(x_n) + g(x_n)\right)\Delta Z_n. \end{aligned} \tag{3}$$

Note that the θ-Taylor method (3) becomes the Taylor method (2) when $\theta = 0$.

Using the idea of the balanced implicit method and combining it with the θ-Taylor method, we have the following balanced θ-Taylor method:

$$\begin{aligned} x_{n+1} = x_n &+ \Delta\left[(1-\theta) f(x_n) + \theta f(x_{n+1})\right] \\ &+ g(x_n)\Delta W_n + \frac{1}{2} g(x_n) g'(x_n)\left[(\Delta W_n)^2 - \Delta\right] \\ &+ \frac{1}{2}(3h(x_n) - h_h(x_n))\Delta N_n \\ &+ (g_h(x_n) - g(x_n))\Delta W_n \Delta N_n \\ &+ \frac{1}{2}(h_h(x_n) - h(x_n))(\Delta N_n)^2 \\ &+ \left(g(x_n)h'(x_n) - g_h(x_n) + g(x_n)\right)\Delta Z_n \\ &+ C_n(x_n - x_{n+1}), \end{aligned} \tag{4}$$

where $C_n = C(x_n) = C_0(x_n)\Delta + C_1(x_n)[(\Delta W_n)^2 - \Delta]$ with $C_0(\cdot)$ and $C_1(\cdot)$ called control functions.

3. Convergence of the Implicit Taylor Methods

Let $|\cdot|$ be the Euclidean norm in $\mathbb{R}^d$. If A is a matrix, $|A| = \sqrt{\operatorname{trace}(A^T A)}$. Denote $\|z\|_{L_2} = (\mathbb{E}|z|^2)^{1/2}$ for $z \in \mathbb{R}^d$. To prove the convergence of the numerical solutions, we make the following assumptions.

Assumption 1. The coefficient functions f, g, and h satisfy the global Lipschitz condition

$$\begin{aligned}&\left|f\left(x_1\right)-f\left(x_2\right)\right|+\left|g\left(x_1\right)-g\left(x_2\right)\right|\\&\quad+\left|h\left(x_1\right)-h\left(x_2\right)\right|\\&\quad+\left|g\left(x_1\right)g'\left(x_1\right)-g\left(x_2\right)g'\left(x_2\right)\right|\\&\quad+\left|g\left(x_1\right)h'\left(x_1\right)-g\left(x_2\right)h'\left(x_2\right)\right|\leq L\left|x_1-x_2\right|\end{aligned}\tag{5}$$

for a positive constant L and any $x_1, x_2 \in \mathbb{R}^d$ and the linear growth condition

$$\begin{aligned}&|f(x)|^2+|g(x)|^2+|h(x)|^2\\&\quad+\left|g(x)g'(x)\right|^2+\left|g(x)h'(x)\right|^2\leq L'\left(1+|x|^2\right)\end{aligned}\tag{6}$$

for a positive constant L' and any $x \in \mathbb{R}^d$.

Assumption 2. The $C_0(\cdot)$ and $C_1(\cdot)$ are bounded $d\times d$-matrix-valued functions. For any real numbers $\alpha_0 \in [0, \overline{\alpha}_0]$ and $\alpha_1 \in [-\overline{\alpha}_1, \overline{\alpha}_1]$ with $\overline{\alpha}_0 \geq \Delta$ and $\overline{\alpha}_1 \geq |(\Delta W_n)^2 - \Delta|$ for all step-size Δ and $x \in \mathbb{R}^d$, the matrix $M(x) = I + \alpha_0 C_0(x) + \alpha_1 C_1(x)$ is reversible and satisfies $|(M(x))^{-1}| \leq B < \infty$, where I is a unit matrix and B is a positive constant.

In what follows, we will derive the strong convergence orders of the implicit Taylor methods for SDEJs (1).

3.1. Convergence of the θ-Taylor Method. Define

$$\begin{aligned}\overline{x}^{\theta}\left(t_{n+1}\right)&=x\left(t_n\right)\\&\quad+\Delta\left[(1-\theta)f\left(x\left(t_n\right)\right)\right.\\&\quad\left.+\theta f\left(x\left(t_{n+1}\right)\right)\right]+g\left(x\left(t_n\right)\right)\Delta W_n\\&\quad+\frac{1}{2}g\left(x\left(t_n\right)\right)g'\left(x\left(t_n\right)\right)\left[\left(\Delta W_n\right)^2-\Delta\right]\\&\quad+\frac{1}{2}\left(3h\left(x\left(t_n\right)\right)-h_h\left(x\left(t_n\right)\right)\right)\Delta N_n\\&\quad+\left(g_h\left(x\left(t_n\right)\right)-g\left(x\left(t_n\right)\right)\right)\Delta W_n\Delta N_n\\&\quad+\frac{1}{2}\left(h_h\left(x\left(t_n\right)\right)-h\left(x\left(t_n\right)\right)\right)\left(\Delta N_n\right)^2\\&\quad+\left(g\left(x\left(t_n\right)\right)h'\left(x\left(t_n\right)\right)\right.\\&\quad\left.-g_h\left(x\left(t_n\right)\right)+g\left(x\left(t_n\right)\right)\right)\Delta Z_n\end{aligned}\tag{7}$$

by replacing the numerical approximations with the exact solution values on the right-hand side of equation (3). Then, the local error of method (3) is defined by $\delta^{\theta}(t_{n+1}) = x(t_{n+1}) - \overline{x}^{\theta}(t_{n+1})$ and the global error of method (3) is defined by $\epsilon_n = x(t_n) - x_n$.

Theorem 3. *Under Assumption 1, the θ-Taylor method* (3) *is consistent with order 2 in the mean and with order 1.5 in the mean square. That is, the local mean error and mean-square error of the θ-Taylor method* (3) *satisfy*

$$\max_{0\leq n\leq N-1}\left\|\mathbb{E}(\delta^{\theta}\left(t_{n+1}\right)\mid\mathcal{F}_{t_n})\right\|_{L_2}\leq H_1\Delta^2\qquad\text{as }\Delta\longrightarrow 0,\tag{8}$$

$$\max_{0\leq n\leq N-1}\left\|\delta^{\theta}(t_{n+1})\right\|_{L_2}\leq H_2\Delta^{3/2}\qquad\text{as }\Delta\longrightarrow 0,\tag{9}$$

where the constants H_1 and H_2 are independent of Δ.

Proof. To obtain the convergence rate of the θ-Taylor method, we firstly introduce the local Taylor numerical approximation $\overline{x}^{A}_{n+1}$ which is defined by

$$\begin{aligned}\overline{x}^{A}_{n+1}&=x\left(t_n\right)+f\left(x\left(t_n\right)\right)\Delta+g\left(x\left(t_n\right)\right)\Delta W_n\\&\quad+\frac{1}{2}g\left(x\left(t_n\right)\right)g'\left(x\left(t_n\right)\right)\left[\left(\Delta W_n\right)^2-\Delta\right]\\&\quad+\frac{1}{2}\left(3h\left(x\left(t_n\right)\right)-h_h\left(x\left(t_n\right)\right)\right)\Delta N_n\\&\quad+\left(g_h\left(x\left(t_n\right)\right)-g\left(x\left(t_n\right)\right)\right)\Delta W_n\Delta N_n\\&\quad+\frac{1}{2}\left(h_h\left(x\left(t_n\right)\right)-h\left(x\left(t_n\right)\right)\right)\left(\Delta N_n\right)^2\\&\quad+\left(g\left(x\left(t_n\right)\right)h'\left(x\left(t_n\right)\right)-g_h\left(x\left(t_n\right)\right)\right.\\&\quad\left.+g\left(x\left(t_n\right)\right)\right)\Delta Z_n.\end{aligned}\tag{10}$$

