Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2014, Article ID 152910, 7 pages
http://dx.doi.org/10.1155/2014/152910

Research Article

Generalized Almost Convergence and Core Theorems of Double Sequences

S. A. Mohiuddine and Abdullah Alotaibi

Department of Mathematics, Faculty of Science, King Abdulaziz University, P.O. Box 80203, Jeddah 21589, Saudi Arabia

Correspondence should be addressed to S. A. Mohiuddine; mohiuddine@gmail.com

Received 15 May 2014; Accepted 15 June 2014; Published 15 July 2014

Academic Editor: Abdul Latif

The idea of $[\lambda,\mu]$-almost convergence (briefly, $\mathcal{F}_{[\lambda,\mu]}$-convergence) has been recently introduced and studied by Mohiuddine and Alotaibi (2014). In this paper first we define a norm on $\mathcal{F}_{[\lambda,\mu]}$ such that it is a Banach space and then we define and characterize those four-dimensional matrices which transform $\mathcal{F}_{[\lambda,\mu]}$-convergence of double sequences $x=(x_{jk})$ into $\mathcal{F}_{[\lambda,\mu]}$-convergence. We also define a $\mathcal{F}_{[\lambda,\mu]}$-core of $x=(x_{jk})$ and determine a Tauberian condition for core inclusions and core equivalence.

1. Background, Notations, and Preliminaries

We begin by recalling the definition of convergence for double sequences which was introduced by Pringsheim [1]. A double sequence $x=(x_{jk})$ is said to be *convergent* to L in the Pringsheim's sense (or P-convergent to L) if for given $\epsilon>0$ there exists an integer N such that $|x_{jk}-L|<\epsilon$ whenever $j,k>N$. We will write this as

$$P\text{-}\lim_{j,k\to\infty} x_{jk}=L, \tag{1}$$

where j and k are tending to infinity independent of each other. We denote by $\mathcal{C}_P$ the space of P-convergent sequences.

We say that a double sequence $x=(x_{jk})$ is *bounded* if

$$\|x\|=\sup_{j,k\geq 0}\left|x_{jk}\right|<\infty. \tag{2}$$

Denote by $\mathcal{L}_\infty$ the space of all bounded double sequences.

If a double sequence $x=(x_{jk})$ in $\mathcal{L}_\infty$ and x is also P-convergent to L, then we say that it is *boundedly P-convergent* to L (or, BP-convergent to L). We denote by $\mathcal{C}_{\mathrm{BP}}$ the space of all boundedly P-convergent double sequences. Note that $\mathcal{C}_{\mathrm{BP}}\subset\mathcal{L}_\infty$.

We remark that, in contrast to the case for single sequences, a P-convergent double sequence need not be bounded.

Let $A=(a_{pqjk}:j,k=0,1,2,\ldots)$ be a four-dimensional infinite matrix of real numbers for all $p,q=0,1,2,\ldots$ and S_1 a space of double sequences. Let S_2 be a double sequences space, converging with respect to a convergence rule $\nu\in\{P,\mathrm{BP}\}$. Define

$$S_1^{A,\nu}=\left\{x=(x_{jk}):Ax=(A_{pq}(x))=\nu-\sum_{j,k}\sum_{k}a_{pqjk}x_{jk}\ \text{exists},\ Ax\in S_1\right\}. \tag{3}$$

Then, we say that a four-dimensional matrix B maps the space S_2 into the space S_1 if $S_2\subset S_1^{B,\nu}$ and is denoted by (S_2,S_1).

The idea of almost convergence of Lorentz [2] is narrowly connected with the limits of Banach (see [3]) as follows. A sequence $x=(x_j)$ in l_∞ is almost convergent to L if all of its Banach limits are equal, where l_∞ denotes the space of all bounded sequences. Mohiuddine [4] applies this concept to established some approximation theorems for sequence of positive linear operator. Móricz and Rhoades [5] extended the notion of almost convergence from single to double sequences as follows.

A double sequence $x = (x_{j,k})$ of real numbers is said to be *almost convergent* to a number L if

$$\lim_{p,q\to\infty} \sup_{m,n>0} \left| \frac{1}{pq} \sum_{j=m}^{m+p-1} \sum_{k=n}^{n+q-1} x_{j,k} - L \right| = 0. \quad (4)$$

For more details on almost convergence for single and double sequences, one can refer to [6–13].

