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Research Article

Algorithms of Common Solutions for Generalized Mixed Equilibria, Variational Inclusions, and Constrained Convex Minimization

Lu-Chuan Ceng[1] and Suliman Al-Homidan[2]

[1] *Department of Mathematics, Shanghai Normal University and Scientific Computing Key Laboratory of Shanghai Universities, Shanghai 200234, China*
[2] *Department of Mathematics and Statistics, King Fahd University of Petroleum and Minerals, Dhahran, Saudi Arabia*

Correspondence should be addressed to Lu-Chuan Ceng; zenglc@hotmail.com

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We introduce new implicit and explicit iterative algorithms for finding a common element of the set of solutions of the minimization problem for a convex and continuously Fréchet differentiable functional, the set of solutions of a finite family of generalized mixed equilibrium problems, and the set of solutions of a finite family of variational inclusions in a real Hilbert space. Under suitable control conditions, we prove that the sequences generated by the proposed algorithms converge strongly to a common element of three sets, which is the unique solution of a variational inequality defined over the intersection of three sets.

1. Introduction

Let C be a nonempty closed convex subset of a real Hilbert space H and let P_C be the metric projection of H onto C. Let $S : C \to C$ be a self-mapping on C. We denote by Fix (S) the set of fixed points of S and by $\mathbf{R}$ the set of all real numbers. A mapping $A : C \to H$ is called L-Lipschitz continuous if there exists a constant $L \geq 0$ such that

$$\|Ax - Ay\| \leq L \|x - y\|, \quad \forall x, y \in C. \tag{1}$$

In particular, if $L = 1$, then A is called a nonexpansive mapping [1]; if $L \in [0, 1)$, then A is called a contraction.

A mapping V is called strongly positive on H if there exists a constant $\mu > 0$ such that

$$\langle Vx, x\rangle \geq \mu\|x\|^2, \quad \forall x \in H. \tag{2}$$

Let $A : C \to H$ be a nonlinear mapping on C. We consider the following variational inequality problem (VIP): find a point $x \in C$ such that

$$\langle Ax, y - x\rangle \geq 0, \quad \forall y \in C. \tag{3}$$

The solution set of VIP (3) is denoted by VI (C, A).

The VIP (3) was first discussed by Lions [2]. There are many applications of VIP (3) in various fields; see, for example, [3–6]. It is well known that if A is a strongly monotone and Lipschitz continuous mapping on C, then VIP (3) has a unique solution. In 1976, Korpelevič [7] proposed an iterative algorithm for solving the VIP (3) in Euclidean space $\mathbf{R}^n$:

$$\begin{aligned} y_n &= P_C\left(x_n - \tau A x_n\right), \\ x_{n+1} &= P_C\left(x_n - \tau A y_n\right), \quad \forall n \geq 0, \end{aligned} \tag{4}$$

with $\tau > 0$ a given number, which is known as the extragradient method (see also [8]). The literature on the VIP is vast and Korpelevich's extragradient method has received great attention given by many authors, who improved it in various ways; see, for example, [9–24] and references therein, to name but a few.

Let $\varphi : C \to \mathbf{R}$ be a real-valued function, $A : H \to H$ a nonlinear mapping, and $\Theta : C \times C \to \mathbf{R}$ a bifunction. In 2008, Peng and Yao [12] introduced the following generalized mixed equilibrium problem (GMEP) of finding $x \in C$ such that

$$\Theta(x, y) + \varphi(y) - \varphi(x) + \langle Ax, y - x\rangle \geq 0, \quad \forall y \in C. \quad (5)$$

We denote the set of solutions of GMEP (5) by GMEP (Θ, φ, A). The GMEP (5) is very general in the sense that it includes, as special cases, optimization problems, variational inequalities, minimax problems, and Nash equilibrium problems in noncooperative games. The GMEP is further considered and studied; see, for example, [11, 14, 23, 25–28]. If $\varphi = 0$ and $A = 0$, then GMEP (5) reduces to the equilibrium problem (EP) which is to find $x \in C$ such that

$$\Theta(x, y) \geq 0, \quad \forall y \in C. \quad (6)$$

It is considered and studied in [29]. The set of solutions of EP is denoted by EP (Θ). It is worth mentioning that the EP is a unified model of several problems, namely, variational inequality problems, optimization problems, saddle point problems, complementarity problems, fixed point problems, Nash equilibrium problems, and so forth.

Throughout this paper, it is assumed as in [12] that $\Theta : C\times C \to \mathbf{R}$ is a bifunction satisfying conditions (A1)–(A4) and $\varphi : C \to \mathbf{R}$ is a lower semicontinuous and convex function with restriction (B1) or (B2), where

(A1) $\Theta(x, x) = 0$ for all $x \in C$;

(A2) Θ is monotone; that is, $\Theta(x, y) + \Theta(y, x) \leq 0$ for any $x, y \in C$;

(A3) Θ is upper-hemicontinuous; that is, for each $x, y, z \in C$,

$$\limsup_{t\to 0^+} \Theta(tz + (1-t)x, y) \leq \Theta(x, y); \quad (7)$$

(A4) $\Theta(x, \cdot)$ is convex and lower semicontinuous for each $x \in C$;

(B1) for each $x \in H$ and $r > 0$, there exists a bounded subset $D_x \subset C$ and $y_x \in C$ such that, for any $z \in C \setminus D_x$,

$$\Theta(z, y_x) + \varphi(y_x) - \varphi(z) + \frac{1}{r}\langle y_x - z, z - x\rangle < 0; \quad (8)$$

(B2) C is a bounded set.

Next we list some elementary results for the MEP.

Proposition 1 (see [26]). *Assume that $\Theta : C \times C \to \mathbf{R}$ satisfies (A1)–(A4) and let $\varphi : C \to \mathbf{R}$ be a proper lower semicontinuous and convex function. Assume that either (B1) or (B2) holds. For $r > 0$ and $x \in H$, define a mapping $T_r^{(\Theta,\varphi)} : H \to C$ as follows:*

$$T_r^{(\Theta,\varphi)}(x) = \Big\{z \in C : \Theta(z, y) + \varphi(y) - \varphi(z) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \forall y \in C\Big\}, \quad (9)$$

for all $x \in H$. Then the following hold:

(i) *for each $x \in H, T_r^{(\Theta,\varphi)}(x)$ is nonempty and single-valued;*

(ii) *$T_r^{(\Theta,\varphi)}$ is firmly nonexpansive; that is, for any $x, y \in H$,*

$$\left\|T_r^{(\Theta,\varphi)}x - T_r^{(\Theta,\varphi)}y\right\|^2 \leq \left\langle T_r^{(\Theta,\varphi)}x - T_r^{(\Theta,\varphi)}y, x - y\right\rangle; \quad (10)$$

(iii) Fix $(T_r^{(\Theta,\varphi)}) = MEP(\Theta, \varphi)$;

(iv) *$MEP(\Theta, \varphi)$ is closed and convex;*

(v) *$\|T_s^{(\Theta,\varphi)}x - T_t^{(\Theta,\varphi)}x\|^2 \leq ((s-t)/s)\langle T_s^{(\Theta,\varphi)}x - T_t^{(\Theta,\varphi)}x, T_s^{(\Theta,\varphi)}x - x\rangle$ for all $s, t > 0$ and $x \in H$.*

Let $\lambda_{n,1}, \lambda_{n,2}, \ldots, \lambda_{n,N} \in (0, 1]$, $n \geq 1$. Given the nonexpansive mappings $S_1, S_2, \ldots, S_N$ on H, for each $n \geq 1$, the mappings $U_{n,1}, U_{n,2}, \ldots, U_{n,N}$ are defined by

$$\begin{aligned}
U_{n,1} &= \lambda_{n,1}S_1 + (1 - \lambda_{n,1}) I,\\
U_{n,2} &= \lambda_{n,2}S_n U_{n,1} + (1 - \lambda_{n,2}) I,\\
U_{n,n-1} &= \lambda_{n-1}T_{n-1}U_{n,n} + (1 - \lambda_{n-1}) I,\\
&\ \ \vdots\\
U_{n,N-1} &= \lambda_{n,N-1}S_{N-1}U_{n,N-2} + (1 - \lambda_{n,N-1}) I,\\
W_n := U_{n,N} &= \lambda_{n,N} S_N U_{n,N-1} + (1 - \lambda_{n,N}) I.
\end{aligned} \quad (11)$$

The W_n is called the W-mapping generated by $S_1, \ldots, S_N$ and $\lambda_{n,1}, \lambda_{n,2}, \ldots, \lambda_{n,N}$. Note that the nonexpansivity of S_i implies the nonexpansivity of W_n.

In 2012, combining the hybrid steepest-descent method in [30] and hybrid viscosity approximation method in [31], Ceng et al. [27] proposed and analyzed the following hybrid iterative method for finding a common element of the set of solutions of GMEP (5) and the set of fixed points of a finite family of nonexpansive mappings $\{S_i\}_{i=1}^N$.

Theorem CGY (see [27, Theorem 3.1]). *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $\Theta : C \times C \to \mathbf{R}$ be a bifunction satisfying assumptions (A1)–(A4) and $\varphi : C \to \mathbf{R}$ a lower semicontinuous and convex function with restriction (B1) or (B2). Let the mapping $A : H \to H$ be δ-inverse-strongly monotone and $\{S_i\}_{i=1}^N$ a finite family of nonexpansive mappings on H such that $\cap_{i=1}^N \mathrm{Fix}(S_i) \cap GMEP(\Theta, \varphi, A) \neq \emptyset$. Let $F : H \to H$ be a κ-Lipschitzian and η-strongly monotone operator with constants $\kappa, \eta > 0$ and $V : H \to H$ a ρ-Lipschitzian mapping with constant $\rho \geq 0$. Let $0 < \mu < 2\eta/\kappa^2$ and $0 \leq \gamma\rho < \tau$, where $\tau = 1-\sqrt{1 - \mu(2\eta - \mu\kappa^2)}$. Suppose $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences in $(0, 1)$, $\{\gamma_n\}$ is a sequence in $(0, 2\delta]$, and $\{\lambda_{n,i}\}_{i=1}^N$ is a sequence in $[a, b]$ with $0 < a \leq b < 1$. For every $n \geq 1$, let W_n be the W-mapping generated by $S_1, \ldots, S_N$ and $\lambda_{n,1}, \lambda_{n,2}, \ldots, \lambda_{n,N}$.*

Given $x_1 \in H$ arbitrarily, suppose that the sequences $\{x_n\}$ and $\{u_n\}$ are generated iteratively by

$$\Theta(u_n, y) + \varphi(y) - \varphi(u_n) + \langle Ax_n, y - u_n \rangle + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq 0, \quad \forall y \in C,$$
$$x_{n+1} = \alpha_n \gamma V x_n + \beta_n x_n + ((1 - \beta_n) I - \alpha_n \mu F) W_n u_n, \quad \forall n \geq 1, \tag{12}$$

where the sequences $\{\alpha_n\}, \{\beta_n\}, \{r_n\}$ and the finite family of sequences $\{\lambda_{n,i}\}_{i=1}^N$ satisfy the conditions:

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(ii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;

(iii) $0 < \liminf_{n\to\infty} r_n \leq \limsup_{n\to\infty} r_n < 2\delta$ *and* $\lim_{n\to\infty}(r_{n+1} - r_n) = 0$;

(iv) $\lim_{n\to\infty}(\lambda_{n+1,i} - \lambda_{n,i}) = 0$ *for all* $i = 1, 2, \ldots, N$.

Then both $\{x_n\}$ and $\{u_n\}$ converge strongly to $x^* \in \cap_{i=1}^N \mathrm{Fix}(S_i) \cap GMEP(\Theta, \varphi, A)$, where $x^* = P_{\cap_{i=1}^N \mathrm{Fix}(S_i) \cap GMEP(\Theta,\varphi,A)}(I - \mu F + \gamma f)x^*$ is a unique solution of the variational inequality problem (VIP):

$$\langle (\mu F - \gamma V) x^*, x^* - x \rangle \leq 0, \quad \forall x \in \bigcap_{i=1}^N \mathrm{Fix}(S_i) \cap GMEP(\Theta, \varphi, A). \tag{13}$$

Let B be a single-valued mapping of C into H and R a multivalued mapping with $D(R) = C$. Consider the following variational inclusion: find a point $x \in C$ such that

$$0 \in Bx + Rx. \tag{14}$$

We denote by $I(B, R)$ the solution set of the variational inclusion (14). In particular, if $B = R = 0$, then $I(B, R) = C$. If $B = 0$, then problem (14) becomes the inclusion problem introduced by Rockafellar [32]. It is known that problem (14) provides a convenient framework for the unified study of optimal solutions in many optimization related areas including mathematical programming, complementarity problems, variational inequalities, optimal control, mathematical economics, and equilibria and game theory.