Then, there exists some constant $K_1 > 0$ such that

$$\begin{aligned}&\mathbb{E}\left[\left|\mathbb{E}\left[\left(\overline{x}^{\theta}_{n+1}-x\left(t_{n+1}\right)\right)\mid\mathcal{F}_{t_n}\right]\right|^2\right]\\&\quad\leq 2\mathbb{E}\left[\left|\mathbb{E}\left[\left(\overline{x}^{A}_{n+1}-x\left(t_{n+1}\right)\right)\mid\mathcal{F}_{t_n}\right]\right|^2\right]\\&\qquad+2\mathbb{E}\left[\left|\mathbb{E}\left[\left(\overline{x}^{\theta}_{n+1}-\overline{x}^{A}_{n+1}\right)\mid\mathcal{F}_{t_n}\right]\right|^2\right]\\&\quad\leq K_1\Delta^4+2\mathbb{E}\left[\left|\mathbb{E}\left[\left(\overline{x}^{\theta}_{n+1}-\overline{x}^{A}_{n+1}\right)\mid\mathcal{F}_{t_n}\right]\right|^2\right].\end{aligned}\tag{11}$$

Since

$$\begin{aligned}&\mathbb{E}\left[\left|\mathbb{E}\left[\left(\overline{x}^{\theta}_{n+1}-\overline{x}^{A}_{n+1}\right)\mid\mathcal{F}_{t_n}\right]\right|^2\right]\\&\quad=\mathbb{E}\left[\left|\mathbb{E}\left[\theta\Delta\left(f\left(x\left(t_{n+1}\right)\right)-f\left(x\left(t_n\right)\right)\right)\mid\mathcal{F}_{t_n}\right]\right|^2\right]\\&\quad\leq L\theta\Delta\mathbb{E}\left[\mathbb{E}\left[\left|x(t_{n+1})-x(t_n)\right|\mid\mathcal{F}_{t_n}\right]\right]^2\\&\quad\leq O\left(\Delta^4\right),\end{aligned}\tag{12}$$

we obtain $(\mathbb{E}[|\mathbb{E}[(\overline{x}^{\theta}_{n+1} - x(t_{n+1})) \mid \mathcal{F}_{t_n}]|^2])^{1/2} \leq O(\Delta^2)$.

On the other hand, since

$$\begin{aligned}\mathbb{E}\left[\left|\bar{x}_{n+1}^{\theta}-\bar{x}_{n+1}^{A}\right|^{2}\mid x_n=x(t_n)\right] &= L^2\theta^2\Delta^2\mathbb{E}\left[\left|x(t_{n+1})-x(t_n)\right|^2\right] \\ &\leq O\left(\Delta^3\right),\end{aligned}\tag{13}$$

we have

$$\begin{aligned}\mathbb{E}\left[\left|\bar{x}_{n+1}^{\theta}-x(t_{n+1})\right|^2\right] &\leq 2\mathbb{E}\left[\left|\bar{x}_{n+1}^{A}-x(t_{n+1})\right|^2\right] \\ &\quad+2\mathbb{E}\left[\left|\bar{x}_{n+1}^{\theta}-\bar{x}_{n+1}^{A}\right|^2\right] \\ &\leq O\left(\Delta^3\right).\end{aligned}\tag{14}$$

Therefore, the result (9) is obtained. □

Theorem 4. *Under Assumption 1, the θ-Taylor method (3) is convergent with order 1 in the mean square. That is, the global error satisfies*

$$\max_{0\leq n\leq N-1}\left\|\epsilon_{n+1}^2\right\|_{L_2}\leq H_3\Delta \quad \text{as } \Delta\longrightarrow 0,\tag{15}$$

where H_3 is independent of Δ.

Proof. From the definitions of δ_n and ϵ_n, we have

$$\epsilon_{n+1}=\epsilon_n+u_n+\delta_{n+1},\tag{16}$$

where

$$\begin{aligned}u_n &= \Delta(1-\theta)\left(f(x(t_n))-f(x_n)\right) \\ &\quad+\Delta\theta\left(f(x(t_{n+1}))-f(x_{n+1})\right) \\ &\quad+\left(g(x(t_n))-g(x_n)\right)\Delta W_n \\ &\quad+\frac{1}{2}\Big(g(x(t_n))g'(x(t_n)) \\ &\qquad-g(x_n)g'(x_n)\Big)\left[(\Delta W)^2-\Delta\right] \\ &\quad+\frac{1}{2}\left[3\left(h(x(t_n))-h(x_n)\right)\right. \\ &\qquad\left.-\left(h_h(x(t_n))-h_h(x_n)\right)\right]\Delta N_n \\ &\quad+\left[\left(g_h(x(t_n))-g_h(x_n)\right)\right. \\ &\qquad\left.-\left(g(x(t_n))-g(x_n)\right)\right]\Delta W_n\Delta N_n \\ &\quad+\frac{1}{2}\left[\left(h_h(x(t_n))-h_h(x_n)\right)\right. \\ &\qquad\left.-\left(h(x(t_n))-h(x_n)\right)\right](\Delta N_n)^2 \\ &\quad+\Big[\left(g(x(t_n))h'(x(t_n))-g(x_n)h'(x_n)\right) \\ &\qquad-\left(g_h(x(t_n))-g_h(x_n)\right) \\ &\qquad+\left(g(x(t_n))-g(x_n)\right)\Big]\Delta Z_n.\end{aligned}\tag{17}$$

Since ϵ_n is $\mathcal{F}_{t_n}$-measurable, we have from Theorem 3 that

$$\begin{aligned}\left|\mathbb{E}\langle\delta_{n+1},\epsilon_n\rangle\right| &= \left|\mathbb{E}\left[\mathbb{E}\left(\langle\delta_{n+1},\epsilon_n\rangle\mid\mathcal{F}_{t_n}\right)\right]\right| \\ &\leq \mathbb{E}\left|\left\langle\mathbb{E}\left(\delta_{n+1}\mid\mathcal{F}_{t_n}\right),\epsilon_n\right\rangle\right| \\ &\leq \left[\Delta^{-1}\left(\mathbb{E}\left|\mathbb{E}\left(\delta_{n+1}\mid\mathcal{F}_{t_n}\right)\right|^2\right)\right]^{1/2} \\ &\qquad\times\left(\Delta\mathbb{E}|\epsilon_n|^2\right)^{1/2} \\ &\leq \Delta^{-1}\left(\mathbb{E}\left|\mathbb{E}\left(\delta_{n+1}\mid\mathcal{F}_{t_n}\right)\right|^2\right) \\ &\qquad+\Delta\mathbb{E}|\epsilon_n|^2 \\ &\leq H_1\Delta^3+\Delta\mathbb{E}|\epsilon_n|^2,\end{aligned}\tag{18}$$

where $\langle\cdot,\cdot\rangle$ indicates the scalar product.

Noting that $\mathbb{E}|\Delta W_n|^2=\Delta$, $\mathbb{E}|\Delta W_n|^4=3\Delta^2$, $\mathbb{E}(\Delta N_n)^2=\lambda\Delta(1+\lambda\Delta)$, $\mathbb{E}(\Delta N_n)^4=\lambda\Delta(1+7\lambda\Delta+6(\lambda\Delta)^2+(\lambda\Delta)^3)$, and ΔW_n is independent of ΔN_n, we have from Assumption 1 that