The two-dimensional analogue of Banach limit has been defined by Mursaleen and Mohiuddine [14] as follows. A linear functional $\mathscr{L}$ on $\mathscr{L}_\infty$ is said to be *Banach limit* if it has the following properties:

(BL_1) $\mathscr{L}(x) \geq 0$ if $x \geq 0$ (i.e., $x_{jk} \geq 0$ for all j, k),

(BL_2) $\mathscr{L}(E) = 1$, where $E = (e_{jk})$ with $e_{jk} = 1$ for all j, k,

(BL_3) $\mathscr{L}(S_{11}x) = \mathscr{L}(x) = \mathscr{L}(S_{10}x) = \mathscr{L}(S_{01}x)$,

where the shift operators S_{01}, S_{10}, and S_{11} are defined by

$$S_{01}x = (x_{j,k+1}), \qquad S_{10}x = (x_{j+1,k}), \qquad S_{11}x = (x_{j+1,k+1}). \quad (5)$$

Denote by $\mathscr{B}$ the set of all Banach limits on $\mathscr{L}_\infty$. Note that if (BL_3) holds, then we may also write $\mathscr{L}(Sx) = \mathscr{L}(x)$. A double sequence $x = (x_{jk})$ is said to be *almost convergent* to a number L if $\mathscr{L}(x) = L$ for all $\mathscr{L} \in \mathscr{B}$.

Let $\lambda = (\lambda_m : m = 0, 1, 2, \ldots)$ and $\mu = (\mu_n : n = 0, 1, 2, \ldots)$ be two nondecreasing sequences of positive reals and each tending to ∞ such that $\lambda_{m+1} \leq \lambda_m + 1$, $\lambda_1 = 0$, $\mu_{n+1} \leq \mu_n + 1$, $\mu_1 = 0$, and

$$\mathfrak{I}_{mn}(x) = \frac{1}{\lambda_m \mu_n} \sum_{j\in J_m} \sum_{k\in I_n} x_{jk}, \quad (6)$$

is called the *double generalized de la Valée-Poussin mean*, where $J_m = [m-\lambda_m+1, m]$ and $I_n = [n-\mu_n+1, n]$. We denote the set of all λ and μ type sequences by using the symbol Λ.

Quite recently, Mohiuddine and Alotaibi [15] presented a generalization of the notion of almost convergent double sequence with the help of de la Vallée-Poussin mean and called it $[\lambda, \mu]$-almost convergent. In the same paper, they also defined and characterized some four-dimensional matrices. For more details on double sequences, four-dimensional matrices, and other related concepts, one can refer to [16–26].

A double sequence $x = (x_{jk})$ of reals is said to be $[\lambda, \mu]$-*almost convergent* (briefly, $\mathscr{F}_{[\lambda,\mu]}$-convergent) [15] to some number L if $x \in \mathscr{F}_{[\lambda,\mu]}$, where

$$\mathscr{F}_{[\lambda,\mu]} = \left\{ x = (x_{jk}) : P\text{-}\lim_{m,n\to\infty} \Omega_{mnst}(x) = L \text{ exists, uniformly in } s, t; L = \mathscr{F}_{[\lambda,\mu]}\text{-}\lim x \right\}, \quad (7)$$

$$\Omega_{mnst}(x) = \frac{1}{\lambda_m \mu_n} \sum_{j\in J_m} \sum_{k\in I_n} x_{j+s,k+t}.$$

Denote by $\mathscr{F}_{[\lambda,\mu]}$ the space of all $[\lambda, \mu]$ almost convergent sequences $(x_{j,k})$. Note that $\mathscr{C}_{BP} \subset \mathscr{F}_{[\lambda,\mu]} \subset \mathscr{L}_\infty$.

We remark that if we take $\lambda_m = m$ and $\mu_n = n$, then the notion of $[\lambda, \mu]$-almost convergence coincides with the notion of almost convergence for double sequences due to Móricz and Rhoades [5].

2. $(\mathscr{F}_{[\lambda,\mu]}, \mathscr{F}_{[\lambda,\mu]})$-Matrices

We will assume throughout this paper that the limit of a double sequence means limit in the Pringsheim sense. We define the following matrix classes and establish interesting results.

Definition 1. A four-dimensional matrix $A = (a_{pqjk})$ is said to be $[\lambda, \mu]$-*almost regular* if $Ax \in \mathscr{F}_{[\lambda,\mu]}$ for all $x = (x_{jk}) \in \mathscr{C}_{BP}$ with $\mathscr{F}_{[\lambda,\mu]}$-lim Ax = lim x, and one denotes this by $A \in (\mathscr{C}_{BP}, \mathscr{F}_{[\lambda,\mu]})_{\text{reg}}$.