In 1998, Huang [33] studied problem (14) in the case where R is maximal monotone and B is strongly monotone and Lipschitz continuous with $D(R) = C = H$. Subsequently, Zeng et al. [34] further studied problem (14) in the case which is more general than Huang's one [33]. Moreover, the authors [34] obtained the same strong convergence conclusion as in Huang's result [33]. In addition, the authors also gave the geometric convergence rate estimate for approximate solutions. Also, various types of iterative algorithms for solving variational inclusions have been further studied and developed; for more details, refer to [35–39] and the references therein.

Let $\{T_n\}_{n=1}^{\infty}$ be an infinite family of nonexpansive self-mappings on C and $\{\lambda_n\}_{n=0}^{\infty}$ a sequence of nonnegative numbers in $[0, 1]$. For any $n \geq 1$, define a self-mapping W_n on C as follows:

$$\begin{aligned}
U_{n,n+1} &= I, \\
U_{n,n} &= \lambda_n T_n U_{n,n+1} + (1 - \lambda_n) I, \\
U_{n,n-1} &= \lambda_{n-1} T_{n-1} U_{n,n} + (1 - \lambda_{n-1}) I, \\
&\vdots \\
U_{n,k} &= \lambda_k T_k U_{n,k+1} + (1 - \lambda_k) I, \\
U_{n,k-1} &= \lambda_{k-1} T_{k-1} U_{n,k} + (1 - \lambda_{k-1}) I, \\
&\vdots \\
U_{n,2} &= \lambda_2 T_2 U_{n,3} + (1 - \lambda_2) I, \\
W_n = U_{n,1} &= \lambda_1 T_1 U_{n,2} + (1 - \lambda_1) I.
\end{aligned} \tag{15}$$

Such a mapping W_n is called the W-mapping generated by $T_n, T_{n-1}, \ldots, T_1$ and $\lambda_n, \lambda_{n-1}, \ldots, \lambda_1$.

Whenever $C = H$ a real Hilbert space, Yao et al. [11] very recently introduced and analyzed an iterative algorithm for finding a common element of the set of solutions of GMEP (5), the set of solutions of the variational inclusion (14), and the set of fixed points of an infinite family of nonexpansive mappings.

Theorem YCL (see [11, Theorem 3.2]). *Let $\varphi : H \to \mathbf{R}$ be a lower semicontinuous and convex function and $\Theta : H \times H \to \mathbf{R}$ a bifunction satisfying conditions (A1)–(A4) and (B1). Let V be a strongly positive bounded linear operator with coefficient $\mu > 0$ and $R : H \to 2^H$ a maximal monotone mapping. Let the mappings $A, B : H \to H$ be α-inverse-strongly monotone and β-inverse-strongly monotone, respectively. Let $f : H \to H$ be a ρ-contraction. Let $r > 0$, $\gamma > 0$, and $\lambda > 0$ be three constants such that $r < 2\alpha$, $\lambda < 2\beta$, and $0 < \gamma < \mu/\rho$. Let $\{\lambda\}_{n=1}^{\infty}$ be a sequence of positive numbers in $(0, b]$ for some $b \in (0, 1)$ and $\{T_n\}_{n=1}^{\infty}$ an infinite family of nonexpansive self-mappings on H such that $\Omega := \cap_{n=1}^{\infty} \mathrm{Fix}(T_n) \cap GMEP(\Theta, \varphi, A) \cap I(B, R) \neq \emptyset$. For arbitrarily given $x_1 \in H$, let the sequence $\{x_n\}$ be generated by*

$$\Theta(u_n, y) + \varphi(y) - \varphi(u_n) + \langle y - u_n, Ax_n \rangle + \frac{1}{r} \langle y - u_n, u_n - x_n \rangle \geq 0, \quad \forall y \in H,$$
$$x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n x_n + [(1 - \beta_n) I - \alpha_n V] W_n J_{R,\lambda}(u_n - \lambda B u_n), \quad \forall n \geq 1, \tag{16}$$

where $\{\alpha_n\}, \{\beta_n\}$ are two real sequences in $[0, 1]$ and W_n is the W-mapping defined by (15) *(with $X = H$ and $C = H$). Assume that the following conditions are satisfied:*

(C1) $\lim_{n\to\infty}\alpha_n = 0$ *and* $\sum_{n=1}^{\infty}\alpha_n = \infty$;

(C2) $0 < \liminf_{n\to\infty}\beta_n \le \limsup_{n\to\infty}\beta_n < 1$.

Then the sequence $\{x_n\}$ *converges strongly to* $x^* \in \Omega$, *where* $x^* = P_\Omega(\gamma f(x^*) + (I - V)x^*)$ *is a unique solution of the VIP:*

$$\langle(\gamma f - V)x^*, y - x^*\rangle \le 0, \quad \forall y \in \Omega. \tag{17}$$

Let $f : C \to \mathbf{R}$ be a convex and continuously Fréchet differentiable functional. Consider the convex minimization problem (CMP) of minimizing f over the constraint set C

$$\min_{x\in C} f(x) \tag{18}$$

(assuming the existence of minimizers). We denote by Γ the set of minimizers of CMP (18). It is well known that the gradient-projection algorithm (GPA) generates a sequence $\{x_n\}$ determined by the gradient ∇f and the metric projection P_C:

$$x_{n+1} := P_C(x_n - \lambda\nabla f(x_n)), \quad \forall n \ge 0, \tag{19}$$

or more generally,

$$x_{n+1} := P_C(x_n - \lambda_n\nabla f(x_n)), \quad \forall n \ge 0, \tag{20}$$

where, in both (19) and (20), the initial guess x_0 is taken from C arbitrarilyn and the parameters λ or λ_n are positive real numbers. The convergence of algorithms (19) and (20) depends on the behavior of the gradient ∇f. As a matter of fact, it is known that if ∇f is α-strongly monotone and L-Lipschitz continuous, then, for $0 < \lambda < 2\alpha/L^2$, the operator $P_C(I - \lambda\nabla f)$ is a contraction; hence, the sequence $\{x_n\}$ defined by the GPA (19) converges in norm to the unique solution of CMP (18). More generally, if the sequence $\{\lambda_n\}$ is chosen to satisfy the property

$$0 < \liminf_{n\to\infty}\lambda_n \le \limsup_{n\to\infty}\lambda_n < \frac{2\alpha}{L^2}, \tag{21}$$

then the sequence $\{x_n\}$ defined by the GPA (20) converges in norm to the unique minimizer of CMP (18). If the gradient ∇f is only assumed to be Lipschitz continuous, then $\{x_n\}$ can only be weakly convergent if H is infinite-dimensional (a counterexample is given in Section 5 of Xu [40]).

Since the Lipschitz continuity of the gradient ∇f implies that it is actually $(1/L)$-inverse-strongly monotone (ism) [41], its complement can be an averaged mapping (i.e., it can be expressed as a proper convex combination of the identity mapping and a nonexpansive mapping). Consequently, the GPA can be rewritten as the composite of a projection and an averaged mapping, which is again an averaged mapping. This shows that averaged mappings play an important role in the GPA. Recently, Xu [40] used averaged mappings to study the convergence analysis of the GPA, which is hence an operator-oriented approach.

Motivated and inspired by the above facts, we in this paper introduce new implicit and explicit iterative algorithms for finding a common element of the set of solutions of the CMP (18) for a convex functional $f : C \to \mathbf{R}$ with L-Lipschitz continuous gradient ∇f, the set of solutions of a finite family of GMEPs, and the set of solutions of a finite family of variational inclusions for maximal monotone and inverse-strong monotone mappings in a real Hilbert space. Under mild control conditions, we prove that the sequences generated by the proposed algorithms converge strongly to a common element of three sets, which is the unique solution of a variational inequality defined over the intersection of three sets. Our iterative algorithms are based on Korpelevich's extragradient method, hybrid steepest-descent method in [30], viscosity approximation method, and averaged mapping approach to the GPA in [40]. The results obtained in this paper improve and extend the corresponding results announced by many others.

2. Preliminaries

Throughout this paper, we assume that H is a real Hilbert space whose inner product and norm are denoted by $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$, respectively. Let C be a nonempty closed convex subset of H. We write $x_n \rightharpoonup x$ to indicate that the sequence $\{x_n\}$ converges weakly to x and $x_n \to x$ to indicate that the sequence $\{x_n\}$ converges strongly to x. Moreover, we use $\omega_w(x_n)$ to denote the weak ω-limit set of the sequence $\{x_n\}$; that is,

$$\omega_w(x_n) := \left\{x \in H : x_{n_i} \rightharpoonup x \text{ for some subsequence } \{x_{n_i}\} \text{ of } \{x_n\}\right\}. \tag{22}$$

Recall that a mapping $A : C \to H$ is called

(i) monotone if

$$\langle Ax - Ay, x - y\rangle \ge 0, \quad \forall x, y \in C; \tag{23}$$

(ii) η-strongly monotone if there exists a constant $\eta > 0$ such that

$$\langle Ax - Ay, x - y\rangle \ge \eta\|x - y\|^2, \quad \forall x, y \in C; \tag{24}$$

(iii) α-inverse-strongly monotone if there exists a constant $\alpha > 0$ such that

$$\langle Ax - Ay, x - y\rangle \ge \alpha\|Ax - Ay\|^2, \quad \forall x, y \in C. \tag{25}$$

It is obvious that if A is α-inverse-strongly monotone, then A is monotone and $(1/\alpha)$-Lipschitz continuous.

The metric (or nearest point) projection from H onto C is the mapping $P_C : H \to C$ which assigns to each point $x \in H$ the unique point $P_C x \in C$ satisfying the property

$$\|x - P_C x\| = \inf_{y\in C}\|x - y\| =: d(x, C). \tag{26}$$

Some important properties of projections are gathered in the following proposition.

Proposition 2. *For given $x \in H$ and $z \in C$,*

(i) $z = P_C x \Leftrightarrow \langle x - z, y - z\rangle \leq 0$, *for all* $y \in C$;

(ii) $z = P_C x \Leftrightarrow \|x - z\|^2 \leq \|x - y\|^2 - \|y - z\|^2$, *for all* $y \in C$;

(iii) $\langle P_C x - P_C y, x - y\rangle \geq \|P_C x - P_C y\|^2$, *for all* $y \in H$.

Consequently, P_C *is nonexpansive and monotone.*

If A is an α-inverse-strongly monotone mapping of C into H, then it is obvious that A is $(1/\alpha)$-Lipschitz continuous. We also have that, for all $u, v \in C$ and $\lambda > 0$,

$$\begin{aligned}\|(I - \lambda A) u - (I - \lambda A) v\|^2 &= \|(u - v) - \lambda (Au - Av)\|^2 \\ &= \|u - v\|^2 - 2\lambda \langle Au - Av, u - v\rangle \\ &\quad + \lambda^2 \|Au - Av\|^2 \\ &\leq \|u - v\|^2 + \lambda (\lambda - 2\alpha) \\ &\quad \times \|Au - Av\|^2.\end{aligned} \tag{27}$$

So, if $\lambda \leq 2\alpha$, then $I - \lambda A$ is a nonexpansive mapping from C to H.

Definition 3. A mapping $T : H \to H$ is said to be

(a) nonexpansive [1] if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in H; \tag{28}$$

(b) firmly nonexpansive if $2T - I$ is nonexpansive or, equivalently, if T is 1-inverse-strongly monotone (1-ism),

$$\langle x - y, Tx - Ty\rangle \geq \|Tx - Ty\|^2, \quad \forall x, y \in H; \tag{29}$$

alternatively, T is firmly nonexpansive if and only if T can be expressed as

$$T = \frac{1}{2}(I + S), \tag{30}$$

where $S : H \to H$ is nonexpansive; projections are firmly nonexpansive.