$$\begin{aligned}&\mathbb{E}\left|\left(g(x(t_n))-g(x_n)\right)\Delta W_n\right|^2 \\ &\leq \Delta\mathbb{E}\left|g(x(t_n))-g(x_n)\right|^2\leq L^2\Delta\mathbb{E}|\epsilon_n|^2; \\ &\mathbb{E}\left|\left(g(x(t_n))g'(x(t_n))-g(x_n)g'(x_n)\right)\left[(\Delta W)^2-\Delta\right]\right|^2 \\ &\leq 2\Delta^2\mathbb{E}\left|g(x(t_n))g'(x(t_n))-g(x_n)g'(x_n)\right| \\ &\leq 2L^2\Delta^2\mathbb{E}|\epsilon_n|^2; \\ &\mathbb{E}\left|\left[3\left(h(x(t_n))-h(x_n)\right)-\left(h_h(x(t_n))-h_h(x_n)\right)\right]\Delta N_n\right|^2 \\ &\leq 2\lambda\Delta(1+\lambda\Delta)\Big[3\mathbb{E}\left|h(x(t_n))-h(x_n)\right|^2 \\ &\qquad+\mathbb{E}\left|h(x(t_n)+h(x(t_n)))-h(x_n+h(x_n))\right|^2\Big] \\ &\leq 2L^2\lambda\Delta(1+\lambda\Delta)\left(5+2L^2\right)\mathbb{E}|\epsilon_n|^2; \\ &\mathbb{E}\left|\left[\left(g_h(x(t_n))-g_h(x_n)\right)-\left(g(x(t_n))-g(x_n)\right)\right]\Delta W_n\Delta N_n\right|^2 \\ &\leq \mathbb{E}\left|\left(g_h(x(t_n))-g_h(x_n)\right)-\left(g(x(t_n))-g(x_n)\right)\right|^2 \\ &\qquad\cdot\mathbb{E}|\Delta W_n|^2\cdot\mathbb{E}(\Delta N_n)^2 \\ &\leq 2L^2\lambda\Delta^2(1+\lambda\Delta)\left(3+2L^2\right)\mathbb{E}|\epsilon_n|^2; \\ &\mathbb{E}\left|\left[\left(h_h(x(t_n))-h_h(x_n)\right)-\left(h(x(t_n))-h(x_n)\right)\right](\Delta N_n)^2\right|^2 \\ &\leq 2\lambda\Delta\left(1+7\lambda\Delta+6(\lambda\Delta)^2+(\lambda\Delta)^3\right)L^2\left(3+2L^2\right)\mathbb{E}|\epsilon_n|^2; \\ &\mathbb{E}\Big|\Big[\left(g(x(t_n))h'(x(t_n))-g(x_n)h'(x_n)\right) \\ &\qquad-\left(g_h(x(t_n))-g_h(x_n)\right)+\left(g(x(t_n))-g(x_n)\right)\Big]\Delta Z_n\Big|^2\end{aligned}$$

$$\leq 6L^2\left(2+L^2\right)\mathbb{E}\left|\epsilon_n\right|^2\cdot\mathbb{E}\left|\Delta Z_n\right|^2$$
$$\leq 6L^2\left(2+L^2\right)\mathbb{E}\left|\epsilon_n\right|^2\cdot\mathbb{E}\left|\Delta W_n-\Delta Z_{1n}(t_{n+1})\right|^2$$
$$\leq 24L^2\left(2+L^2\right)\Delta\mathbb{E}\left|\epsilon_n\right|^2. \tag{19}$$

Hence,

$$\mathbb{E}\left|u_n\right|^2\leq J_1\Delta\mathbb{E}\left|\epsilon_n\right|^2+8L^2\Delta^2\theta^2\mathbb{E}\left|\epsilon_{n+1}\right|^2, \tag{20}$$

where $J_1=8[4L^2+2L^2\lambda(1+\lambda)(5+2L^2)+2L^2\lambda(1+\lambda)(3+2L^2)+2\lambda(1+7\lambda+6\lambda^2+\lambda^3)L^2(3+2L^2)+24L^2(2+L^2)$.

Noting that $x(t_n)$ and x_n are $\mathcal{F}_{t_n}$-measurable and ΔW_n and ΔN_n are independent of $\mathcal{F}_{t_n}$, we have

$$\begin{aligned}
&\mathbb{E}\left[\left(g\left(x\left(t_n\right)\right)-g\left(x_n\right)\right)\Delta W_n\mathcal{F}_{t_n}\right]=0;\\
&\mathbb{E}\left[\left(g\left(x\left(t_n\right)\right)g'\left(x\left(t_n\right)\right)-g\left(x_n\right)g'\left(x_n\right)\right)\right.\\
&\qquad\left.\times\left((\Delta W)^2-\Delta\right)\mid\mathcal{F}_{t_n}\right]=0;\\
&\mathbb{E}\left[\left[\left(g_h\left(x\left(t_n\right)\right)-g_h\left(x_n\right)\right)-\left(g\left(x\left(t_n\right)\right)-g\left(x_n\right)\right)\right]\right.\\
&\qquad\left.\times\Delta W_n\Delta N_n\mid\mathcal{F}_{t_n}\right]=0;\\
&\mathbb{E}\left[\left[\left(g\left(x\left(t_n\right)\right)h'\left(x\left(t_n\right)\right)-g\left(x_n\right)h'\left(x_n\right)\right)\right.\right.\\
&\qquad-\left(g_h\left(x\left(t_n\right)\right)-g_h\left(x_n\right)\right)\\
&\qquad\left.\left.+\left(g\left(x\left(t_n\right)\right)-g\left(x_n\right)\right)\right]\Delta Z_n\mid\mathcal{F}_{t_n}\right]=0.
\end{aligned} \tag{21}$$

Therefore,

$$\begin{aligned}
&\left|\mathbb{E}\left(u_n\mid\mathcal{F}_{t_n}\right)\right|^2\\
&\quad\leq\left|\Delta(1-\theta)\mathbb{E}\left[f\left(x\left(t_n\right)\right)-f\left(x_n\right)\mid\mathcal{F}_{t_n}\right]\right.\\
&\qquad+\theta\Delta\mathbb{E}\left[f\left(x\left(t_{n+1}\right)\right)-f\left(x_{n+1}\right)\mid\mathcal{F}_{t_n}\right]\\
&\qquad+\frac{3}{2}\mathbb{E}\left[\left(h\left(x\left(t_n\right)\right)-h\left(x_n\right)\right)\Delta N_n\mid\mathcal{F}_{t_n}\right]\\
&\qquad-\frac{1}{2}\mathbb{E}\left[\left(h_h\left(x\left(t_n\right)\right)-h_h\left(x_n\right)\right)\Delta N_n\mid\mathcal{F}_{t_n}\right]\\
&\qquad+\frac{1}{2}\mathbb{E}\left[\left(h_h\left(x\left(t_n\right)\right)-h_h\left(x_n\right)\right)\left(\Delta N_n\right)^2\mid\mathcal{F}_{t_n}\right]\\
&\qquad\left.-\frac{1}{2}\mathbb{E}\left[\left(h\left(x\left(t_n\right)\right)-h\left(x_n\right)\right)\left(\Delta N_n\right)^2\mid\mathcal{F}_{t_n}\right]\right|^2\\
&\quad\leq J_2\Delta^2\mathbb{E}\left|\epsilon_n\right|^2+6L^2\theta^2\Delta^2\mathbb{E}\left[\left|\epsilon_{n+1}\right|^2\mid\mathcal{F}_{t_n}\right],
\end{aligned} \tag{22}$$

where $J_2=3L^2[2+3\lambda^2+(1+L)^2\lambda^2+(1+L)^2(1+\lambda)^2+\lambda^2(1+\lambda)^2]$. Thus,

$$\begin{aligned}
\left|\mathbb{E}\left\langle\epsilon_n,u_n\right\rangle\right|&\leq\Delta^{-1}\left(\mathbb{E}\left|\mathbb{E}\left(u_n\mid\mathcal{F}_{t_n}\right)\right|^2\right)+\Delta\mathbb{E}\left|\epsilon_n\right|^2\\
&\leq\left(1+J_2\right)\Delta\mathbb{E}\left|\epsilon_n\right|^2+6L^2\theta^2\Delta\mathbb{E}\left|\epsilon_{n+1}\right|^2.
\end{aligned} \tag{23}$$