Definition 2. A matrix $A = (a_{pqjk})$ is said to be of class $(\mathscr{F}_{[\lambda,\mu]}, \mathscr{F}_{[\lambda,\mu]})$ if it maps every $\mathscr{F}_{[\lambda,\mu]}$-convergent double sequence into $\mathscr{F}_{[\lambda,\mu]}$-convergent double sequence; that is, $Ax \in \mathscr{F}_{[\lambda,\mu]}$ for all $x = (x_{jk}) \in \mathscr{F}_{[\lambda,\mu]}$. In addition, if $\mathscr{F}_{[\lambda,\mu]}$-lim $Ax = \mathscr{F}_{[\lambda,\mu]}$-lim x, then A is $\mathscr{F}_{[\lambda,\mu]}$-regular and, in symbol, one will write $A \in (\mathscr{F}_{[\lambda,\mu]}, \mathscr{F}_{[\lambda,\mu]})_{\text{reg}}$.

Now we define the norm on $\mathscr{F}_{[\lambda,\mu]}$ as follows.

Theorem 3. *$\mathscr{F}_{[\lambda,\mu]}$ is a Banach space normed by*

$$\|x\| = \sup_{m,n,s,t} |\Omega_{mnst}(x)|. \quad (8)$$

Proof. It can be easily verified that (8) defines a norm on $\mathscr{F}_{[\lambda,\mu]}$. We show that $\mathscr{F}_{[\lambda,\mu]}$ is complete. Now, let (x^b) be a Cauchy sequence in $\mathscr{F}_{[\lambda,\mu]}$. Then for each j, k, (x^b_{jk}) is a Cauchy sequence in $\mathbb{R}$. Therefore $x^b_{jk} \to x_{jk}$ (say). Put $x = (x_{jk})$; given ϵ there exists an integer $N(\epsilon) = N$, say, such that, for each $b, d > N$,

$$\|x^b - x^d\| < \frac{\epsilon}{2}. \quad (9)$$

Hence

$$\sup_{m,n,s,t} |\Omega_{mnst}(x^b - x^d)| < \frac{\epsilon}{2}; \quad (10)$$

then, for each m, n, s, t and $b, d > N$, we have

$$|\Omega_{mnst}(x^b - x^d)| < \frac{\epsilon}{2}. \quad (11)$$

Taking limit $d \to \infty$, we have for $b > N$ and for each of m, n, s, t

$$|\Omega_{mnst}(x^b - x)| < \frac{\epsilon}{2}. \quad (12)$$

Now for fixed b, the above inequality holds. Since for fixed b, $x^b \in \mathscr{F}_{[\lambda,\mu]}$, we get

$$\lim_{m,n\to\infty} \Omega_{mnst}(x^b) = \ell \quad (13)$$

uniformly in s, t. For given $\epsilon > 0$, there exist positive integers m_0, n_0 such that

$$\left|\Omega_{mnst}\left(x^b\right) - \ell\right| < \frac{\epsilon}{2}, \tag{14}$$

for $m \geq m_0$, $n \geq n_0$ and for all s, t. Here m_0, n_0 are independent of s, t but depend upon ϵ. Now by using (12) and (14), we get

$$\begin{aligned} &\left|\Omega_{mnst}(x) - \ell\right| \\ &\quad = \left|\Omega_{mnst}(x) - \Omega_{mnst}\left(x^b\right) + \Omega_{mnst}\left(x^b\right) - \ell\right| \\ &\quad \leq \left|\Omega_{mnst}(x) - \Omega_{mnst}\left(x^b\right)\right| + \left|\Omega_{mnst}\left(x^b\right) - \ell\right| \\ &\quad < \epsilon, \end{aligned} \tag{15}$$

for $m \geq m_0, n \geq n_0$ and for all s, t. Hence $x = (x_{jk}) \in \mathscr{F}_{[\lambda,\mu]}$ and $\mathscr{F}_{[\lambda,\mu]}$ is complete. □

Now we characterize the matrix class $(\mathscr{F}_{[\lambda,\mu]}, \mathscr{F}_{[\lambda,\mu]})$ as well as $(\mathscr{F}_{[\lambda,\mu]}, \mathscr{F}_{[\lambda,\mu]})_{\text{reg}}$. Let $\mathscr{M}_{[\lambda,\mu]}$ be the subspace of $\mathscr{F}_{[\lambda,\mu]}$ such that $P\text{-}\lim_{m,n\to\infty}\Omega_{mnst}(x) = 0$, uniformly in s, t; that is

$$\mathscr{M}_{[\lambda,\mu]} = \left\{x = \left(x_{jk}\right) \in \mathscr{F}_{[\lambda,\mu]} : P\text{-}\lim_{m,n\to\infty}\Omega_{mnst}(x) = 0, \text{ uniformly in } s, t\right\}. \tag{16}$$

Note that every $y \in \mathscr{F}_{[\lambda,\mu]}$ can be written as

$$y = x + \ell E, \tag{17}$$

where $x \in \mathscr{M}_{[\lambda,\mu]}$, $\ell = P\text{-}\lim_{m,n}\Omega_{mnst}(y)$ uniformly in s, t, and $E = (e_{jk})$ with $e_{jk} = 1$ for all j, k.