It can be easily seen that if T is nonexpansive, then $I - T$ is monotone. It is also easy to see that a projection P_C is 1-ism. Inverse-strongly monotone (also referred to as co-coercive) operators have been applied widely in solving practical problems in various fields.

Definition 4. A mapping $T : H \to H$ is said to be an averaged mapping if it can be written as the average of the identity I and a nonexpansive mapping; that is,

$$T \equiv (1 - \alpha) I + \alpha S, \tag{31}$$

where $\alpha \in (0, 1)$ and $S : H \to H$ is nonexpansive. More precisely, when the last equality holds, we say that T is α-averaged. Thus, firmly nonexpansive mappings (in particular, projections) are $(1/2)$-averaged mappings.

Proposition 5 (see [42]). *Let $T : H \to H$ be a given mapping.*

(i) *T is nonexpansive if and only if the complement $I - T$ is $(1/2)$-ism.*

(ii) *If T is ν-ism, then, for $\gamma > 0$, γT is (ν/γ)-ism.*

(iii) *T is averaged if and only if the complement $I - T$ is ν-ism for some $\nu > 1/2$. Indeed, for $\alpha \in (0, 1)$, T is α-averaged if and only if $I - T$ is $(1/2\alpha)$-ism.*

Proposition 6 (see [42, 43]). *Let $S, T, V : H \to H$ be given operators.*

(i) *If $T = (1 - \alpha)S + \alpha V$ for some $\alpha \in (0, 1)$ and if S is averaged and V is nonexpansive, then T is averaged.*

(ii) *T is firmly nonexpansive if and only if the complement $I - T$ is firmly nonexpansive.*

(iii) *If $T = (1 - \alpha)S + \alpha V$ for some $\alpha \in (0, 1)$ and if S is firmly nonexpansive and V is nonexpansive, then T is averaged.*

(iv) *The composite of finitely many averaged mappings is averaged. That is, if each of the mappings $\{T_i\}_{i=1}^N$ is averaged, then so is the composite $T_1 \cdots T_N$. In particular, if T_1 is α_1-averaged and T_2 is α_2-averaged, where $\alpha_1, \alpha_2 \in (0, 1)$, then the composite $T_1 T_2$ is α-averaged, where $\alpha = \alpha_1 + \alpha_2 - \alpha_1\alpha_2$.*

(v) *If the mappings $\{T_i\}_{i=1}^N$ are averaged and have a common fixed point, then*

$$\bigcap_{i=1}^{N} \operatorname{Fix}(T_i) = \operatorname{Fix}(T_1 \cdots T_N). \tag{32}$$

The notation $\operatorname{Fix}(T)$ *denotes the set of all fixed points of the mapping T; that is,* $\operatorname{Fix}(T) = \{x \in H : Tx = x\}$.

We need some facts and tools in a real Hilbert space H which are listed as lemmas below.

Lemma 7. *Let X be a real inner product space. Then the following inequality holds:*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle, \quad \forall x, y \in X. \tag{33}$$

Lemma 8. *Let $A : C \to H$ be a monotone mapping. In the context of the variational inequality problem the characterization of the projection (see Proposition 2(i)) implies*

$$u \in \operatorname{VI}(C, A) \Longleftrightarrow u = P_C(u - \lambda Au), \quad \text{for some } \lambda > 0. \tag{34}$$

Lemma 9 (see [44, Demiclosedness principle]). *Let C be a nonempty closed convex subset of a real Hilbert space H. Let T be a nonexpansive self-mapping on C with* $\operatorname{Fix}(T) \neq \emptyset$. *Then $I - T$ is demiclosed. That is, whenever $\{x_n\}$ is a sequence in C weakly converging to some $x \in C$ and the sequence $\{(I - T)x_n\}$ strongly converges to some y, it follows that $(I - T)x = y$. Here I is the identity operator of H.*

Lemma 10 (see [45]). *Let $\{s_n\}$ be a sequence of nonnegative numbers satisfying the conditions*

$$s_{n+1} \leq (1-\alpha_n) s_n + \alpha_n \beta_n, \quad \forall n \geq 1, \tag{35}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences of real numbers such that

(i) $\{\alpha_n\} \subset [0,1]$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$ *or, equivalently,*

$$\prod_{n=1}^{\infty} (1-\alpha_n) := \lim_{n\to\infty} \prod_{k=1}^{n} (1-\alpha_k) = 0; \tag{36}$$

(ii) $\limsup_{n\to\infty} \beta_n \leq 0$, *or* $\sum_{n=1}^{\infty} |\alpha_n \beta_n| < \infty$.

Then $\lim_{n\to\infty} s_n = 0$.

Lemma 11 (see [46]). *Let $\{x_n\}$ and $\{z_n\}$ be bounded sequences in a Banach space X and $\{\beta_n\}$ a sequence in $[0,1]$ with*

$$0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1. \tag{37}$$

Suppose that $x_{n+1} = (1-\beta_n) z_n + \beta_n x_n$ for each $n \geq 1$ and

$$\limsup_{n\to\infty} (\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \leq 0. \tag{38}$$

Then $\lim_{n\to\infty} \|z_n - x_n\| = 0$.

The following lemma can be easily proven and, therefore, we omit the proof.

Lemma 12. *Let $V : H \to H$ be an l-Lipschitzian mapping with constant $l \geq 0$, and let $F : H \to H$ be a κ-Lipschitzian and η-strongly monotone operator with positive constants $\kappa, \eta > 0$. Then, for $0 \leq \gamma l < \mu\eta$,*

$$\langle (\mu F - \gamma V) x - (\mu F - \gamma V) y, x - y \rangle \geq (\mu\eta - \gamma l) \|x - y\|^2, \quad \forall x, y \in H. \tag{39}$$

That is, $\mu F - \gamma V$ is strongly monotone with constant $\mu\eta - \gamma l$.

Let C be a nonempty closed convex subset of a real Hilbert space H. We introduce some notations. Let λ be a number in $(0,1]$ and let $\mu > 0$. Associating with a nonexpansive mapping $T : C \to H$, we define the mapping $T^\lambda : C \to H$ by

$$T^\lambda x := Tx - \lambda\mu F(Tx), \quad \forall x \in C, \tag{40}$$

where $F : H \to H$ is an operator such that, for some positive constants $\kappa, \eta > 0$, F is κ-Lipschitzian and η-strongly monotone on H; that is, F satisfies the conditions:

$$\|Fx - Fy\| \leq \kappa \|x - y\|, \qquad \langle Fx - Fy, x - y \rangle \geq \eta \|x - y\|^2, \tag{41}$$

for all $x, y \in H$.

Lemma 13 (see [45, Lemma 3.1]). *T^λ is a contraction provided $0 < \mu < 2\eta/\kappa^2$; that is,*

$$\|T^\lambda x - T^\lambda y\| \leq (1 - \lambda\tau) \|x - y\|, \quad \forall x, y \in C, \tag{42}$$

where $\tau = 1 - \sqrt{1 - \mu(2\eta - \mu\kappa^2)} \in (0,1]$.

Recall that a set-valued mapping $R : D(R) \subset H \to 2^H$ is called monotone if for all $x, y \in D(R)$, $f \in R(x)$ and $g \in R(y)$ imply

$$\langle f - g, x - y \rangle \geq 0. \tag{43}$$

A set-valued mapping R is called maximal monotone if R is monotone and $(I + \lambda R) D(R) = H$ for each $\lambda > 0$, where I is the identity mapping of H. We denote by $G(R)$ the graph of R. It is known that a monotone mapping R is maximal if and only if, for $(x, f) \in H \times H$, $\langle f - g, x - y \rangle \geq 0$ for every $(y, g) \in G(R)$ implies $f \in R(x)$.

Let $A : C \to H$ be a monotone, k-Lipschitz continuous mapping and let $N_C v$ be the normal cone to C at $v \in C$; that is,

$$N_C v = \{w \in H : \langle v - u, w \rangle \geq 0, \ \forall u \in C\}. \tag{44}$$

Define

$$Tv = \begin{cases} Av + N_C v, & \text{if } v \in C, \\ \emptyset, & \text{if } v \notin C. \end{cases} \tag{45}$$

Then, T is maximal monotone and $0 \in Tv$ if and only if $v \in VI(C, A)$; see [32].

Assume that $R : D(R) \subset H \to 2^H$ is a maximal monotone mapping. Then, for $\lambda > 0$, associated with R, the resolvent operator $J_{R,\lambda}$ can be defined as

$$J_{R,\lambda} x = (I + \lambda R)^{-1} x, \quad \forall x \in H. \tag{46}$$

In terms of Huang [33] (see also [34]), the following property holds for the resolvent operator $J_{R,\lambda} : H \to \overline{D(R)}$.

Lemma 14. *$J_{R,\lambda}$ is single-valued and firmly nonexpansive; that is,*

$$\langle J_{R,\lambda} x - J_{R,\lambda} y, x - y \rangle \geq \|J_{R,\lambda} x - J_{R,\lambda} y\|^2, \quad \forall x, y \in H. \tag{47}$$

Consequently, $J_{R,\lambda}$ is nonexpansive and monotone.

Lemma 15 (see [39]). *Let R be a maximal monotone mapping with $D(R) = C$. Then, for any given $\lambda > 0$, $u \in C$ is a solution of problem* (14) *if and only if $u \in C$ satisfies*

$$u = J_{R,\lambda} (u - \lambda B u). \tag{48}$$

Lemma 16 (see [34]). *Let R be a maximal monotone mapping with $D(R) = C$ and let $B : C \to H$ be a strongly monotone, continuous, and single-valued mapping. Then, for each $z \in H$, the equation $z \in (B + \lambda R) x$ has a unique solution x_λ for $\lambda > 0$.*

Lemma 17 (see [39]). *Let R be a maximal monotone mapping with $D(R) = C$ and let $B : C \to H$ be a monotone, continuous, and single-valued mapping. Then $(I + \lambda(R + B))C = H$ for each $\lambda > 0$. In this case, $R + B$ is maximal monotone.*

3. Implicit Iterative Algorithm and Its Convergence Criteria

We now state and prove the first main result of this paper.

Theorem 18. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $f : C \to \mathbf{R}$ be a convex functional with L-Lipschitz continuous gradient ∇f. Let M, N be two integers. Let Θ_k be a bifunction from $C \times C$ to $\mathbf{R}$ satisfying (A1)–(A4) and let $\varphi_k : C \to \mathbf{R} \cup \{+\infty\}$ be a proper lower semicontinuous and convex function, where $k \in \{1, 2, \ldots, M\}$. Let $R_i : C \to 2^H$ be a maximal monotone mapping and let $A_k : H \to H$ and $B_i : C \to H$ be μ_k-inverse-strongly monotone and η_i-inverse-strongly monotone, respectively, where $k \in \{1, 2, \ldots, M\}$, $i \in \{1, 2, \ldots, N\}$. Let $F : H \to H$ be a κ-Lipschitzian and η-strongly monotone operator with positive constants $\kappa, \eta > 0$. Let $V : H \to H$ be an l-Lipschitzian mapping with constant $l \geq 0$. Let $0 < \mu < 2\eta/\kappa^2$ and $0 \leq \gamma l < \tau$, where $\tau = 1 - \sqrt{1 - \mu(2\eta - \mu\kappa^2)}$. Assume that $\Omega := \cap_{k=1}^{M} GMEP(\Theta_k, \varphi_k, A_k) \cap \cap_{i=1}^{N} I(B_i, R_i) \cap \Gamma \neq \emptyset$ and that either (B1) or (B2) holds. Let $\{x_n\}$ be a sequence generated by*

$$\begin{aligned} u_n &= T_{r_{M,n}}^{(\Theta_M,\varphi_M)} \left(I - r_{M,n} A_M\right) T_{r_{M-1,n}}^{(\Theta_{M-1},\varphi_{M-1})} \\ &\quad \times \left(I - r_{M-1,n} A_{M-1}\right) \cdots T_{r_{1,n}}^{(\Theta_1,\varphi_1)} \left(I - r_{1,n} A_1\right) x_n, \\ v_n &= J_{R_N,\lambda_{N,n}} \left(I - \lambda_{N,n} B_N\right) J_{R_{N-1},\lambda_{N-1,n}} \\ &\quad \times \left(I - \lambda_{N-1,n} B_{N-1}\right) \cdots J_{R_1,\lambda_{1,n}} \left(I - \lambda_{1,n} B_1\right) u_n, \\ x_n &= s_n \gamma V x_n + \left(I - s_n \mu F\right) T_n v_n, \\ &\qquad \forall n \geq 1, \end{aligned} \tag{49}$$

where $P_C(I - \lambda_n \nabla f) = s_n I + (1 - s_n) T_n$ (here T_n is nonexpansive and $s_n = (2 - \lambda_n L)/4 \in (0, 1/2)$ for each $\lambda_n \in (0, 2/L)$). Assume that the following conditions hold:

(i) *$s_n \in (0, 1/2)$ for each $\lambda_n \in (0, 2/L)$, $\lim_{n\to\infty} s_n = 0$ ($\Leftrightarrow$ $\lim_{n\to\infty} \lambda_n = 2/L$);*

(ii) *$\{\lambda_{i,n}\} \subset [a_i, b_i] \subset (0, 2\eta_i)$, for all $i \in \{1, 2, \ldots, N\}$;*

(iii) *$\{r_{k,n}\} \subset [e_k, f_k] \subset (0, 2\mu_k)$, for all $k \in \{1, 2, \ldots, M\}$.*

Then $\{x_n\}$ converges strongly as $\lambda_n \to 2/L$ ($\Leftrightarrow s_n \to 0$) to a point $q \in \Omega$, which is a unique solution of the VIP:

$$\left\langle \left(\mu F - \gamma V\right) q, p - q \right\rangle \geq 0, \quad \forall p \in \Omega. \tag{50}$$

Equivalently, $q = P_\Omega(I - \mu F + \gamma V) q$.