From the above arguments, we obtain

$$\begin{aligned}
&\mathbb{E}\left|\epsilon_{n+1}\right|^2\\
&\quad\leq\mathbb{E}\left|\epsilon_n\right|^2+\mathbb{E}\left|\delta_{n+1}\right|^2+\mathbb{E}\left|u_n\right|^2\\
&\qquad+2\left|\mathbb{E}\left\langle\delta_{n+1},\epsilon_n\right\rangle\right|+2\left|\mathbb{E}\left\langle\epsilon_n,u_n\right\rangle\right|\\
&\quad\leq\left[1+2\left(2+J_1+J_2\right)\Delta\right]\mathbb{E}\left|\epsilon_n\right|^2\\
&\qquad+2L^2\theta^2\Delta\left(\Delta+6\right)\mathbb{E}\left|\epsilon_{n+1}\right|^2\\
&\qquad+\left(H_2+2H_1\right)\Delta^3.
\end{aligned} \tag{24}$$

Because $\Delta\to 0$, we can assume $1-2L^2\theta^2\Delta(\Delta+6)>0$ without loss of generality. Let $J_3=2(2+J_1+J_2)$. Then,

$$\begin{aligned}
\mathbb{E}\left|\epsilon_{n+1}\right|^2&\leq\left(1+J_3\Delta\right)\mathbb{E}\left|\epsilon_n\right|^2+\left(H_2+2H_1\right)\Delta^3\\
&=\left(H_2+2H_1\right)\Delta^2\frac{\left(1+J_3\Delta\right)^{n+1}-1}{J_3}\leq J_4\Delta^2,
\end{aligned} \tag{25}$$

where $J_4=(H_2+2H_1)((e^{J_3T}-1)/J_3)$. □

3.2. Convergence of the Balanced θ-Taylor Method. Define

$$\begin{aligned}
&\overline{x}^B\left(t_{n+1}\right)\\
&\quad=x\left(t_n\right)+\Delta\left[(1-\theta)f\left(x\left(t_n\right)\right)+\theta f\left(x\left(t_{n+1}\right)\right)\right]\\
&\qquad+g\left(x\left(t_n\right)\right)\Delta W_n\\
&\qquad+\frac{1}{2}g\left(x\left(t_n\right)\right)g'\left(x\left(t_n\right)\right)\left[\left(\Delta W_n\right)^2-\Delta\right]\\
&\qquad+\frac{1}{2}\left(3h\left(x\left(t_n\right)\right)-h_h\left(x\left(t_n\right)\right)\right)\Delta N_n\\
&\qquad+\left(g_h\left(x\left(t_n\right)\right)-g\left(x\left(t_n\right)\right)\right)\Delta W_n\Delta N_n\\
&\qquad+\frac{1}{2}\left(h_h\left(x\left(t_n\right)\right)-h\left(x\left(t_n\right)\right)\right)\left(\Delta N_n\right)^2\\
&\qquad+\left(g\left(x\left(t_n\right)\right)h'\left(x\left(t_n\right)\right)-g_h\left(x\left(t_n\right)\right)\right.\\
&\qquad\quad\left.+g\left(x\left(t_n\right)\right)\right)\Delta Z_n+C_n\left(x\left(t_n\right)-\overline{x}^B\left(t_{n+1}\right)\right)
\end{aligned} \tag{26}$$

by replacing the numerical approximations with the exact solution values on the right-hand side of (4). Then, the local error of method (4) is $\delta^B(t_{n+1})=x(t_{n+1})-\overline{x}^B(t_{n+1})$ and the global error of method (4) is $\epsilon_n=x(t_n)-x_n$.

Theorem 5. *Under Assumptions 1 and 2, the balanced θ-Taylor method (4) is consistent with order 2 in the mean and with* *order 1.5 in the mean square. That is, the local mean error and mean-square error of the balanced θ-Taylor method* (4) *satisfy*

$$\max_{0\le n\le N-1}\left\|\mathbb{E}\left(\delta^{B}\left(t_{n+1}\right)\mid\mathscr{F}_{t_n}\right)\right\|_{L_2}\le H_4\Delta^2\qquad as\ \Delta\longrightarrow 0,$$
$$\max_{0\le n\le N-1}\left\|\delta^{B}(t_{n+1})\right\|_{L_2}\le H_5\Delta^{3/2}\qquad as\ \Delta\longrightarrow 0,\tag{27}$$

where the constants H_4 *and* H_5 *are independent of* Δ.

Proof. From Theorem 3, we have

$$\begin{aligned}&\mathbb{E}\left[\left|\mathbb{E}\left[\left(\overline{x}^{B}_{n+1}-x\left(t_{n+1}\right)\right)\mid\mathscr{F}_{t_n}\right]\right|^2\right]\\&\le 2\mathbb{E}\left[\left|\mathbb{E}\left[\left(\overline{x}^{\theta}_{n+1}-x\left(t_{n+1}\right)\right)\mid\mathscr{F}_{t_n}\right]\right|^2\right]\\&+2\mathbb{E}\left[\left|\mathbb{E}\left[\left(\overline{x}^{B}_{n+1}-\overline{x}^{\theta}_{n+1}\right)\mid\mathscr{F}_{t_n}\right]\right|^2\right]\\&\le H_1^2\Delta^4+2\mathbb{E}\left[\left|\mathbb{E}\left[\left(\overline{x}^{B}_{n+1}-\overline{x}^{\theta}_{n+1}\right)\mid\mathscr{F}_{t_n}\right]\right|^2\right].\end{aligned}\tag{28}$$

From the definitions of $\overline{x}^{\theta}_{n+1}$ and $\overline{x}^{B}_{n+1}$ in (7) and (26), we can write

$$\begin{aligned}&\overline{x}^{B}_{n+1}-\overline{x}^{\theta}_{n+1}\\&=-C_n\left(x\left(t_n\right)-\overline{x}^{B}\left(t_{n+1}\right)\right)\\&=C_n\Big[\Delta\left[\left(1-\theta\right)f\left(x\left(t_n\right)\right)+\theta f\left(x\left(t_{n+1}\right)\right)\right]\\&\quad+g\left(x\left(t_n\right)\right)\Delta W_n\\&\quad+\frac{1}{2}g\left(x\left(t_n\right)\right)g'\left(x\left(t_n\right)\right)\left[\left(\Delta W_n\right)^2-\Delta\right]\\&\quad+\frac{1}{2}\left(3h\left(x\left(t_n\right)\right)-h_h\left(x\left(t_n\right)\right)\right)\Delta N_n\\&\quad+\left(g_h\left(x\left(t_n\right)\right)-g\left(x\left(t_n\right)\right)\right)\Delta W_n\Delta N_n\\&\quad+\frac{1}{2}\left(h_h\left(x\left(t_n\right)\right)-h\left(x\left(t_n\right)\right)\right)\left(\Delta N_n\right)^2\\&\quad+\Big(g\left(x\left(t_n\right)\right)h'\left(x\left(t_n\right)\right)-g_h\left(x\left(t_n\right)\right)\\&\qquad+g\left(x\left(t_n\right)\right)\Big)\Delta Z_n+C_n\left(x_n-x^{B}_{n+1}\right)\Big]\\&=\left(I+C_n\right)^{-1}C_n\\&\quad\times\left[\Delta\left[\left(1-\theta\right)f\left(x\left(t_n\right)\right)+\theta f\left(x\left(t_{n+1}\right)\right)\right]\right.\\&\qquad+g\left(x\left(t_n\right)\right)\Delta W_n+\frac{1}{2}g\left(x\left(t_n\right)\right)\\&\qquad\times g'\left(x\left(t_n\right)\right)\left[\left(\Delta W_n\right)^2-\Delta\right]\\&\qquad+\frac{1}{2}\left(3h\left(x\left(t_n\right)\right)-h_h\left(x\left(t_n\right)\right)\right)\Delta N_n\\&\qquad+\left(g_h\left(x\left(t_n\right)\right)-g\left(x\left(t_n\right)\right)\right)\Delta W_n\Delta N_n\\&\qquad+\frac{1}{2}\left(h_h\left(x\left(t_n\right)\right)-h\left(x\left(t_n\right)\right)\right)\left(\Delta N_n\right)^2\\&\qquad+\Big(g\left(x\left(t_n\right)\right)h'\left(x\left(t_n\right)\right)-g_h\left(x\left(t_n\right)\right)\\&\qquad\left.+g\left(x\left(t_n\right)\right)\Big)\Delta Z_n\right].\end{aligned}\tag{29}$$