Theorem 4. *A matrix $A = (a_{pqjk}) \in (\mathscr{F}_{[\lambda,\mu]}, \mathscr{F}_{[\lambda,\mu]})$ if and only if*

(AC_1) $\|A\| = \sup_{p,q}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}|a_{pqjk}| < \infty$,

(AC_2) $a = (\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}a_{pqjk})_{p,q=1}^{\infty} \in \mathscr{F}_{[\lambda,\mu]}$,

(AC_3) $A(S - I) \in (\mathscr{L}_\infty, \mathscr{F}_{[\lambda,\mu]})$,

where S is the shift operator.

Proof.
Necessity. Let $A \in (\mathscr{F}_{[\lambda,\mu]}, \mathscr{F}_{[\lambda,\mu]})$. We know that $\mathscr{C}_{\text{BP}} \subset \mathscr{F}_{[\lambda,\mu]} \subset \mathscr{L}_\infty$, so we have $A \in (\mathscr{C}_{\text{BP}}, \mathscr{L}_\infty)$. Hence the necessity of (AC_1) follows. Since $\in \mathscr{F}_{[\lambda,\mu]}$, then $AE \in \mathscr{F}_{[\lambda,\mu]}$. This is equivalent to

$$\left(\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}a_{pqjk}\right)_{p,q=1}^{\infty} \in \mathscr{F}_{[\lambda,\mu]}; \tag{18}$$

that is, (AC_2) holds. For each $x = (x_{jk}) \in \mathscr{L}_\infty$, we have $Sx - x \in \mathscr{F}_{[\lambda,\mu]}$ because

$$\mathscr{L}(Sx - x) = \mathscr{L}(Sx) - \mathscr{L}(x) = 0 \tag{19}$$

for all Banach limit $\mathscr{L}$. Hence $A(Sx - x) \in \mathscr{F}_{[\lambda,\mu]}$; that is, ($AC_3$) holds.

Sufficiency. Let conditions (AC_1)–(AC_3) hold and $y = (y_{jk}) \in \mathscr{F}_{[\lambda,\mu]}$. Then

$$y = x + \ell E, \tag{20}$$

where $x = (x_{jk}) \in \mathscr{M}_{[\lambda,\mu]}$, $\ell = P\text{-}\lim_{m,n\to\infty}\Omega_{mnst}(y)$, uniformly in s, t and $E = (e_{jk})$ with $e_{jk} = 1$ for all j, k. Taking A-transform in (20), we obtain

$$\begin{aligned} Ay &= Ax + \ell AE \\ &= Ax + \ell\left(\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}a_{pqjk}\right)_{p,q=1}^{\infty}. \end{aligned} \tag{21}$$

If $x = (x_{jk}) \in \mathscr{L}_\infty$, then by ($AC_3$) we have $A(Sx - x) \in \mathscr{F}_{[\lambda,\mu]}$. Since by ($AC_1$), A is bounded linear operator on $\mathscr{L}_\infty$, we get $A\mathscr{M}_{[\lambda,\mu]} \subset \mathscr{F}_{[\lambda,\mu]}$. This yields $Ax \in \mathscr{F}_{[\lambda,\mu]}$. Now from condition ($AC_2$) and (21), $Ay \in \mathscr{F}_{[\lambda,\mu]}$. Therefore $A \in (\mathscr{F}_{[\lambda,\mu]}, \mathscr{F}_{[\lambda,\mu]})$. □

Corollary 5. *A matrix $A = (a_{pqjk}) \in (\mathscr{F}_{[\lambda,\mu]}, \mathscr{F}_{[\lambda,\mu]})_{reg}$ if and only if conditions (AC_1) and (AC_2) with $\mathscr{F}_{[\lambda,\mu]}$-$\lim a = 1$ and (AC_3) hold.*

3. Some Core Theorems

The core or Knopp core of a real number single sequence x is the closed interval $[\liminf x, \limsup x]$ (see [27, 28]). In 1999, Patterson [29] extended the Knopp core from single sequences to double sequences and called it Pringsheim core (shortly, P-core) which is given by [P-lim inf x, P-lim sup x]. In the recent past, the M-core and σ-core for double sequences have been defined and studied by Mursaleen and Edely [30] and Mursaleen and Mohiuddine [31, 32], respectively, while the σ-core for single sequences is given by Mishra et al. [33]. In 2011, Kayaduman and Çakan [34] presented the concept of Cesáro core of double sequences.