Proof. First of all, let us show that the sequence $\{x_n\}$ is well defined. Indeed, since ∇f is L-Lipschitzian, it follows that ∇f is $1/L$-ism; see [41]. By Proposition 5(ii) we know that, for $\lambda > 0$, $\lambda \nabla f$ is $(1/\lambda L)$-ism. So by Proposition 5(iii) we deduce that $I - \lambda \nabla f$ is $(\lambda L/2)$-averaged. Now since the projection P_C is $(1/2)$-averaged, it is easy to see from Proposition 6(iv) that the composite $P_C(I - \lambda \nabla f)$ is $(2 + \lambda L)/4$-averaged for $\lambda \in (0, 2/L)$. Hence, we obtain that, for each $n \geq 1$, $P_C(I - \lambda_n \nabla f)$ is $((2 + \lambda_n L)/4)$-averaged for each $\lambda_n \in (0, 2/L)$. Therefore, we can write

$$P_C\left(I - \lambda_n \nabla f\right) = \frac{2 - \lambda_n L}{4} I + \frac{2 + \lambda_n L}{4} T_n = s_n I + \left(1 - s_n\right) T_n, \tag{51}$$

where T_n is nonexpansive and $s_n := s_n(\lambda_n) = (2 - \lambda_n L)/4 \in (0, 1/2)$ for each $\lambda_n \in (0, 2/L)$. It is clear that

$$\lambda_n \longrightarrow \frac{2}{L} \Longleftrightarrow s_n \longrightarrow 0. \tag{52}$$

Put

$$\begin{aligned} \Delta_n^k &= T_{r_{k,n}}^{(\Theta_k,\varphi_k)} \left(I - r_{k,n} A_k\right) T_{r_{k-1,n}}^{(\Theta_{k-1},\varphi_{k-1})} \\ &\quad \times \left(I - r_{k-1,n} A_{k-1}\right) \cdots T_{r_{1,n}}^{(\Theta_1,\varphi_1)} \left(I - r_{1,n} A_1\right) x_n, \end{aligned} \tag{53}$$

for all $k \in \{1, 2, \ldots, M\}$ and $n \geq 1$,

$$\begin{aligned} \Lambda_n^i &= J_{R_i,\lambda_{i,n}} \left(I - \lambda_{i,n} B_i\right) J_{R_{i-1},\lambda_{i-1,n}} \\ &\quad \times \left(I - \lambda_{i-1,n} B_{i-1}\right) \cdots J_{R_1,\lambda_{1,n}} \left(I - \lambda_{1,n} B_1\right), \end{aligned} \tag{54}$$

for all $i \in \{1, 2, \ldots, N\}$ and $n \geq 1$, and $\Delta_n^0 = \Lambda_n^0 = I$, where I is the identity mapping on H. Then we have that $u_n = \Delta_n^M x_n$ and $v_n = \Lambda_n^N u_n$.

Consider the following mapping G_n on H defined by

$$G_n x = s_n \gamma V x + \left(I - s_n \mu F\right) T_n \Lambda_n^N \Delta_n^M x, \quad \forall x \in H, \ n \geq 1, \tag{55}$$

where $s_n = (2 - \lambda_n L)/4 \in (0, 1/2)$ for each $\lambda_n \in (0, 2/L)$. By Proposition 1(ii) and Lemma 13 we obtain from (27) that for all $x, y \in H$

$$\begin{aligned} &\left\| G_n x - G_n y \right\| \\ &\leq s_n \gamma \left\| V x - V y \right\| + \left\| \left(I - s_n \mu F\right) T_n \Lambda_n^N \Delta_n^M x \right. \\ &\qquad \left. - \left(I - s_n \mu F\right) T_n \Lambda_n^N \Delta_n^M y \right\| \\ &\leq s_n \gamma l \left\| x - y \right\| + \left(1 - s_n \tau\right) \left\| \Lambda_n^N \Delta_n^M x - \Lambda_n^N \Delta_n^M y \right\| \\ &\leq s_n \gamma l \left\| x - y \right\| + \left(1 - s_n \tau\right) \left\| \left(I - \lambda_{N,n} B_N\right) \Lambda_n^{N-1} \Delta_n^M x \right. \\ &\qquad \left. - \left(I - \lambda_{N,n} B_N\right) \Lambda_n^{N-1} \Delta_n^M y \right\| \\ &\leq s_n \gamma l \left\| x - y \right\| + \left(1 - s_n \tau\right) \left\| \Lambda_n^{N-1} \Delta_n^M x - \Lambda_n^{N-1} \Delta_n^M y \right\| \\ &\ \vdots \\ &\leq s_n \gamma l \left\| x - y \right\| + \left(1 - s_n \tau\right) \left\| \Lambda_n^0 \Delta_n^M x - \Lambda_n^0 \Delta_n^M y \right\| \end{aligned}$$

$$\begin{aligned}
&= s_n\gamma l\|x-y\| + (1-s_n\tau)\left\|\Delta_n^M x - \Delta_n^M y\right\| \\
&\le s_n\gamma l\|x-y\| + (1-s_n\tau)\left\|(I-r_{M,n}A_M)\Delta_n^{M-1}x\right. \\
&\qquad\left. - (I-r_{M,n}A_M)\Delta_n^{M-1}y\right\| \\
&\le s_n\gamma l\|x-y\| + (1-s_n\tau)\left\|\Delta_n^{M-1}x - \Delta_n^{M-1}y\right\| \\
&\;\;\vdots \\
&\le s_n\gamma l\|x-y\| + (1-s_n\tau)\left\|\Delta_n^0 x - \Delta_n^0 y\right\| \\
&= s_n\gamma l\|x-y\| + (1-s_n\tau)\|x-y\| \\
&= (1-s_n(\tau-\gamma l))\|x-y\|.
\end{aligned} \tag{56}$$

Since $0 < 1 - s_n(\tau - \gamma l) < 1$, $G_n : H \to H$ is a contraction. Therefore, by the Banach contraction principle, G_n has a unique fixed point $x_n \in H$, which uniquely solves the fixed point equation:

$$x_n = s_n\gamma V x_n + (I - s_n\mu F)T_n\Lambda_n^N\Delta_n^M x_n. \tag{57}$$

This shows that the sequence $\{x_n\}$ is defined well.

Note that $0 \le \gamma l < \tau$ and $\mu\eta \ge \tau \Leftrightarrow \kappa \ge \eta$. Hence, by Lemma 12 we know that

$$\langle(\mu F - \gamma V)x - (\mu F - \gamma V)y, x - y\rangle \ge (\mu\eta - \gamma l)\|x-y\|^2, \quad \forall x, y \in H. \tag{58}$$

That is, $\mu F - \gamma V$ is strongly monotone for $0 \le \gamma l < \tau \le \mu\eta$. Moreover, it is clear that $\mu F - \gamma V$ is Lipschitz continuous. So the VIP (50) has only one solution. Below we use $q \in \Omega$ to denote the unique solution of the VIP (50).

Now, let us show that $\{x_n\}$ is bounded. In fact, take $p \in \Omega$ arbitrarily. Then from (27) and Proposition 1(ii) we have

$$\begin{aligned}
\|u_n - p\| &= \left\|T_{r_{M,n}}^{(\Theta_M,\varphi_M)}(I-r_{M,n}A_M)\Delta_n^{M-1}x_n\right. \\
&\qquad\left. - T_{r_{M,n}}^{(\Theta_M,\varphi_M)}(I-r_{M,n}A_M)\Delta_n^{M-1}p\right\| \\
&\le \left\|(I-r_{M,n}A_M)\Delta_n^{M-1}x_n\right. \\
&\qquad\left. - (I-r_{M,n}A_M)\Delta_n^{M-1}p\right\| \\
&\le \left\|\Delta_n^{M-1}x_n - \Delta_n^{M-1}p\right\| \\
&\;\;\vdots \\
&\le \left\|\Delta_n^0 x_n - \Delta_n^0 p\right\| \\
&= \|x_n - p\|.
\end{aligned} \tag{59}$$

Similarly, we have

$$\begin{aligned}
\|v_n - p\| &= \left\|J_{R_N,\lambda_{N,n}}(I-\lambda_{N,n}B_N)\Lambda_n^{N-1}u_n\right. \\
&\qquad\left. - J_{R_N,\lambda_{N,n}}(I-\lambda_{N,n}B_N)\Lambda_n^{N-1}p\right\| \\
&\le \left\|(I-\lambda_{N,n}B_N)\Lambda_n^{N-1}u_n - (I-\lambda_{N,n}B_N)\Lambda_n^{N-1}p\right\| \\
&\le \left\|\Lambda_n^{N-1}u_n - \Lambda_n^{N-1}p\right\| \\
&\;\;\vdots \\
&\le \left\|\Lambda_n^0 u_n - \Lambda_n^0 p\right\| \\
&= \|u_n - p\|.
\end{aligned} \tag{60}$$

Combining (59) and (60), we have

$$\|v_n - p\| \le \|x_n - p\|. \tag{61}$$

Since

$$p = P_C(I - \lambda_n\nabla f)p = s_n p + (1-s_n)T_n p, \quad \forall \lambda_n \in \left(0, \frac{2}{L}\right), \tag{62}$$

where $s_n := s_n(\lambda_n) = (2 - \lambda_n L)/4 \in (0, 1/2)$, it is clear that $T_n p = p$ for each $\lambda_n \in (0, 2/L)$. Thus, utilizing Lemma 13 and the nonexpansivity of T_n, we obtain from (61) that

$$\begin{aligned}
\|x_n - p\| &= \|s_n(\gamma V x_n - \mu F p) + (I - s_n\mu F)T_n v_n \\
&\qquad - (I - s_n\mu F)T_n p\| \\
&\le \|(I - s_n\mu F)T_n v_n - (I - s_n\mu F)T_n p\| \\
&\quad + s_n\|\gamma V x_n - \mu F p\| \\
&\le (1 - s_n\tau)\|v_n - p\| \\
&\quad + s_n(\gamma\|V x_n - V p\| + \|\gamma V p - \mu F p\|) \\
&\le (1 - s_n\tau)\|x_n - p\| \\
&\quad + s_n(\gamma l\|x_n - p\| + \|\gamma V p - \mu F p\|) \\
&= (1 - s_n(\tau - \gamma l))\|x_n - p\| + s_n\|\gamma V p - \mu F p\|.
\end{aligned} \tag{63}$$

This implies that $\|x_n - p\| \le \|\gamma V p - \mu F p\|/(\tau - \gamma l)$. Hence, $\{x_n\}$ is bounded. So, according to (59) and (61) we know that $\{u_n\}$, $\{v_n\}$, $\{T_n v_n\}$, $\{V x_n\}$, and $\{F T_n v_n\}$ are bounded.