Since the components of the matrices $C_0(\cdot)$ and $C_1(\cdot)$ in $C_n(\cdot)$ are bounded, there exists a positive constant M such that $|C_i|\le M\ (i=0,1)$. Under Assumptions 1 and 2, we have

$$\begin{aligned}&\left|\mathbb{E}\left[\left(I+C_n\right)^{-1}C_n\Delta f\left(x\left(t_n\right)\right)\mid\mathscr{F}_{t_n}\right]\right|\\&\quad\le M\Delta\left|f\left(x_n\right)\right|\mathbb{E}\left[\left|C_0\Delta+C_1\left(\left(\Delta W_n\right)^2-\Delta\right)\right|\mid\mathscr{F}_{t_n}\right]\\&\quad\le L'MB\left(1+\left|x\left(t_n\right)\right|^2\right)^{1/2}\Delta^2;\\&\left|\mathbb{E}\left[\left(I+C_n\right)^{-1}C_n g\left(x\left(t_n\right)\right)\Delta W_n\mid\mathscr{F}_{t_n}\right]\right|=0;\\&\left|\mathbb{E}\left[\left(I+C_n\right)^{-1}C_n g\left(x\left(t_n\right)\right)g'\left(x\left(t_n\right)\right)\right.\right.\\&\quad\left.\left.\times\left[\left(\Delta W_n\right)^2-\Delta\right]\mid\mathscr{F}_{t_n}\right]\right|\\&\quad\le BM\left|g\left(x\left(t_n\right)\right)g'\left(x\left(t_n\right)\right)\right|\\&\quad\mathbb{E}\left[\left|\left(\left(\Delta W_n\right)^2-\Delta\right)^2\right|\mid\mathscr{F}_{t_n}\right]\\&\quad\le 2L'MB\left(1+\left|x_n\right|^2\right)^{1/2}\Delta^2;\\&\left|\left[\mathbb{E}\left[\left(I+C_n\right)^{-1}C_n\left(3h\left(x\left(t_n\right)\right)-h_h\left(x\left(t_n\right)\right)\right)\Delta N_n\right]\mid\mathscr{F}_{t_n}\right]\right|\\&\quad\le\lambda BM\left|3h\left(x\left(t_n\right)\right)-h_h\left(x\left(t_n\right)\right)\right|\Delta^2\\&\quad\le\left(2+L\right)L'BM\left(1+\left|x\left(t_n\right)\right|^2\right)^{1/2}\Delta^2,\\&\left|\mathbb{E}\left[\left(I+C_n\right)^{-1}C_n\left(g_h\left(x\left(t_n\right)\right)-g\left(x\left(t_n\right)\right)\right)\right.\right.\\&\quad\left.\left.\times\Delta W_n\Delta N_n\mid\mathscr{F}_{t_n}\right]\right|=0;\\&\left|\mathbb{E}\left[\left(I+C_n\right)^{-1}C_n\left(h_h\left(x\left(t_n\right)\right)-h\left(x\left(t_n\right)\right)\right)\right.\right.\\&\quad\left.\left.\times\left(\Delta N_n\right)^2\mid\mathscr{F}_{t_n}\right]\right|\\&\quad\le BM\left|h_h\left(x\left(t_n\right)\right)-h\left(x\left(t_n\right)\right)\right|\lambda\Delta^2\left(1+\lambda\Delta\right)\\&\quad\le L'BM\lambda\left(1+\left|x\left(t_n\right)\right|^2\right)^{1/2}\Delta^2\left(1+\lambda\Delta\right);\\&\left|\mathbb{E}\left[\left(I+C_n\right)^{-1}C_n\left(g\left(x\left(t_n\right)\right)h'\left(x\left(t_n\right)\right)\right.\right.\right.\\&\quad\left.\left.\left.-g_h\left(x\left(t_n\right)\right)+g\left(x\left(t_n\right)\right)\right)\Delta Z_n\mid\mathscr{F}_{t_n}\right]\right|\\&=\left|\mathbb{E}\left[\left(I+C_n\right)^{-1}C_n\left(g\left(x\left(t_n\right)\right)h'\left(x\left(t_n\right)\right)\right.\right.\right.\\&\quad\left.-g_h\left(x\left(t_n\right)\right)+g\left(x\left(t_n\right)\right)\right)\\&\quad\left.\left.\times\left(\Delta W_n-\Delta Z_{1k}\left(t_{n+1}\right)\right)\mid\mathscr{F}_{t_n}\right]\right|\\&=0.\end{aligned}\tag{30}$$

Therefore,

$$\mathbb{E}\left[\left|\mathbb{E}\left[\left(\overline{x}_{n+1}^{B}-x\left(t_{n+1}\right)\right)\mid\mathcal{F}_{t_n}\right]\right|^2\right]\leq O\left(\Delta^4\right). \tag{31}$$

On the other hand, since

$$\begin{aligned}
&\mathbb{E}\left[\left|(I+C_n)^{-1}C_n\Delta\left[(1-\theta)f\left(x\left(t_n\right)\right)+\theta f\left(x\left(t_{n+1}\right)\right)\right]\right|^2\right]\\
&\quad\leq O\left(\Delta^4\right);\\
&\quad\mathbb{E}\left[\left|(I+C_n)^{-1}C_n g\left(x_n\right)\Delta W_n\right|^2\right]\leq O\left(\Delta^3\right);\\
&\mathbb{E}\left[\left|(I+C_n)^{-1}C_n g\left(x\left(t_n\right)\right)g'\left(x\left(t_n\right)\right)\left[\left(\Delta W_n\right)^2-\Delta\right]\right|^2\right]\\
&\quad\leq O\left(\Delta^3\right);\\
&\mathbb{E}\left[\left|(I+C_n)^{-1}C_n\left(3h\left(x\left(t_n\right)\right)-h_h\left(x\left(t_n\right)\right)\right)\Delta N_n\right|^2\right]\\
&\quad\leq O\left(\Delta^3\right);\\
&\mathbb{E}\left[\left|(I+C_n)^{-1}C_n\left(g_h\left(x\left(t_n\right)\right)-g\left(x\left(t_n\right)\right)\right)\Delta W_n\Delta N_n\right|^2\right]\\
&\quad\leq O\left(\Delta^4\right);\\
&\mathbb{E}\left[\left|(I+C_n)^{-1}C_n\left(h_h\left(x\left(t_n\right)\right)-h\left(x\left(t_n\right)\right)\right)\left(\Delta N_n\right)^2\right|^2\right]\\
&\quad\leq O\left(\Delta^3\right);\\
&\mathbb{E}\left[\left|(I+C_n)^{-1}C_n\left(g\left(x\left(t_n\right)\right)h'\left(x\left(t_n\right)\right)-g_h\left(x\left(t_n\right)\right)\right.\right.\right.\\
&\qquad\left.\left.\left.+g\left(x\left(t_n\right)\right)\right)\left(\Delta Z_n\right)\right|^2\right]\\
&\quad\leq O\left(\Delta^3\right),
\end{aligned} \tag{32}$$

we have

$$\begin{aligned}
&\mathbb{E}\left[\left|x_{n+1}^{B}-x\left(t_{n+1}\right)\right|^2\right]\\
&\leq 2\mathbb{E}\left[\left|x\left(t_{n+1}\right)-x_{n+1}^{\theta}\right|^2\right]+2\mathbb{E}\left[\left|x_{n+1}^{\theta}-x_{n+1}^{B}\right|^2\right]\\
&\leq O\left(\Delta^3\right).
\end{aligned} \tag{33}$$

□

Theorem 6. *Under Assumptions 1 and 2, the balanced θ-Taylor method (4) is convergent with order 1 in the mean square. That is, the global error satisfies*

$$\max_{0\leq n\leq N-1}\left\|\epsilon_{n+1}^2\right\|_{L_2}\leq H_6\Delta \qquad as\ \Delta\longrightarrow 0, \tag{34}$$

where H_6 is independent of Δ.