We define the following sublinear functional on $\mathscr{L}_\infty$:

$$\Gamma(x) = \limsup_{m,n\to\infty}\sup_{s,t}\frac{1}{\lambda_m\mu_n}\sum_{j\in J_m}\sum_{k\in I_n}x_{j+s,k+t}. \tag{22}$$

Then we define the $\mathscr{F}_{[\lambda,\mu]}$-core of a real-valued bounded double sequence (x_{jk}) to be the closed interval $[-\Gamma(-x), \Gamma(x)]$.

Since every BP-convergent double sequence is $\mathscr{F}_{[\lambda,\mu]}$-convergent, we have

$$\Gamma(x) \leq L(x), \tag{23}$$

where $L(x) = P$-lim sup x, and hence it follows that $\mathscr{F}_{[\lambda,\mu]}$-core$\{x\} \subseteq P$-core$\{x\}$ for all $x \in \mathscr{L}_\infty$.

Theorem 6. *For every $x = (x_{jk}) \in \mathscr{F}_{[\lambda,\mu]}$,*

$$\Gamma(Ax) \leq \Gamma(x) \quad \left(\text{or } \mathscr{F}_{[\lambda,\mu]}\text{-core }\{Ax\} \subset \mathscr{F}_{[\lambda,\mu]}\text{-core }\{x\}\right) \tag{24}$$

if and only if

(CR_1) *A is $\mathcal{F}_{[\lambda,\mu]}$-regular,*

(CR_2) $\lim\sup_{m,n\to\infty}\sup_{s,t}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}|\alpha(m,n,s,t,j,k)| = 1$,

where

$$\alpha(m,n,s,t,j,k) = \frac{1}{\lambda_m\mu_n}\sum_{p\in J_m}\sum_{q\in I_n} a_{p+s,q+t,j,k}. \quad (25)$$

Proof.

Necessity. Suppose that (24) holds for all $x = (x_{jk}) \in \mathcal{F}_{[\lambda,\mu]}$. One obtains

$$-\Gamma(-x) \le -\Gamma(-Ax) \le \Gamma(Ax) \le \Gamma(x); \quad (26)$$

that is,

$$\begin{aligned}\mathcal{F}_{[\lambda,\mu]}\text{-lim inf } x &\le -\Gamma(-Ax) \le \Gamma(Ax)\\ &\le \mathcal{F}_{[\lambda,\mu]}\text{-lim sup } x.\end{aligned} \quad (27)$$

If $x = (x_{jk}) \in \mathcal{F}_{[\lambda,\mu]}$, then

$$-\Gamma(-Ax) = \Gamma(Ax) = \mathcal{F}_{[\lambda,\mu]}\text{-lim } x; \quad (28)$$

that is,

$$\mathcal{F}_{[\lambda,\mu]}\text{-lim}(Ax) = \mathcal{F}_{[\lambda,\mu]}\text{-lim } x. \quad (29)$$

Therefore A is $\mathcal{F}_{[\lambda,\mu]}$-regular. This yields the necessity of (CR_1).

Now, with the help of Lemma 2.1 of [35], there is a double sequence $x = (x_{jk}) \in \mathcal{L}_\infty$ such that $\|x\| \le 1$ and

$$\begin{aligned}&\lim\sup_{m,n\to\infty}\sup_{s,t}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\alpha(m,n,s,t,j,k)x_{jk}\\ &\quad = \lim\sup_{m,n\to\infty}\sup_{s,t}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}|\alpha(m,n,s,t,j,k)|.\end{aligned} \quad (30)$$

If a double sequence $x = (x_{jk})$ defined by

$$x_{jk} = \begin{cases}1; & \text{if } j = k,\\ 0; & \text{otherwise},\end{cases} \quad (31)$$

then

$$\begin{aligned}1 = \Gamma'(Ax) &= \lim\inf_{m,n\to\infty}\sup_{s,t}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}|\alpha(m,n,s,t,j,k)|\\ &\le \Gamma(Ax) \le \Gamma(x) \le \|x\| \le 1,\end{aligned} \quad (32)$$

where

$$\Gamma'(x) = \lim\inf_{m,n\to\infty}\sup_{s,t}\frac{1}{\lambda_m\mu_n}\sum_{j=0}^{p}\sum_{k=0}^{q}x_{j+s,k+t}. \quad (33)$$

This yields the necessity of (CR_2).