Next let us show that $\|u_n - x_n\| \to 0$, $\|v_n - u_n\| \to 0$, and $\|x_n - T_n x_n\| \to 0$ as $n \to \infty$.

Indeed, from (27) it follows that for all $i \in \{1,2,\ldots,N\}$ and $k \in \{1,2,\ldots,M\}$

$$\begin{aligned}
\|v_n - p\|^2 &= \left\|\Lambda_n^N u_n - p\right\|^2 \\
&\le \left\|\Lambda_n^i u_n - p\right\|^2 \\
&= \Big\|J_{R_i,\lambda_{i,n}}\left(I-\lambda_{i,n}B_i\right)\Lambda_n^{i-1}u_n \\
&\quad - J_{R_i,\lambda_{i,n}}\left(I-\lambda_{i,n}B_i\right)p\Big\|^2 \\
&\le \left\|\left(I-\lambda_{i,n}B_i\right)\Lambda_n^{i-1}u_n - \left(I-\lambda_{i,n}B_i\right)p\right\|^2 \\
&\le \left\|\Lambda_n^{i-1}u_n - p\right\|^2 + \lambda_{i,n}\left(\lambda_{i,n}-2\eta_i\right) \\
&\quad \times \left\|B_i\Lambda_n^{i-1}u_n - B_i p\right\|^2 \\
&\le \|u_n - p\|^2 + \lambda_{i,n}\left(\lambda_{i,n}-2\eta_i\right) \\
&\quad \times \left\|B_i\Lambda_n^{i-1}u_n - B_i p\right\|^2 \\
&\le \|x_n - p\|^2 + \lambda_{i,n}\left(\lambda_{i,n}-2\eta_i\right) \\
&\quad \times \left\|B_i\Lambda_n^{i-1}u_n - B_i p\right\|^2, \\
\|u_n - p\|^2 &= \left\|\Delta_n^M x_n - p\right\|^2 \\
&\le \left\|\Delta_n^k x_n - p\right\|^2 \\
&= \Big\|T_{r_{k,n}}^{(\Theta_k,\varphi_k)}\left(I-r_{k,n}A_k\right)\Delta_n^{k-1}x_n \\
&\quad - T_{r_{k,n}}^{(\Theta_k,\varphi_k)}\left(I-r_{k,n}A_k\right)p\Big\|^2 \\
&\le \left\|\left(I-r_{k,n}A_k\right)\Delta_n^{k-1}x_n - \left(I-r_{k,n}A_k\right)p\right\|^2 \\
&\le \left\|\Delta_n^{k-1}x_n - p\right\|^2 + r_{k,n}\left(r_{k,n}-2\mu_k\right) \\
&\quad \times \left\|A_k\Delta_n^{k-1}x_n - A_k p\right\|^2 \\
&\le \|x_n - p\|^2 + r_{k,n}\left(r_{k,n}-2\mu_k\right) \\
&\quad \times \left\|A_k\Delta_n^{k-1}x_n - A_k p\right\|^2.
\end{aligned} \tag{64}$$

Thus, utilizing Lemma 7, from (49) and (64) we have

$$\begin{aligned}
&\|x_n - p\|^2 \\
&= \big\|s_n\left(\gamma V x_n - \mu F p\right) + \left(I - s_n\mu F\right)T_n v_n \\
&\quad - \left(I - s_n\mu F\right)T_n p\big\|^2 \\
&\le \left\|\left(I - s_n\mu F\right)T_n v_n - \left(I - s_n\mu F\right)T_n p\right\|^2 \\
&\quad + 2s_n\left\langle \gamma V x_n - \mu F p, x_n - p\right\rangle \\
&\le \left(1 - s_n\tau\right)^2\|v_n - p\|^2 + 2s_n\left\langle \gamma V x_n - \mu F p, x_n - p\right\rangle \\
&= \left(1 - s_n\tau\right)^2\|v_n - p\|^2 + 2s_n\gamma\left\langle V x_n - V p, x_n - p\right\rangle \\
&\quad + 2s_n\left\langle \gamma V p - \mu F p, x_n - p\right\rangle \\
&\le \left(1 - s_n\tau\right)^2\|v_n - p\|^2 + 2s_n\gamma l\|x_n - p\|^2 \\
&\quad + 2s_n\|\gamma V p - \mu F p\|\,\|x_n - p\| \\
&\le \left(1 - s_n\tau\right)^2\Big[\|u_n - p\|^2 + \lambda_{i,n}\left(\lambda_{i,n}-2\eta_i\right) \\
&\qquad \times \left\|B_i\Lambda_n^{i-1}u_n - B_i p\right\|^2\Big] \\
&\quad + 2s_n\gamma l\|x_n - p\|^2 + 2s_n\|\gamma V p - \mu F p\|\,\|x_n - p\| \\
&\le \left(1 - s_n\tau\right)^2\Big[\|x_n - p\|^2 + r_{k,n}\left(r_{k,n}-2\mu_k\right) \\
&\qquad \times \left\|A_k\Delta_n^{k-1}x_n - A_k p\right\|^2 \\
&\qquad + \lambda_{i,n}\left(\lambda_{i,n}-2\eta_i\right)\left\|B_i\Lambda_n^{i-1}u_n - B_i p\right\|^2\Big] \\
&\quad + 2s_n\gamma l\|x_n - p\|^2 + 2s_n\|\gamma V p - \mu F p\|\,\|x_n - p\| \\
&= \left[1 - 2s_n\left(\tau - \gamma l\right) + s_n^2\tau^2\right]\|x_n - p\|^2 \\
&\quad - \left(1 - s_n\tau\right)^2\Big[r_{k,n}\left(2\mu_k - r_{k,n}\right)\left\|A_k\Delta_n^{k-1}x_n - A_k p\right\|^2 \\
&\qquad + \lambda_{i,n}\left(2\eta_i - \lambda_{i,n}\right)\left\|B_i\Lambda_n^{i-1}u_n - B_i p\right\|^2\Big] \\
&\quad + 2s_n\|\gamma V p - \mu F p\|\,\|x_n - p\| \\
&\le \|x_n - p\|^2 + s_n^2\tau^2\|x_n - p\|^2 - \left(1 - s_n\tau\right)^2 \\
&\quad \times \Big[r_{k,n}\left(2\mu_k - r_{k,n}\right)\left\|A_k\Delta_n^{k-1}x_n - A_k p\right\|^2 \\
&\qquad + \lambda_{i,n}\left(2\eta_i - \lambda_{i,n}\right)\left\|B_i\Lambda_n^{i-1}u_n - B_i p\right\|^2\Big] \\
&\quad + 2s_n\|\gamma V p - \mu F p\|\,\|x_n - p\|,
\end{aligned} \tag{65}$$

which implies that

$$\begin{aligned}
&\left(1 - s_n\tau\right)^2\Big[r_{k,n}\left(2\mu_k - r_{k,n}\right)\left\|A_k\Delta_n^{k-1}x_n - A_k p\right\|^2 \\
&\qquad + \lambda_{i,n}\left(2\eta_i - \lambda_{i,n}\right)\left\|B_i\Lambda_n^{i-1}u_n - B_i p\right\|^2\Big] \\
&\quad \le s_n^2\tau^2\|x_n - p\|^2 + 2s_n\|\gamma V p - \mu F p\|\,\|x_n - p\|.
\end{aligned} \tag{66}$$

Since $\{\lambda_{i,n}\} \subset [a_i, b_i] \subset (0, 2\eta_i)$ and $\{r_{k,n}\} \subset [e_k, f_k] \subset (0, 2\mu_k)$ for all $i \in \{1,2,\ldots,N\}$ and $k \in \{1,2,\ldots,M\}$, from $s_n \to 0$ we conclude immediately that

$$\begin{aligned}
\lim_{n\to\infty}\left\|A_k\Delta_n^{k-1}x_n - A_k p\right\| &= 0, \\
\lim_{n\to\infty}\left\|B_i\Lambda_n^{i-1}u_n - B_i p\right\| &= 0,
\end{aligned} \tag{67}$$

for all $k \in \{1,2,\ldots,M\}$ and $i \in \{1,2,\ldots,N\}$.

Furthermore, by Proposition 1(ii) we obtain that for each $k \in \{1, 2, \ldots, M\}$

$$\begin{aligned}
&\left\|\Delta_n^k x_n - p\right\|^2 \\
&= \left\|T_{r_{k,n}}^{(\Theta_k,\varphi_k)} (I - r_{k,n}A_k) \Delta_n^{k-1} x_n - T_{r_{k,n}}^{(\Theta_k,\varphi_k)} (I - r_{k,n}A_k) p\right\|^2 \\
&\leq \left\langle (I - r_{k,n}A_k) \Delta_n^{k-1} x_n - (I - r_{k,n}A_k) p, \Delta_n^k x_n - p \right\rangle \\
&= \frac{1}{2} \left(\left\|(I - r_{k,n}A_k) \Delta_n^{k-1} x_n - (I - r_{k,n}A_k) p\right\|^2 + \left\|\Delta_n^k x_n - p\right\|^2 - \left\|(I - r_{k,n}A_k) \Delta_n^{k-1} x_n - (I - r_{k,n}A_k) p - (\Delta_n^k x_n - p)\right\|^2 \right) \\
&\leq \frac{1}{2} \left(\left\|\Delta_n^{k-1} x_n - p\right\|^2 + \left\|\Delta_n^k x_n - p\right\|^2 - \left\|\Delta_n^{k-1} x_n - \Delta_n^k x_n - r_{k,n} (A_k \Delta_n^{k-1} x_n - A_k p)\right\|^2 \right),
\end{aligned} \tag{68}$$

which implies that

$$\begin{aligned}
&\left\|\Delta_n^k x_n - p\right\|^2 \\
&\leq \left\|\Delta_n^{k-1} x_n - p\right\|^2 - \left\|\Delta_n^{k-1} x_n - \Delta_n^k x_n - r_{k,n} (A_k \Delta_n^{k-1} x_n - A_k p)\right\|^2 \\
&= \left\|\Delta_n^{k-1} x_n - p\right\|^2 - \left\|\Delta_n^{k-1} x_n - \Delta_n^k x_n\right\|^2 - r_{k,n}^2 \left\|A_k \Delta_n^{k-1} x_n - A_k p\right\|^2 + 2r_{k,n} \left\langle \Delta_n^{k-1} x_n - \Delta_n^k x_n, A_k \Delta_n^{k-1} x_n - A_k p \right\rangle \\
&\leq \left\|\Delta_n^{k-1} x_n - p\right\|^2 - \left\|\Delta_n^{k-1} x_n - \Delta_n^k x_n\right\|^2 + 2r_{k,n} \left\|\Delta_n^{k-1} x_n - \Delta_n^k x_n\right\| \left\|A_k \Delta_n^{k-1} x_n - A_k p\right\| \\
&\leq \left\|x_n - p\right\|^2 - \left\|\Delta_n^{k-1} x_n - \Delta_n^k x_n\right\|^2 + 2r_{k,n} \left\|\Delta_n^{k-1} x_n - \Delta_n^k x_n\right\| \times \left\|A_k \Delta_n^{k-1} x_n - A_k p\right\|.
\end{aligned} \tag{69}$$