Proof. From the definitions of δ_n and ϵ_n, we have

$$\epsilon_{n+1}=\epsilon_n+P_n+\delta_{n+1}, \tag{35}$$

where

$$\begin{aligned}
P_n&=u_n+C\left(x\left(t_n\right)\right)\left(x\left(t_n\right)-\overline{x}^{B}\left(t_{n+1}\right)\right)\\
&\qquad-C\left(x_n\right)\left(x_n-x_{n+1}\right)\\
&=u_n+C\left(x\left(t_n\right)\right)\left(x\left(t_n\right)-x_n\right)\\
&\qquad-C\left(x\left(t_n\right)\right)\left(\overline{x}^{B}\left(t_{n+1}\right)-x_{n+1}\right)\\
&\qquad+\left(C\left(x\left(t_n\right)\right)-C\left(x_n\right)\right)\left(x_n-x_{n+1}\right)\\
&=u_n-C\left(x\left(t_n\right)\right)P_n\\
&\qquad+\left(C\left(x\left(t_n\right)\right)-C\left(x_n\right)\right)\left(x_n-x_{n+1}\right)\\
&=\left(I+C\left(x\left(t_n\right)\right)\right)^{-1}\\
&\qquad\times\left[u_n+\left(C\left(x\left(t_n\right)\right)-C\left(x_n\right)\right)\left(x_n-x_{n+1}\right)\right].
\end{aligned} \tag{36}$$

Thus, there exists a constant J_5 such that

$$\begin{aligned}
&\mathbb{E}\left|P_n\right|^2\\
&\leq 2B^2\mathbb{E}\left|u_n\right|^2+2B^2\mathbb{E}\left|\left[\left(C_0\left(x\left(t_n\right)\right)-C_0\left(x_n\right)\right)\Delta\right.\right.\\
&\qquad\left.\left.+\left(C_1\left(x\left(t_n\right)\right)-C_1\left(x_n\right)\right)\left(\left(\Delta W_n\right)^2-\Delta\right)\right]\left(x_n-x_{n+1}\right)\right|^2\\
&\leq 2B^2\mathbb{E}\left|u_n\right|^2+4B^2M^2\Delta^2\mathbb{E}\left|\left(x_n-x_{n+1}\right)\right|^2\\
&\leq 2B^2J_1\Delta\mathbb{E}\left|\epsilon_n\right|^2\\
&\qquad+2B^2L^2\Delta^2\theta^2\mathbb{E}\left|\epsilon_{n+1}\right|^2+J_5\Delta^3
\end{aligned} \tag{37}$$

and there exists a constant J_6 such that

$$\begin{aligned}
&\left|\mathbb{E}\left(P_n\mid\mathcal{F}_{t_n}\right)\right|^2\\
&\quad=\left|\mathbb{E}\left(\left(I+C\left(x\left(t_n\right)\right)\right)^{-1}\right.\right.\\
&\qquad\times\left[u_n+\left(C\left(x\left(t_n\right)\right)-C\left(x_n\right)\right)\right.\\
&\qquad\left.\left.\left.\times\left(x_n-x_{n+1}\right)\right]\mid\mathcal{F}_{t_n}\right)\right|^2\\
&\quad\leq 2B^2\left|\mathbb{E}\left(u_n\mid\mathcal{F}_{t_n}\right)\right|^2+2B^2\\
&\qquad\times\left|\mathbb{E}\left(\left(C\left(x\left(t_n\right)\right)-C\left(x_n\right)\right)\left(x_n-x_{n+1}\right)\mid\mathcal{F}_{t_n}\right)\right|^2\\
&\quad\leq 2B^2\left(J_2\Delta^2\mathbb{E}\left|\epsilon_n\right|^2+L^2\Delta^2\theta^2\mathbb{E}\left(\left|\epsilon_{n+1}\right|^2\mid\mathcal{F}_{t_n}\right)\right)\\
&\qquad+2B^2\Delta\left|\mathbb{E}\left(\left(C_0\left(x\left(t_n\right)\right)-C_0\left(x_n\right)\right)\left(x_n-x_{n+1}\right)\mid\mathcal{F}_{t_n}\right)\right|^2\\
&\quad\leq 2B^2J_2\Delta^2\mathbb{E}\left|\epsilon_n\right|^2\\
&\qquad+2B^2L^2\Delta^2\theta^2\mathbb{E}\left(\left|\epsilon_{n+1}\right|^2\mid\mathcal{F}_{t_n}\right)+J_6\Delta^4.
\end{aligned} \tag{38}$$

Thus,

$$\begin{aligned} &\left|\mathbb{E}\left\langle \epsilon_n, P_n\right\rangle\right| \\ &\quad\leq \Delta^{-1}\left(\mathbb{E}\left|\mathbb{E}\left(P_n \mid \mathscr{F}_{t_n}\right)\right|^2\right) + \Delta\mathbb{E}\left|\epsilon_n\right|^2 \\ &\quad\leq \left(1 + 2B^2 J_2\right)\Delta\mathbb{E}\left|\epsilon_n\right|^2 \\ &\qquad + 12B^2L^2\theta^2\Delta\mathbb{E}\left|\epsilon_{n+1}\right|^2 + J_6\Delta^3. \end{aligned} \tag{39}$$

From Theorem 5, we have

$$\left|\mathbb{E}\left\langle \delta_{n+1}, \epsilon_n\right\rangle\right| \leq H_4\Delta^3 + \Delta\mathbb{E}\left|\epsilon_n\right|^2. \tag{40}$$

Therefore,

$$\begin{aligned} &\mathbb{E}\left|\epsilon_{n+1}\right|^2 \\ &\quad\leq \left(1 + J_7\Delta\right)\mathbb{E}\left|\epsilon_n\right|^2 \\ &\qquad + 2B^2L^2\theta^2\Delta\left(\Delta + 12\right)\mathbb{E}\left|\epsilon_{n+1}\right|^2 \\ &\qquad + \left(J_5 + H_5 + 2H_4 + 2J_6\right)\Delta^3, \end{aligned} \tag{41}$$

where $J_7 = 2(B^2J_1 + 2 + 2B^2J_2)$. Because $\Delta \to 0$, we can assume $1 - 2B^2L^2\theta^2\Delta(\Delta + 12) > 0$ without loss of generality. Let $J_8 = J_5 + H_5 + 2H_4 + 2J_6$. Then,

$$\begin{aligned} \mathbb{E}\left|\epsilon_{n+1}\right|^2 &\leq \left(1 + J_7\Delta\right)\mathbb{E}\left|\epsilon_n\right|^2 + J_8\Delta^3 \\ &= J_8\Delta^2\frac{\left(1 + J_7\Delta\right)^{n+1} - 1}{J_7} \leq J_9\Delta^2, \end{aligned} \tag{42}$$

where $J_9 = J_8((e^{J_7T} - 1)/J_7)$. □

4. Stability of the Implicit Taylor Methods

In this section, we will discuss the stability properties of the numerical methods introduced in Section 2. Consider a scalar linear test equation,

$$\begin{aligned} \mathrm{d}x(t) = ax(t)\,\mathrm{d}t + bx(t)\,\mathrm{d}W(t) + cx(t)\,\mathrm{d}N(t), \\ x(t_0) = x_0, \end{aligned} \tag{43}$$

where a, b, and c are real constants. The solution of (43) is $x(t) = x_0e^{(a-(1/2)b^2)t+bW(t)}(1+c)^{N(t)}$ and is mean-square (MS) stable if $2a + b^2 + \lambda c(2 + c) < 0$ [2].