Sufficiency. We know that $\mathcal{C}_{BP} \subset \mathcal{F}_{[\lambda,\mu]}$. Following the lines of Theorem 2 of [31] for translation mapping, one obtains

$$\Gamma(Ax) \le L(x). \quad (34)$$

For any $x' \in \mathcal{M}_{[\lambda,\mu]}$, we have

$$\Gamma(Ax + Ax') \le L(x + x'). \quad (35)$$

Taking infimum over $x' \in \mathcal{M}_{[\lambda,\mu]}$, we obtain

$$\begin{aligned}\inf_{x'\in\mathcal{M}_{[\lambda,\mu]}}\Gamma(Ax + Ax') &\le \inf_{x'\in\mathcal{M}_{[\lambda,\mu]}}\lim\sup_{m,n\to\infty}(x_{mn} + x'_{mn})\\ &= w(x), \quad \text{say.}\end{aligned} \quad (36)$$

Thus

$$\begin{aligned}&\sup_{s,t}\lim\sup_{m,n\to\infty}\Omega_{mnst}(Ax) + \inf_{x'\in\mathcal{M}_{[\lambda,\mu]}}\inf_{s,t}\lim\inf_{m,n\to\infty}\Omega_{mnst}(Ax')\\ &\quad \le w(x).\end{aligned} \quad (37)$$

Since $Ax' \in \mathcal{F}_{[\lambda,\mu]}$, we can write

$$Ax' = \overline{x} + \ell E, \quad (38)$$

where $\overline{x} \in \mathcal{M}_{[\lambda,\mu]}$, $\ell = \mathcal{F}_{[\lambda,\mu]}\text{-lim}Ax'$ $(= \mathcal{F}_{[\lambda,\mu]}\text{-lim}x'$, since A is $\mathcal{F}_{[\lambda,\mu]}$-regular). Operating Ω_{mnst} to (38), one obtains

$$\Omega_{mnst}(Ax') = \Omega_{mnst}(\overline{x}) + \Omega_{mnst}(\ell E). \quad (39)$$

By $[\lambda,\mu]$-almost regularity, we have

$$\begin{aligned}\lim\inf_{m,n\to\infty}\Omega_{mnst}(Ax') &= \lim_{m,n\to\infty}\Omega_{mnst}(\overline{x})\\ &\quad + \ell\lim_{m,n\to\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\alpha(m,n,s,t,j,k).\end{aligned} \quad (40)$$

From the definition of $\mathcal{M}_{[\lambda,\mu]}$, we get

$$\lim_{m,n\to\infty}\Omega_{mnst}(\overline{x}) = 0 \quad (41)$$

uniformly in s, t. Also

$$\lim_{m,n\to\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\alpha(m,n,s,t,j,k) = 1. \quad (42)$$

Therefore we obtain from (40) that

$$\lim\inf_{m,n\to\infty}\Omega_{mnst}(Ax') = 1 \quad (43)$$

uniformly in s, t. Equations (37) and (43) give that

$$\Gamma(Ax) + 1 \le w(x); \quad (44)$$

that is,

$$\Gamma(Ax) \le w(x). \quad (45)$$

As $w(x) = \Gamma(x)$, one obtains $\Gamma(Ax) \le \Gamma(x)$. □

Note that $\mathcal{F}_{[\lambda,\mu]}$-core$\{x\} \subseteq P$-core$\{x\}$. This motivates us to prove the following result by adding a condition to get a more general result.

Theorem 7. *For $x = (x_{jk}) \in \mathcal{L}_\infty$, if*

$$\lim_{s,t} (x_{st} - x_{s+1,t+1}) = 0 \tag{46}$$

holds, then P-core{x} $\subseteq \mathcal{F}_{[\lambda,\mu]}$-core{x}.