Also, by Lemma 14, we obtain that for each $i \in \{1, 2, \ldots, N\}$

$$\begin{aligned}
&\left\|\Lambda_n^i u_n - p\right\|^2 \\
&= \left\|J_{R_i,\lambda_{i,n}} (I - \lambda_{i,n}B_i) \Lambda_n^{i-1} u_n - J_{R_i,\lambda_{i,n}} (I - \lambda_{i,n}B_i) p\right\|^2 \\
&\leq \left\langle (I - \lambda_{i,n}B_i) \Lambda_n^{i-1} u_n - (I - \lambda_{i,n}B_i) p, \Lambda_n^i u_n - p \right\rangle \\
&= \frac{1}{2} \left(\left\|(I - \lambda_{i,n}B_i) \Lambda_n^{i-1} u_n - (I - \lambda_{i,n}B_i) p\right\|^2 + \left\|\Lambda_n^i u_n - p\right\|^2 - \left\|(I - \lambda_{i,n}B_i) \Lambda_n^{i-1} u_n - (I - \lambda_{i,n}B_i) p - (\Lambda_n^i u_n - p)\right\|^2 \right) \\
&\leq \frac{1}{2} \left(\left\|\Lambda_n^{i-1} u_n - p\right\|^2 + \left\|\Lambda_n^i u_n - p\right\|^2 - \left\|\Lambda_n^{i-1} u_n - \Lambda_n^i u_n - \lambda_{i,n} (B_i \Lambda_n^{i-1} u_n - B_i p)\right\|^2 \right) \\
&\leq \frac{1}{2} \left(\left\|u_n - p\right\|^2 + \left\|\Lambda_n^i u_n - p\right\|^2 - \left\|\Lambda_n^{i-1} u_n - \Lambda_n^i u_n - \lambda_{i,n} (B_i \Lambda_n^{i-1} u_n - B_i p)\right\|^2 \right),
\end{aligned} \tag{70}$$

which implies

$$\begin{aligned}
\left\|\Lambda_n^i u_n - p\right\|^2 &\leq \left\|u_n - p\right\|^2 - \left\|\Lambda_n^{i-1} u_n - \Lambda_n^i u_n - \lambda_{i,n} (B_i \Lambda_n^{i-1} u_n - B_i p)\right\|^2 \\
&= \left\|u_n - p\right\|^2 - \left\|\Lambda_n^{i-1} u_n - \Lambda_n^i u_n\right\|^2 - \lambda_{i,n}^2 \left\|B_i \Lambda_n^{i-1} u_n - B_i p\right\|^2 + 2\lambda_{i,n} \left\langle \Lambda_n^{i-1} u_n - \Lambda_n^i u_n, B_i \Lambda_n^{i-1} u_n - B_i p \right\rangle \\
&\leq \left\|u_n - p\right\|^2 - \left\|\Lambda_n^{i-1} u_n - \Lambda_n^i u_n\right\|^2 + 2\lambda_{i,n} \left\|\Lambda_n^{i-1} u_n - \Lambda_n^i u_n\right\| \times \left\|B_i \Lambda_n^{i-1} u_n - B_i p\right\|.
\end{aligned} \tag{71}$$

Thus, utilizing Lemma 7, from (49), (69), and (71) we have

$$\begin{aligned}
&\left\|x_n - p\right\|^2 \\
&= \left\|s_n (\gamma V x_n - \mu F p) + (I - s_n\mu F) T_n v_n - (I - s_n\mu F) T_n p\right\|^2 \\
&\leq \left\|(I - s_n\mu F) T_n v_n - (I - s_n\mu F) T_n p\right\|^2 + 2s_n \langle \gamma V x_n - \mu F p, x_n - p \rangle \\
&\leq (1 - s_n\tau)^2 \left\|v_n - p\right\|^2 + 2s_n\gamma l \left\|x_n - p\right\|^2 + 2s_n \left\|\gamma V p - \mu F p\right\| \left\|x_n - p\right\|
\end{aligned}$$

$$
\begin{aligned}
&= (1 - s_n\tau)^2 \left\|\Lambda_n^N u_n - p\right\|^2 + 2s_n\gamma l \|x_n - p\|^2 \\
&\quad + 2s_n \|\gamma V p - \mu F p\| \|x_n - p\| \\
&\leq (1 - s_n\tau)^2 \left\|\Lambda_n^i u_n - p\right\|^2 + 2s_n\gamma l \|x_n - p\|^2 \\
&\quad + 2s_n \|\gamma V p - \mu F p\| \|x_n - p\| \\
&\leq (1 - s_n\tau)^2 \Big[\|u_n - p\|^2 - \left\|\Lambda_n^{i-1} u_n - \Lambda_n^i u_n\right\|^2 \\
&\qquad + 2\lambda_{i,n} \left\|\Lambda_n^{i-1} u_n - \Lambda_n^i u_n\right\| \\
&\qquad \times \left\|B_i \Lambda_n^{i-1} u_n - B_i p\right\| \Big] \\
&\quad + 2s_n\gamma l \|x_n - p\|^2 + 2s_n \|\gamma V p - \mu F p\| \|x_n - p\| \\
&= (1 - s_n\tau)^2 \Big[\left\|\Delta_n^M x_n - p\right\|^2 - \left\|\Lambda_n^{i-1} u_n - \Lambda_n^i u_n\right\|^2 \\
&\qquad + 2\lambda_{i,n} \left\|\Lambda_n^{i-1} u_n - \Lambda_n^i u_n\right\| \\
&\qquad \times \left\|B_i \Lambda_n^{i-1} u_n - B_i p\right\| \Big] \\
&\quad + 2s_n\gamma l \|x_n - p\|^2 + 2s_n \|\gamma V p - \mu F p\| \|x_n - p\| \\
&\leq (1 - s_n\tau)^2 \Big[\left\|\Delta_n^k x_n - p\right\|^2 - \left\|\Lambda_n^{i-1} u_n - \Lambda_n^i u_n\right\|^2 \\
&\qquad + 2\lambda_{i,n} \left\|\Lambda_n^{i-1} u_n - \Lambda_n^i u_n\right\| \\
&\qquad \times \left\|B_i \Lambda_n^{i-1} u_n - B_i p\right\| \Big] \\
&\quad + 2s_n\gamma l \|x_n - p\|^2 + 2s_n \|\gamma V p - \mu F p\| \|x_n - p\| \\
&\leq (1 - s_n\tau)^2 \Big[\|x_n - p\|^2 - \left\|\Delta_n^{k-1} x_n - \Delta_n^k x_n\right\|^2 \\
&\qquad + 2r_{k,n} \left\|\Delta_n^{k-1} x_n - \Delta_n^k x_n\right\| \\
&\qquad \times \left\|A_k \Delta_n^{k-1} x_n - A_k p\right\| \\
&\qquad - \left\|\Lambda_n^{i-1} u_n - \Lambda_n^i u_n\right\|^2 \\
&\qquad + 2\lambda_{i,n} \left\|\Lambda_n^{i-1} u_n - \Lambda_n^i u_n\right\| \\
&\qquad \times \left\|B_i \Lambda_n^{i-1} u_n - B_i p\right\| \Big] \\
&\quad + 2s_n\gamma l \|x_n - p\|^2 \\
&\quad + 2s_n \|\gamma V p - \mu F p\| \|x_n - p\| \\
&\leq \left(1 - 2s_n(\tau - \gamma l) + s_n^2\tau^2\right) \|x_n - p\|^2 \\
&\quad - (1 - s_n\tau)^2 \Big(\left\|\Delta_n^{k-1} x_n - \Delta_n^k x_n\right\|^2 \\
&\qquad + \left\|\Lambda_n^{i-1} u_n - \Lambda_n^i u_n\right\|^2 \Big) \\
&\quad + 2r_{k,n} \left\|\Delta_n^{k-1} x_n - \Delta_n^k x_n\right\| \left\|A_k \Delta_n^{k-1} x_n - A_k p\right\| \\
&\quad + 2\lambda_{i,n} \left\|\Lambda_n^{i-1} u_n - \Lambda_n^i u_n\right\| \left\|B_i \Lambda_n^{i-1} u_n - B_i p\right\| \\
&\quad + 2s_n \|\gamma V p - \mu F p\| \|x_n - p\| \\
&\leq \|x_n - p\|^2 + s_n^2\tau^2 \|x_n - p\|^2 - (1 - s_n\tau)^2 \\
&\quad \times \left(\left\|\Delta_n^{k-1} x_n - \Delta_n^k x_n\right\|^2 + \left\|\Lambda_n^{i-1} u_n - \Lambda_n^i u_n\right\|^2 \right) \\
&\quad + 2r_{k,n} \left\|\Delta_n^{k-1} x_n - \Delta_n^k x_n\right\| \left\|A_k \Delta_n^{k-1} x_n - A_k p\right\| \\
&\quad + 2\lambda_{i,n} \left\|\Lambda_n^{i-1} u_n - \Lambda_n^i u_n\right\| \left\|B_i \Lambda_n^{i-1} u_n - B_i p\right\| \\
&\quad + 2s_n \|\gamma V p - \mu F p\| \|x_n - p\| .
\end{aligned} \tag{72}
$$

It immediately follows that

$$
\begin{aligned}
&(1 - s_n\tau)^2 \left(\left\|\Delta_n^{k-1} x_n - \Delta_n^k x_n\right\|^2 + \left\|\Lambda_n^{i-1} u_n - \Lambda_n^i u_n\right\|^2 \right) \\
&\quad \leq s_n^2\tau^2 \|x_n - p\|^2 + 2r_{k,n} \left\|\Delta_n^{k-1} x_n - \Delta_n^k x_n\right\| \\
&\qquad \times \left\|A_k \Delta_n^{k-1} x_n - A_k p\right\| + 2\lambda_{i,n} \left\|\Lambda_n^{i-1} u_n - \Lambda_n^i u_n\right\| \\
&\qquad \times \left\|B_i \Lambda_n^{i-1} u_n - B_i p\right\| + 2s_n \|\gamma V p - \mu F p\| \|x_n - p\| .
\end{aligned} \tag{73}
$$

Since $\{\lambda_{i,n}\} \subset [a_i, b_i] \subset (0, 2\eta_i)$ and $\{r_{k,n}\} \subset [e_k, f_k] \subset (0, 2\mu_k)$ for all $i \in \{1, 2, \ldots, N\}$ and $k \in \{1, 2, \ldots, M\}$, from (67) and $s_n \to 0$ we deduce that

$$
\lim_{n \to \infty} \left\|\Delta_n^{k-1} x_n - \Delta_n^k x_n\right\| = 0, \qquad \lim_{n \to \infty} \left\|\Lambda_n^{i-1} u_n - \Lambda_n^i u_n\right\| = 0, \tag{74}
$$

for all $k \in \{1, 2, \ldots, M\}$ and $i \in \{1, 2, \ldots, N\}$. Hence, we get

$$
\begin{aligned}
\|x_n - u_n\| &= \left\|\Delta_n^0 x_n - \Delta_n^M x_n\right\| \\
&\leq \left\|\Delta_n^0 x_n - \Delta_n^1 x_n\right\| + \left\|\Delta_n^1 x_n - \Delta_n^2 x_n\right\| \\
&\quad + \cdots + \left\|\Delta_n^{M-1} x_n - \Delta_n^M x_n\right\| \longrightarrow 0 \\
&\qquad \text{as } n \longrightarrow \infty,
\end{aligned} \tag{75}
$$

$$
\begin{aligned}
\|u_n - v_n\| &= \left\|\Lambda_n^0 u_n - \Lambda_n^N u_n\right\| \\
&\leq \left\|\Lambda_n^0 u_n - \Lambda_n^1 u_n\right\| + \left\|\Lambda_n^1 u_n - \Lambda_n^2 u_n\right\| \\
&\quad + \cdots + \left\|\Lambda_n^{N-1} u_n - \Lambda_n^N u_n\right\| \longrightarrow 0 \\
&\qquad \text{as } n \longrightarrow \infty.
\end{aligned} \tag{76}
$$

So, taking into account that $\|x_n - v_n\| \leq \|x_n - u_n\| + \|u_n - v_n\|$, we have

$$
\lim_{n \to \infty} \|x_n - v_n\| = 0. \tag{77}
$$

Thus, from (77) and $s_n \to 0$ we have

$$
\begin{aligned}
\|v_n - T_n v_n\| &\leq \|v_n - x_n\| + \|x_n - T_n v_n\| \\
&= \|v_n - x_n\| + s_n \|\gamma V x_n - \mu F T_n v_n\| \longrightarrow 0 \\
&\qquad \text{as } n \longrightarrow \infty.
\end{aligned} \tag{78}
$$

Now we show that $\|x_n - T_n x_n\| \to 0$ as $n \to \infty$. In fact, from the nonexpansivity of T_n, we have

$$\begin{aligned}\|x_n - T_n x_n\| &\le \|x_n - v_n\| + \|v_n - T_n v_n\| \\ &\quad + \|T_n v_n - T_n x_n\| \\ &\le 2\|x_n - v_n\| + \|v_n - T_n v_n\|.\end{aligned} \tag{79}$$