The one-step scheme of the test equation (43) is

$$x_{n+1} = R\left(a, b, c, \Delta, \Delta W_n, \Delta N_n\right)x_n. \tag{44}$$

The numerical method is MS-stable if

$$\overline{R}(a, b, c, \Delta, \lambda) = \mathbb{E}\left(R^2\left(a, b, c, \Delta, \Delta W_n, \Delta N_n\right)\right) < 1, \tag{45}$$

where $\overline{R}(a, b, c, \Delta, \lambda)$ is called the MS-stability function of the numerical method.

If the Taylor method (2) is applied to the test equation (43), we obtain

$$x_{n+1} = R_1\left(a, b, c, \Delta, \Delta W_n, \Delta N_n\right)x_n, \tag{46}$$

where

$$\begin{aligned} &R_1\left(a, b, c, \Delta, \Delta W_n, \Delta N_n\right) \\ &\quad= 1 + \left(a - \frac{1}{2}b^2\right)\Delta + b\Delta W_n + \frac{1}{2}b^2(\Delta W_n)^2 \\ &\qquad + \frac{1}{2}\left(2c - c^2\right)\Delta N_n + bc\Delta W_n\Delta N_n \\ &\qquad + \frac{1}{2}c^2(\Delta N_n)^2. \end{aligned} \tag{47}$$

Let $p = a\Delta$, $q = b\sqrt{\Delta}$, and $z = c\lambda\Delta$. Then the MS-stability function of the Taylor method is

$$\begin{aligned} &\overline{R}_1(p, q, z, c) \\ &\quad= \mathbb{E}\left(R_1^2\left(a, b, c, \Delta, \Delta W_n, \Delta N_n\right)\right) \\ &\quad= 1 + 2p + q^2 + p^2 + \frac{1}{2}pq^2 + \frac{1}{2}q^4 \\ &\qquad + \left(2 + c + 2p + cq^2 + 2q^2\right)z \\ &\qquad + \left(2 + 2c + \frac{1}{2}c^2 + q^2 + p\right)z^2 \\ &\qquad + (c + 1)z^3 + \frac{1}{4}z^4. \end{aligned} \tag{48}$$

Thus, the strong Taylor method (2) for the linear test equation (43) is MS-stable if $\overline{R}_1(p, q, z, c) < 1$.

Applying the θ-Taylor method (3) to the test equation (43), we obtain

$$x_{n+1} = R_2\left(a, b, c, \Delta, \Delta W_n, \Delta N_n, \theta\right)x_n, \tag{49}$$

where

$$\begin{aligned} &R_2\left(a, b, c, \Delta, \Delta W_n, \Delta N_n, \theta\right) \\ &\quad= \frac{1}{1 - a\theta\Delta}\left[1 + \left((1 - \theta)a - \frac{1}{2}b^2\right)\Delta\right. \\ &\qquad + b\Delta W_n + \frac{1}{2}b^2(\Delta W_n)^2 + \frac{1}{2}\left(2c - c^2\right)\Delta N_n \\ &\qquad \left. + bc\Delta W_n\Delta N_n + \frac{1}{2}c^2(\Delta N_n)^2\right]. \end{aligned} \tag{50}$$

Then the MS-stability function of the θ-Taylor method is

$$\begin{aligned} &\overline{R}_2(p, q, z, c, \theta) \\ &\quad= \frac{1}{(1 - p\theta)^2}\left[\overline{R}_1(p, q, z, c)\right. \\ &\qquad - 2p\theta + p^2\theta^2 - 2p^2\theta \\ &\qquad \left. - 2pz\theta - pz^2\theta\right]. \end{aligned} \tag{51}$$

Thus, the θ-Taylor method (3) for the linear test equation (43) is MS-stable if $\overline{R}_2(p,q,z,c) < 1$.

Applying the balanced θ-Taylor method (4) to the test equation (43), we obtain

$$x_{n+1} = R_3(a,b,c,\Delta,\Delta W_n,\Delta N_n)\,x_n, \tag{52}$$

where

$$\begin{aligned} &R_3(a,b,c,\Delta,\Delta W_n,\Delta N_n) \\ &\quad= \left((1-a\theta\Delta)I + C_n\right)^{-1} \\ &\qquad\times\left[1 + \left((1-\theta)a - \frac{1}{2}b^2\right)\Delta \right.\\ &\qquad\quad + b\Delta W_n + \frac{1}{2}b^2(\Delta W_n)^2 + \frac{1}{2}\left(2c-c^2\right)\Delta N_n \\ &\qquad\quad \left. + bc\Delta W_n\Delta N_n + \frac{1}{2}c^2(\Delta N_n)^2 + C_n\right]. \end{aligned} \tag{53}$$

Since $\mathbb{E}(R_3^2)$ is rather complex in the general case, we try to investigate the stability of balanced θ-method (4) for the following two typical cases.

Case 1. Let $C_0 = -a$ and $C_1 = 0$. Then, applying the balanced θ-Taylor method (4) with $C_n = -a\Delta$ to the test equation (43), we obtain

$$x_{n+1} = R_3'(a,b,c,\Delta,\Delta W_n,\Delta N_n)\,x_n, \tag{54}$$

where

$$\begin{aligned} &R_3'(a,b,c,\Delta,\Delta W_n,\Delta N_n) \\ &\quad= (1-a(1+\theta)\Delta)^{-1}\left[1 - a\theta\Delta - \frac{1}{2}b^2\Delta + b\Delta W_n \right.\\ &\qquad + \frac{1}{2}b^2(\Delta W_n)^2 + \frac{1}{2}\left(2c-c^2\right)\Delta N_n \\ &\qquad \left. + bc\Delta W_n\Delta N_n + \frac{1}{2}c^2(\Delta N_n)^2\right]. \end{aligned} \tag{55}$$

Then the MS-stability function of the balanced θ-Taylor method (4) with $C_n = -a\Delta$ is

$$\begin{aligned} &\overline{R}_3'(p,q,z,c) \\ &\quad= \frac{1}{(1-p(1+\theta))^2} \\ &\qquad\times\left[1 + q^2 + \frac{1}{2}pq^2 + \frac{1}{2}q^4 - 2p\theta + p^2\theta^2 \right.\\ &\qquad\quad + \left(2 + c + cq^2 + 2q^2 - 2p\theta\right)z \\ &\qquad\quad + \left(2 + 2c + \frac{1}{2}c^2 + q^2 - p\theta\right)z^2 \\ &\qquad\quad \left. + (c+1)z^3 + \frac{1}{4}z^4\right]. \end{aligned} \tag{56}$$

Thus, the balanced θ-Taylor method (4) with $C_n = -a\Delta$ for the linear test equation (43) is MS-stable if $\overline{R}_3'(p,q,z,c) < 1$.