Proof. By the definition of P-core and $\mathcal{F}_{[\lambda,\mu]}$-core, we have to show that $L(x) \le \Gamma(x)$. Let $\Gamma(x) = \ell$. Then, for given $\epsilon > 0$, for all j, k, s, t and for large m, n it follows from the definition of Γ that

$$\frac{1}{\lambda_m \mu_n} \sum_{j\in J_m} \sum_{k\in I_n} x_{j+s,k+t} < \ell + \frac{\epsilon}{2}. \tag{47}$$

We can write

$$\begin{aligned} x_{st} &= x_{st} - \frac{1}{\lambda_m \mu_n} \sum_{j\in J_m} \sum_{k\in I_n} x_{j+s,k+t} + \frac{1}{\lambda_m \mu_n} \sum_{j\in J_m} \sum_{k\in I_n} x_{j+s,k+t} \\ &\le \left| x_{st} - \frac{1}{\lambda_m \mu_n} \sum_{j\in J_m} \sum_{k\in I_n} x_{j+s,k+t} \right| + \ell + \frac{\epsilon}{2}. \end{aligned} \tag{48}$$

Since (46) holds, for given $\epsilon > 0$, we get that

$$\left| x_{st} - x_{j+s,k+t} \right| < \frac{\epsilon}{2}, \tag{49}$$

for all $j, k \ge 0$. Thus we have

$$\begin{aligned} &\left| x_{st} - \frac{1}{\lambda_m \mu_n} \sum_{j\in J_m} \sum_{k\in I_n} x_{j+s,k+t} \right| \\ &\quad = \frac{1}{\lambda_m \mu_n} \left| \lambda_m \mu_n x_{st} - \sum_{j\in J_m} \sum_{k\in I_n} x_{j+s,k+t} \right| \\ &\quad \le \frac{1}{\lambda_m \mu_n} \lambda_m \mu_n \left| x_{st} - x_{j+s,k+t} \right|, \quad j, k \ge 0. \end{aligned} \tag{50}$$

Equation (49) yields

$$\left| x_{st} - \frac{1}{\lambda_m \mu_n} \sum_{j\in J_m} \sum_{k\in I_n} x_{j+s,k+t} \right| < \frac{\epsilon}{2}. \tag{51}$$

Taking $\limsup_{s,t}$ in (48) and using (51), one obtains $L(x) \le \ell + \epsilon$. Since ϵ is arbitrary, we obtain $L(x) \le \Gamma(x)$. □

Corollary 8. *If (46) holds and $x = (x_{jk})$ is $\mathcal{F}_{[\lambda,\mu]}$-convergent, then x is convergent.*

Finally, we define the concepts of $[\lambda, \mu]$-almost uniformly positive and $[\lambda, \mu]$-almost absolutely equivalent and establish a theorem related to these concepts.

Definition 9. A matrix $A = (a_{pqjk})$ is said to be $[\lambda, \mu]$-*almost uniformly positive*, denoted by $\mathcal{F}_{[\lambda,\mu]}$-uniformly positive, if

$$\lim_{m,n\to\infty} \sup_{s,t} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \frac{1}{\lambda_m \mu_n} \left| \sum_{p\in J_m} \sum_{q\in I_n} a_{p+s,q+t,j,k} \right| = 1. \tag{52}$$

Definition 10. Let $A = (a_{pqjk})$ and $B = (b_{pqjk})$ be two $\mathcal{F}_{[\lambda,\mu]}$-regular matrices and

$$y_{pq} = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} a_{pqjk} x_{jk}, \qquad y'_{pq} = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} b_{pqjk} x_{jk}. \tag{53}$$

Then A and B are said to be $[\lambda, \mu]$-*almost absolutely equivalent*, denoted by $\mathcal{F}_{[\lambda,\mu]}$-absolutely equivalent, on $\mathcal{L}_\infty$ whenever $\mathcal{F}_{[\lambda,\mu]}$-lim$(y_{pq} - y'_{pq}) = 0$; that is, either (y_{pq}) and (y'_{pq}) both tend to the same $\mathcal{F}_{[\lambda,\mu]}$-limit or neither of them tends to a $\mathcal{F}_{[\lambda,\mu]}$-limit, but their difference tends to $\mathcal{F}_{[\lambda,\mu]}$-limit zero.

Before proceeding further, first we state the following lemma which we will use to our next result.

Lemma 11. *For $x, y \in \mathcal{L}_\infty$, if $\mathcal{F}_{[\lambda,\mu]}$-lim|x − y| = 0, then $\mathcal{F}_{[\lambda,\mu]}$-core{x} = $\mathcal{F}_{[\lambda,\mu]}$-core{y}.*

Proof of the lemma is straightforward and thus omitted.