By (77) and (78), we get

$$\lim_{n\to\infty} \|x_n - T_n x_n\| = 0. \tag{80}$$

From (78) it is easy to see that

$$\lim_{n\to\infty} \|x_n - T_n v_n\| = 0. \tag{81}$$

Observe that

$$\begin{aligned}&\|P_C(I - \lambda_n \nabla f)v_n - v_n\| \\ &\quad = \|s_n v_n + (1 - s_n)T_n v_n - v_n\| \\ &\quad = (1 - s_n)\|T_n v_n - v_n\| \\ &\quad \le \|T_n v_n - v_n\|,\end{aligned} \tag{82}$$

where $s_n = (2 - \lambda_n L)/4 \in (0, 1/2)$ for each $\lambda_n \in (0, 2/L)$. Hence, we have

$$\begin{aligned}&\left\|P_C\left(I - \frac{2}{L}\nabla f\right)v_n - v_n\right\| \\ &\quad \le \left\|P_C\left(I - \frac{2}{L}\nabla f\right)v_n - P_C(I - \lambda_n\nabla f)v_n\right\| \\ &\qquad + \|P_C(I - \lambda_n\nabla f)v_n - v_n\| \\ &\quad \le \left\|\left(I - \frac{2}{L}\nabla f\right)v_n - (I - \lambda_n\nabla f)v_n\right\| \\ &\qquad + \|P_C(I - \lambda_n\nabla f)v_n - v_n\| \\ &\quad \le \left(\frac{2}{L} - \lambda_n\right)\|\nabla f(v_n)\| + \|T_n v_n - v_n\|.\end{aligned} \tag{83}$$

From the boundedness of $\{v_n\}$, $s_n \to 0$ ($\Leftrightarrow \lambda_n \to 2/L$) and $\|T_n v_n - v_n\| \to 0$ (due to (78)), it follows that

$$\lim_{n\to\infty} \left\|v_n - P_C\left(I - \frac{2}{L}\nabla f\right)v_n\right\| = 0. \tag{84}$$

Further, we show that $\omega_w(x_n) \subset \Omega$. Indeed, since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ which converges weakly to some w. Note that $\lim_{n\to\infty}\|x_n - u_n\| = 0$ (due to (75)). Hence, $x_{n_i} \rightharpoonup w$. Since C is closed and convex, C is weakly closed. So, we have $w \in C$. From (74) and (75), we have that $\Delta^k_{n_i} x_{n_i} \rightharpoonup w$, $\Lambda^m_{n_i} u_{n_i} \rightharpoonup w$, $u_{n_i} \rightharpoonup w$, $v_{n_i} \rightharpoonup w$, where $k \in \{1, 2, \ldots, M\}$, $m \in \{1, 2, \ldots, N\}$. First, we prove that $w \in \cap_{m=1}^N I(B_m, R_m)$. As a matter of fact, since B_m is η_m-inverse-strongly monotone, B_m is a monotone and Lipschitz continuous mapping. It follows from Lemma 17 that $R_m + B_m$ is maximal monotone. Let $(v, g) \in G(R_m + B_m)$; that is, $g - B_m v \in R_m v$. Again, since $\Lambda^m_n u_n = J_{R_m,\lambda_{m,n}}(I - \lambda_{m,n}B_m)\Lambda^{m-1}_n u_n$, $n \ge 1$, $m \in \{1, 2, \ldots, N\}$, we have

$$\Lambda^{m-1}_n u_n - \lambda_{m,n}B_m\Lambda^{m-1}_n u_n \in (I + \lambda_{m,n}R_m)\Lambda^m_n u_n; \tag{85}$$

that is,

$$\frac{1}{\lambda_{m,n}}\left(\Lambda^{m-1}_n u_n - \Lambda^m_n u_n - \lambda_{m,n}B_m\Lambda^{m-1}_n u_n\right) \in R_m\Lambda^m_n u_n. \tag{86}$$

In terms of the monotonicity of R_m, we get

$$\begin{aligned}&\left\langle v - \Lambda^m_n u_n, g - B_m v - \frac{1}{\lambda_{m,n}}\right. \\ &\qquad \left.\times\left(\Lambda^{m-1}_n u_n - \Lambda^m_n u_n - \lambda_{m,n}B_m\Lambda^{m-1}_n u_n\right)\right\rangle \ge 0\end{aligned} \tag{87}$$

and hence

$$\begin{aligned}&\langle v - \Lambda^m_n u_n, g\rangle \\ &\quad \ge \left\langle v - \Lambda^m_n u_n, B_m v + \frac{1}{\lambda_{m,n}}\right. \\ &\qquad \left.\times\left(\Lambda^{m-1}_n u_n - \Lambda^m_n u_n - \lambda_{m,n}B_m\Lambda^{m-1}_n u_n\right)\right\rangle \\ &\quad = \left\langle v - \Lambda^m_n u_n, B_m v - B_m\Lambda^m_n u_n + B_m\Lambda^m_n u_n\right. \\ &\qquad \left. - B_m\Lambda^{m-1}_n u_n + \frac{1}{\lambda_{m,n}}\left(\Lambda^{m-1}_n u_n - \Lambda^m_n u_n\right)\right\rangle \\ &\quad \ge \left\langle v - \Lambda^m_n u_n, B_m\Lambda^m_n u_n - B_m\Lambda^{m-1}_n u_n\right\rangle \\ &\qquad + \left\langle v - \Lambda^m_n u_n, \frac{1}{\lambda_{m,n}}\left(\Lambda^{m-1}_n u_n - \Lambda^m_n u_n\right)\right\rangle.\end{aligned} \tag{88}$$

In particular,

$$\begin{aligned}&\left\langle v - \Lambda^m_{n_i} u_{n_i}, g\right\rangle \\ &\quad \ge \left\langle v - \Lambda^m_{n_i} u_{n_i}, B_m\Lambda^m_{n_i} u_{n_i} - B_m\Lambda^{m-1}_{n_i} u_{n_i}\right\rangle \\ &\qquad + \left\langle v - \Lambda^m_{n_i} u_{n_i}, \frac{1}{\lambda_{m,n_i}}\left(\Lambda^{m-1}_{n_i} u_{n_i} - \Lambda^m_{n_i} u_{n_i}\right)\right\rangle.\end{aligned} \tag{89}$$

Since $\|\Lambda^m_n u_n - \Lambda^{m-1}_n u_n\| \to 0$ (due to (74)) and $\|B_m\Lambda^m_n u_n - B_m\Lambda^{m-1}_n u_n\| \to 0$ (due to the Lipschitz continuity of B_m), we conclude from $\Lambda^m_{n_i} u_{n_i} \rightharpoonup w$ and condition (ii) that

$$\lim_{i\to\infty}\left\langle v - \Lambda^m_{n_i} u_{n_i}, g\right\rangle = \langle v - w, g\rangle \ge 0. \tag{90}$$

It follows from the maximal monotonicity of $B_m + R_m$ that $0 \in (R_m + B_m)w$; that is, $w \in I(B_m, R_m)$. Therefore, $w \in \cap_{m=1}^N I(B_m, R_m)$. Next we prove that

$w \in \cap_{k=1}^{M} \mathrm{GMEP}(\Theta_k, \varphi_k, A_k)$. Since $\Delta_n^k x_n = T_{r_{k,n}}^{(\Theta_k,\varphi_k)}(I - r_{k,n}A_k)\Delta_n^{k-1}x_n$, $n \geq 1$, $k \in \{1, 2, \ldots, M\}$, we have

$$\begin{aligned}&\Theta_k\left(\Delta_n^k x_n, y\right) + \varphi_k(y) - \varphi_k\left(\Delta_n^k x_n\right)\\&\quad+ \left\langle A_k\Delta_n^{k-1}x_n, y - \Delta_n^k x_n\right\rangle\\&\quad+ \frac{1}{r_{k,n}}\left\langle y - \Delta_n^k x_n, \Delta_n^k x_n - \Delta_n^{k-1}x_n\right\rangle \geq 0.\end{aligned} \tag{91}$$

By (A2), we have

$$\begin{aligned}&\varphi_k(y) - \varphi_k\left(\Delta_n^k x_n\right) + \left\langle A_k\Delta_n^{k-1}x_n, y - \Delta_n^k x_n\right\rangle\\&\quad+ \frac{1}{r_{k,n}}\left\langle y - \Delta_n^k x_n, \Delta_n^k x_n - \Delta_n^{k-1}x_n\right\rangle \geq \Theta_k\left(y, \Delta_n^k x_n\right).\end{aligned} \tag{92}$$

Let $z_t = ty + (1-t)w$ for all $t \in (0, 1]$ and $y \in C$. This implies that $z_t \in C$. Then, we have

$$\begin{aligned}&\left\langle z_t - \Delta_n^k x_n, A_k z_t\right\rangle\\&\geq \varphi_k\left(\Delta_n^k x_n\right) - \varphi_k(z_t) + \left\langle z_t - \Delta_n^k x_n, A_k z_t\right\rangle\\&\quad- \left\langle z_t - \Delta_n^k x_n, A_k\Delta_n^{k-1}x_n\right\rangle\\&\quad- \left\langle z_t - \Delta_n^k x_n, \frac{\Delta_n^k x_n - \Delta_n^{k-1}x_n}{r_{k,n}}\right\rangle + \Theta_k\left(z_t, \Delta_n^k x_n\right)\\&= \varphi_k\left(\Delta_n^k x_n\right) - \varphi_k(z_t)\\&\quad+ \left\langle z_t - \Delta_n^k x_n, A_k z_t - A_k\Delta_n^k x_n\right\rangle\\&\quad+ \left\langle z_t - \Delta_n^k x_n, A_k\Delta_n^k x_n - A_k\Delta_n^{k-1}x_n\right\rangle\\&\quad- \left\langle z_t - \Delta_n^k x_n, \frac{\Delta_n^k x_n - \Delta_n^{k-1}x_n}{r_{k,n}}\right\rangle + \Theta_k\left(z_t, \Delta_n^k x_n\right).\end{aligned} \tag{93}$$

By (74), we have $\|A_k\Delta_n^k x_n - A_k\Delta_n^{k-1}x_n\| \to 0$ (due to the Lipschitz continuity of A_k). Furthermore, by the monotonicity of A_k, we obtain $\langle z_t - \Delta_n^k x_n, A_k z_t - A_k\Delta_n^k x_n\rangle \geq 0$. Then, by (A4) we obtain

$$\left\langle z_t - w, A_k z_t\right\rangle \geq \varphi_k(w) - \varphi_k(z_t) + \Theta_k(z_t, w). \tag{94}$$

Utilizing (A1), (A4), and (94), we obtain

$$\begin{aligned}0 &= \Theta_k(z_t, z_t) + \varphi_k(z_t) - \varphi_k(z_t)\\&\leq t\Theta_k(z_t, y) + (1-t)\Theta_k(z_t, w) + t\varphi_k(y)\\&\quad+ (1-t)\varphi_k(w) - \varphi_k(z_t)\\&\leq t\left[\Theta_k(z_t, y) + \varphi_k(y) - \varphi_k(z_t)\right]\\&\quad+ (1-t)\left\langle z_t - w, A_k z_t\right\rangle\\&= t\left[\Theta_k(z_t, y) + \varphi_k(y) - \varphi_k(z_t)\right]\\&\quad+ (1-t)t\left\langle y - w, A_k z_t\right\rangle,\end{aligned} \tag{95}$$

and hence

$$0 \leq \Theta_k(z_t, y) + \varphi_k(y) - \varphi_k(z_t) + (1-t)\left\langle y - w, A_k z_t\right\rangle. \tag{96}$$

Letting $t \to 0$, we have, for each $y \in C$,

$$0 \leq \Theta_k(w, y) + \varphi_k(y) - \varphi_k(w) + \left\langle y - w, A_k w\right\rangle. \tag{97}$$

This implies that $w \in \mathrm{GMEP}(\Theta_k, \varphi_k, A_k)$ and hence $w \in \cap_{k=1}^{M} \mathrm{GMEP}(\Theta_k, \varphi_k, A_k)$. Further, let us show that $w \in \Gamma$. As a matter of fact, from (84), $v_{n_i} \rightharpoonup w$, and Lemma 9, we conclude that

$$w = P_C\left(I - \frac{2}{L}\nabla f\right)w. \tag{98}$$

So, $w \in \mathrm{VI}(C, \nabla f) = \Gamma$. Therefore, $w \in \cap_{k=1}^{M} \mathrm{GMEP}(\Theta_k, \varphi_k, A_k) \cap \cap_{i=1}^{N} I(B_i, R_i) \cap \Gamma =: \Omega$. This shows that $\omega_w(x_n) \subset \Omega$.