Case 2. Let $C_0 = -a$ and $C_1 = b^2$. Then, applying the balanced θ-Taylor method (4) with $C_n = -a\Delta + b^2((\Delta W_n)^2 - \Delta)$ to the test equation (43), we have

$$x_{n+1} = R_3''(p,q,c,J,\Delta N_n)\,x_n, \tag{57}$$

where

$$\begin{aligned} &R_3''(p,q,c,J,\Delta N_n) \\ &\quad= \left(1 - p\theta + q^2\left(J^2-1\right)\right)^{-1} \\ &\qquad\times\left[1 - p\theta - \frac{3}{2}q^2 + qJ + \frac{3}{2}q^2J^2 \right.\\ &\qquad\quad + \frac{1}{2}\left(2c-c^2\right)\Delta N_n + qcJ\Delta N_n \\ &\qquad\quad \left. + \frac{1}{2}c^2(\Delta N_n)^2\right] \end{aligned} \tag{58}$$

and J is the standard Gaussian random variable $J = \Delta W_n/\sqrt{\Delta} \sim N(0,1)$. Then the MS-stability function of the balanced θ-Taylor method (5) with $C_n = -a\Delta + b^2((\Delta W_n)^2 - \Delta)$ is

$$\begin{aligned} &\overline{R}_3''(p,q,z,c,J,\Delta N_n) \\ &\quad= \frac{1}{\sqrt{2\pi}}\sum_{-\infty}^{+\infty}\frac{(\lambda\Delta)^m e^{-\lambda\Delta}}{m!} \\ &\qquad\times\int_{-\infty}^{+\infty}\left(R_3''(p,q,c,x,m)\right)^2 e^{-(x^2/2)}\,\mathrm{d}x. \end{aligned} \tag{59}$$

Thus, the balanced θ-Taylor method (4) with $C_n = -a\Delta + b^2((\Delta W_n)^2 - \Delta)$ for the linear test equation (43) is MS-stable if $\overline{R}_3''(p,q,z,c) < 1$.

For the case of $c = -1$ and $z = -1$, the MS-stable regions of the numerical methods for the test equation are plotted in Figures 1 and 2. Figure 1 shows the MS-stable regions of Taylor method, the θ-Taylor method, and the balanced θ-Taylor method with $C_0 = -a$ and $C_1 = 0$ when $\theta = 1/2$ and $\theta = 1$. Figure 2 shows the MS-stable regions of Taylor method, the θ-Taylor method, and the balanced θ-Taylor method with $C_0 = -a$ and $C_1 = b^2$ when $\theta = 1/4$ and $\theta = 1/2$. It should be noted that the MS-stable regions are the areas below the plotted curves and symmetric about the p-axis. From Figures 1 and 2, it is observed that the MS-stable regions of the θ-Taylor method and the balanced θ-Taylor method increase as the parameter θ increases. The MS-stable properties of the θ-Taylor method and the balanced θ-Taylor method are better than the Taylor method. Furthermore, the MS-stable properties of the balanced θ-Taylor method with $C_0 = -a$ and $C_1 = 0$ are better than those of the θ-Taylor method for all $\theta \in [0,1]$. In addition, the MS-stable properties of the balanced θ-Taylor method with $C_0 = -a$ and $C_1 = b^2$ are better than those of the θ-Taylor method when $\theta \le 1/4$, and the MS-stable properties of the θ-Taylor method are better than those of the balanced θ-Taylor method with $C_0 = -a$ and $C_1 = b^2$ when $\theta \ge 1/2$.

Table 1: Mean of the absolute errors for different values of Δ and different methods.

Methods	Δ	2^{-8}	2^{-7}	2^{-6}	2^{-5}	2^{-4}	2^{-3}	2^{-2}	2^{-1}
θ-Taylor	$\theta = 0$	0.0024	0.0048	0.0099	0.0186	0.0389	0.0911	0.6853	3.0473
	$\theta = 1/2$	0.0022	0.0043	0.0090	0.0177	0.0368	0.0805	0.2548	0.7407
	$\theta = 1$	0.0042	0.0083	0.0170	0.0342	0.0699	0.1454	0.3025	0.5568
Balanced θ-Taylor	$\theta = 0$	0.0021	0.0041	0.0083	0.0152	0.0317	0.0665	0.3852	1.5570
	$\theta = 1/2$	0.0076	0.0134	0.0223	0.0361	0.0619	0.1504	0.7024	3.1984
	$\theta = 1$	0.0075	0.0132	0.0217	0.0347	0.0582	0.1360	0.6285	2.6648

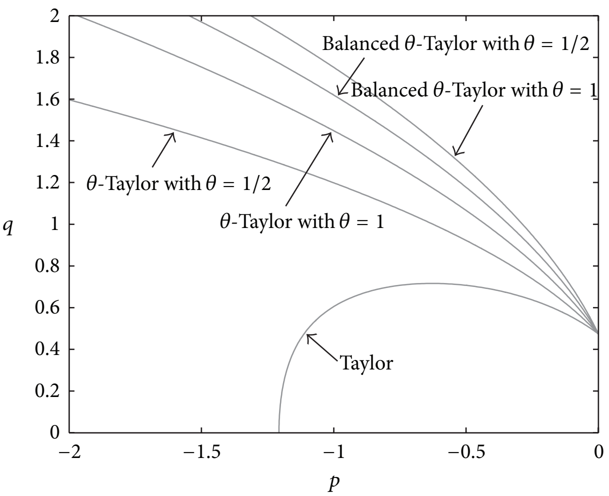

Figure 1: MS-stable regions of the θ-Taylor methods and the balanced θ-Taylor method with $C_0 = -a$ and $C_1 = 0$.

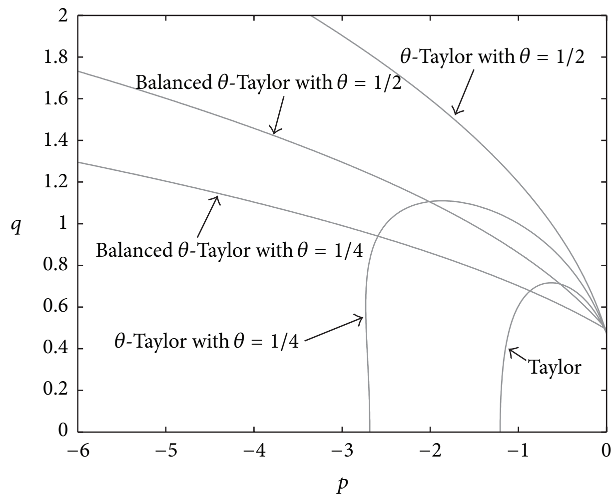

Figure 2: MS-stable regions of the θ-Taylor methods and the balanced θ-Taylor method with $C_0 = -a$ and $C_1 = b^2$.

5. Numerical Examples

In this section, we conduct some simulation to demonstrate the convergence of the proposed implicit Taylor numerical solutions (3) and (4) for the equation system (43) with the coefficients $a = -4,\ b = 1,\ c = -0.5$ and the jump intensity $\lambda = 2$. We compare the explicit solutions with the numerical approximations for the step-sizes $\Delta = 2^{-1}, 2^{-2}, \ldots, 2^{-8}$. To measure the accuracy and convergence property of the proposed methods, we compute mean of the absolute errors as

$$e = \frac{1}{2000} \sum_{i=1}^{2000} \left| x_N^{(i)} - x^{(i)}(t_N) \right|. \qquad (60)$$

In Table 1, we report the simulated errors of the θ-Taylor method and the balanced θ-Taylor method with $C_0 = 1$ and $C_1 = 1$ for different values of θ and Δ. Note that the Taylor method is a special case of the θ-Taylor method with $\theta = 0$. From Table 1, we know that the accuracy of the θ-Taylor method with $\theta = 1/2$ and the balanced θ-Taylor method with $\theta = 0$ is higher than that of the Taylor method. The accuracy of the balanced θ-Taylor method with $\theta = 0$ is the highest for $\Delta \le 2^{-2}$. When $\theta \ge 1/2$, the accuracy of the θ-Taylor method is higher than that of the balanced θ-Taylor method with $C_0 = 1$ and $C_1 = 1$.

6. Conclusions

In this paper, we introduce two kinds of the implicit methods, the θ-Taylor method and the balanced θ-Taylor method, for solving stochastic differential equations with Poisson jumps. It is proved that the proposed numerical methods have a strong convergence order of 1.0. Moreover, the MS-stable regions of the proposed numerical methods are derived for a linear scalar test equation and it is demonstrated that the θ-Taylor method and the balanced θ-Taylor method have better stable properties than the Taylor method. As has been confirmed by the theoretical and the numerical results, the proposed numerical methods perform satisfactorily in solving SDEJs.

Conflict of Interests

The authors declare that they have no conflict of interests regarding to the publication of this paper.

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