Theorem 12. *Let $A = (a_{pqjk})$ be a $\mathcal{F}_{[\lambda,\mu]}$-regular matrix. Then, $\Gamma(Ax) \le \Gamma(x)$ for all $x = (x_{jk}) \in \mathcal{L}_\infty$ if and only if there is a $\mathcal{F}_{[\lambda,\mu]}$-regular matrix $B = (b_{pqjk})$ such that B is $\mathcal{F}_{[\lambda,\mu]}$-uniformly positive and $\mathcal{F}_{[\lambda,\mu]}$-absolutely equivalent with A on $\mathcal{L}_\infty$.*

Proof. Let there be a $\mathcal{F}_{[\lambda,\mu]}$-regular matrix B such that B is $\mathcal{F}_{[\lambda,\mu]}$-uniformly positive and $\mathcal{F}_{[\lambda,\mu]}$-absolutely equivalent with A on $\mathcal{L}_\infty$. Then, by (53) and $\mathcal{F}_{[\lambda,\mu]}$-absolutely equivalent of A and B, we have

$$\begin{aligned} &\mathcal{F}_{[\lambda,\mu]}\text{-lim} \left| y_{mn} - y'_{mn} \right| \\ &\quad = \lim_{m,n\to\infty} \sup_{s,t} \left| \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \frac{1}{\lambda_m \mu_n} \right. \\ &\qquad \left. \times \sum_{p\in J_m} \sum_{q\in I_n} \left[a_{p+s,q+t,j,k} - b_{p+s,q+t,j,k} \right] x_{jk} \right| \\ &\quad \le \|x\| \lim_{m,n\to\infty} \sup_{s,t} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \frac{1}{\lambda_m \mu_n} \\ &\qquad \times \left| \sum_{p\in J_m} \sum_{q\in I_n} \left[a_{p+s,q+t,j,k} - b_{p+s,q+t,j,k} \right] \right| \\ &\quad = 0, \end{aligned} \tag{54}$$

uniformly in s, t. Now, by Lemma 11, $\mathcal{F}_{[\lambda,\mu]}$-core$\{Ax\}$ = $\mathcal{F}_{[\lambda,\mu]}$-core$\{Bx\}$ for all $x \in \mathcal{L}_\infty$. By Theorem 6, we have $\Gamma(Ax) \le \Gamma(x)$, since x is arbitrary.

Conversely, let $\Gamma(Ax) \leq \gamma(x)$ for all $x \in \mathscr{L}_{\infty}$. Then by Theorem 6, A is $\mathscr{F}_{[\lambda,\mu]}$-uniformly positive. Now we define a matrix $B = (b_{pqjk})$ as

$$b_{pqjk} = \frac{1}{2}\left(a_{pqjk} + a_{p,q,j+1,k+1}\right) \tag{55}$$

for all $p, q, j, k \in \mathbb{N}$. Then it is easy to see that B is $\mathscr{F}_{[\lambda,\mu]}$-regular since A is $\mathscr{F}_{[\lambda,\mu]}$-regular, and

$$\mathscr{F}_{[\lambda,\mu]}\text{-}\lim(Ax) = \mathscr{F}_{[\lambda,\mu]}\text{-}\lim(Bx). \tag{56}$$

Further

$$\begin{aligned} &\lim_{m,n\to\infty}\sup_{s,t}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{1}{\lambda_m\mu_n}\left|\sum_{p\in J_m}\sum_{q\in I_n}b_{p+s,q+t,j,k}\right| \\ &\quad\leq\frac{1}{2}\left[\lim_{m,n\to\infty}\sup_{s,t}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{1}{\lambda_m\mu_n}\left|\sum_{p\in J_m}\sum_{q\in I_n}a_{p+s,q+t,j,k}\right|\right. \\ &\qquad+\lim_{m,n\to\infty}\sup_{s,t}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{1}{\lambda_m\mu_n} \\ &\qquad\qquad\left.\times\left|\sum_{p\in J_m}\sum_{q\in I_n}a_{p+s,q+t,j+1,k+1}\right|\right]. \end{aligned} \tag{57}$$

Since B is $\mathscr{F}_{[\lambda,\mu]}$-regular, we have by (57) that

$$\lim_{m,n\to\infty}\sup_{s,t}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{1}{\lambda_m\mu_n}\left|\sum_{p\in J_m}\sum_{q\in I_n}b_{p+s,q+t,j,k}\right| = 1. \tag{58}$$

Thus B is $\mathscr{F}_{[\lambda,\mu]}$-uniformly positive. Further, it follows from (56) that A and B are $\mathscr{F}_{[\lambda,\mu]}$-absolutely equivalent. □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgment

This work was funded by the Deanship of Scientific Research (DSR), King Abdulaziz University, Jeddah, under Grant no. (130-265-D1435). The authors, therefore, acknowledge with thanks DSR technical and financial support.

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