Finally, let us show that $x_n \to q$ as $n \to \infty$ where q is the unique solution of the VIP (50). Indeed, we note that, for $w \in \Omega$ with $x_{n_i} \rightharpoonup w$,

$$\begin{aligned}x_n - w &= s_n(\gamma V x_n - \mu F w) + (I - s_n\mu F)T_n v_n\\&\quad- (I - s_n\mu F)w.\end{aligned} \tag{99}$$

By (61) and Lemma 13, we obtain that

$$\begin{aligned}\|x_n - w\|^2 &= s_n\langle \gamma V x_n - \mu F w, x_n - w\rangle\\&\quad+ \langle (I - s_n\mu F)T_n v_n - (I - s_n\mu F)w, x_n - w\rangle\\&= s_n\langle \gamma V x_n - \mu F w, x_n - w\rangle\\&\quad+ \|(I - s_n\mu F)T_n v_n - (I - s_n\mu F)w\|\,\|x_n - w\|\\&\leq s_n\langle \gamma V x_n - \mu F w, x_n - w\rangle\\&\quad+ (1 - s_n\tau)\|v_n - w\|\,\|x_n - w\|\\&\leq s_n\langle \gamma V x_n - \mu F w, x_n - w\rangle + (1 - s_n\tau)\|x_n - w\|^2.\end{aligned} \tag{100}$$

Hence, it follows that

$$\begin{aligned}\|x_n - w\|^2 &\leq \frac{1}{\tau}\langle \gamma V x_n - \mu F w, x_n - w\rangle\\&= \frac{1}{\tau}\left(\gamma\langle V x_n - V w, x_n - w\rangle\right.\\&\quad\left.+ \langle \gamma V w - \mu F w, x_n - w\rangle\right)\\&\leq \frac{1}{\tau}\left(\gamma l\|x_n - w\|^2 + \langle \gamma V w - \mu F w, x_n - w\rangle\right),\end{aligned} \tag{101}$$

which hence leads to

$$\|x_n - w\|^2 \leq \frac{\langle \gamma V w - \mu F w, x_n - w\rangle}{\tau - \gamma l}. \tag{102}$$

In particular, we have

$$\left\|x_{n_i} - w\right\|^2 \le \frac{\left\langle \gamma V w - \mu F w, x_{n_i} - w\right\rangle}{\tau - \gamma l}. \tag{103}$$

Since $x_{n_i} \rightharpoonup w$, it follows from (103) that $x_{n_i} \to w$ as $i \to \infty$.

Now we show that w solves the VIP (50). Since $x_n = s_n\gamma V x_n + (I - s_n\mu F)T_n v_n$, we have

$$(\mu F - \gamma V) x_n = -\frac{1}{s_n}\left((I - s_n\mu F) x_n - (I - s_n\mu F) T_n v_n\right). \tag{104}$$

It follows that, for each $p \in \Omega$,

$$\begin{aligned}
&\langle(\mu F - \gamma V) x_n, x_n - p\rangle \\
&= -\frac{1}{s_n}\langle(I - s_n\mu F) x_n - (I - s_n\mu F) T_n v_n, x_n - p\rangle \\
&= -\frac{1}{s_n}\langle(I - s_n\mu F) x_n - (I - s_n\mu F) T_n v_n, x_n - p\rangle \\
&= -\frac{1}{s_n}\left\langle(I - s_n\mu F) x_n - (I - s_n\mu F) T_n\Lambda_n^N\Delta_n^M x_n, x_n - p\right\rangle \\
&= -\frac{1}{s_n}\left\langle\left(I - T_n\Lambda_n^N\Delta_n^M\right) x_n - \left(I - T_n\Lambda_n^N\Delta_n^M\right) p, x_n - p\right\rangle \\
&\quad + \left\langle\mu F x_n - \mu F T_n\Lambda_n^N\Delta_n^M x_n, x_n - p\right\rangle \\
&\le \left\langle\mu F x_n - \mu F T_n\Lambda_n^N\Delta_n^M x_n, x_n - p\right\rangle,
\end{aligned} \tag{105}$$

since $I - T_n\Lambda_n^N\Delta_n^M$ is monotone (i.e., $\langle(I - T_n\Lambda_n^N\Delta_n^M)x - (I - T_n\Lambda_n^N\Delta_n^M)y, x - y\rangle \ge 0$ for all $x, y \in H$. This is due to the nonexpansivity of $T_n\Lambda_n^N\Delta_n^M$). Since $\|x_n - T_n v_n\| = \|(I - T_n\Lambda_n^N\Delta_n^M)x_n\| \to 0$ as $n \to \infty$, by replacing n in (105) with n_i and letting $i \to \infty$, we get

$$\begin{aligned}
&\langle(\mu F - \gamma V) w, w - p\rangle \\
&= \lim_{i\to\infty}\left\langle(\mu F - \gamma V) x_{n_i}, x_{n_i} - p\right\rangle \\
&\le \lim_{i\to\infty}\left\langle\mu F x_{n_i} - \mu F T_{n_i} v_{n_i}, x_{n_i} - p\right\rangle = 0.
\end{aligned} \tag{106}$$

That is, $w \in \Omega$ is a solution of VIP (50).

Finally, in terms of the uniqueness of solutions of VIP (50), we deduce that $w = q$ and $x_{n_i} \to q$ as $n \to \infty$. So, every weak convergence subsequence of $\{x_n\}$ converges strongly to the unique solution q of VIP (50). Therefore, $\{x_n\}$ converges strongly to the unique solution q of VIP (50). In addition, the VIP (50) can be rewritten as

$$\left\langle(I - \mu F + \gamma V) q - q, q - p\right\rangle \ge 0, \quad \forall p \in \Omega. \tag{107}$$

By Proposition 2(i), this is equivalent to the fixed point equation:

$$P_\Omega (I - \mu F + \gamma V) q = q. \tag{108}$$

This completes the proof. □

Corollary 19. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $f : C \to \mathbf{R}$ be a convex functional with L-Lipschitz continuous gradient ∇f. Let Θ be a bifunction from $C \times C$ to $\mathbf{R}$ satisfying (A1)–(A4) and let $\varphi : C \to \mathbf{R} \cup \{+\infty\}$ be a proper lower semicontinuous and convex function. Let $R_i : C \to 2^H$ be a maximal monotone mapping and let $A : H \to H$ and $B_i : C \to H$ be ζ-inverse-strongly monotone and η_i-inverse-strongly monotone, respectively, for $i = 1, 2$. Let $F : H \to H$ be a κ-Lipschitzian and η-strongly monotone operator with positive constants $\kappa, \eta > 0$. Let $V : H \to H$ be an l-Lipschitzian mapping with constant $l \ge 0$. Let $0 < \mu < 2\eta/\kappa^2$ and $0 \le \gamma l < \tau$, where $\tau = 1 - \sqrt{1 - \mu(2\eta - \mu\kappa^2)}$. Assume that $\Omega := GMEP(\Theta, \varphi, A) \cap I(B_1, R_1) \cap I(B_2, R_2) \cap \Gamma \ne \emptyset$ and that either (B1) or (B2) holds. Let $\{x_n\}$ be a sequence generated by*

$$\begin{aligned}
&\Theta(u_n, y) + \varphi(y) - \varphi(u_n) + \langle A x_n, y - u_n\rangle \\
&\quad + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \ge 0, \quad \forall y \in C, \\
&v_n = J_{R_2,\lambda_{2,n}}(I - \lambda_{2,n}B_2) J_{R_1,\lambda_{1,n}}(I - \lambda_{1,n}B_1) u_n, \\
&x_n = s_n\gamma V x_n + (I - s_n\mu F) T_n v_n, \quad \forall n \ge 1,
\end{aligned} \tag{109}$$

where $P_C(I - \lambda_n\nabla f) = s_n I + (1 - s_n)T_n$ (here T_n is nonexpansive and $s_n = (2 - \lambda_n L)/4 \in (0, 1/2)$ for each $\lambda_n \in (0, 2/L)$). Assume that the following conditions hold:

(i) *$s_n \in (0, 1/2)$ for each $\lambda_n \in (0, 2/L)$, $\lim_{n\to\infty} s_n = 0$ ($\Leftrightarrow \lim_{n\to\infty}\lambda_n = 2/L$);*

(ii) *$\{\lambda_{i,n}\} \subset [a_i, b_i] \subset (0, 2\eta_i)$ for $i = 1, 2$;*

(iii) *$\{r_n\} \subset [e, f] \subset (0, 2\zeta)$.*

Then $\{x_n\}$ converges strongly as $\lambda_n \to 2/L$ ($\Leftrightarrow s_n \to 0$) to a point $q \in \Omega$, which is a unique solution of the VIP:

$$\left\langle(\mu F - \gamma V) q, p - q\right\rangle \ge 0, \quad \forall p \in \Omega. \tag{110}$$

Corollary 20. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $f : C \to \mathbf{R}$ be a convex functional with L-Lipschitz continuous gradient ∇f. Let Θ be a bifunction from $C \times C$ to $\mathbf{R}$ satisfying (A1)–(A4) and let $\varphi : C \to \mathbf{R} \cup \{+\infty\}$ be a proper lower semicontinuous and convex function. Let $R : C \to 2^H$ be a maximal monotone mapping and let $A : H \to H$ and $B : C \to H$ be ζ-inverse-strongly monotone and ξ-inverse-strongly monotone, respectively. Let $F : H \to H$ be a κ-Lipschitzian and η-strongly monotone operator with positive constants $\kappa, \eta > 0$. Let $V : H \to H$ be an l-Lipschitzian mapping with constant $l \ge 0$. Let $0 < \mu < 2\eta/\kappa^2$ and $0 \le \gamma l < \tau$, where $\tau = 1 - \sqrt{1 - \mu(2\eta - \mu\kappa^2)}$. Assume that $\Omega := GMEP(\Theta, \varphi, A) \cap I(B, R) \cap \Gamma \ne \emptyset$ and that either (B1) or (B2) holds. Let $\{x_n\}$ be a sequence generated by*

$$\begin{aligned}
&\Theta(u_n, y) + \varphi(y) - \varphi(u_n) + \langle A x_n, y - u_n\rangle \\
&\quad + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \ge 0, \quad \forall y \in C, \\
&x_n = s_n\gamma V x_n + (I - s_n\mu F) T_n J_{R,\rho_n}(u_n - \rho_n B u_n), \\
&\qquad \forall n \ge 1,
\end{aligned} \tag{111}$$

where $P_C(I-\lambda_n\nabla f) = s_nI+(1-s_n)T_n$ *(here* T_n *is nonexpansive and* $s_n = (2-\lambda_nL)/4 \in (0,1/2)$ *for each* $\lambda_n \in (0,2/L)$*). Assume that the following conditions hold:*

(i) $s_n \in (0,1/2)$ *for each* $\lambda_n \in (0,2/L)$, $\lim_{n\to\infty}s_n = 0$ ($\Leftrightarrow \lim_{n\to\infty}\lambda_n = 2/L$);

(ii) $\{\rho_n\} \subset [a,b] \subset (0,2\xi)$;

(iii) $\{r_n\} \subset [e,f] \subset (0,2\zeta)$.

Then $\{x_n\}$ *converges strongly as* $\lambda_n \to 2/L$ ($\Leftrightarrow s_n \to 0$) *to a point* $q \in \Omega$*, which is a unique solution of the VIP:*

$$\langle(\mu F-\gamma V)q, p-q\rangle \geq 0, \quad \forall p \in \Omega. \tag{112}$$

4. Explicit Iterative Algorithm and Its Convergence Criteria

We next state and prove the second main result of this paper.

Theorem 21. *Let* C *be a nonempty closed convex subset of a real Hilbert space* H*. Let* $f : C \to \mathbf{R}$ *be a convex functional with L-Lipschitz continuous gradient* ∇f*. Let* M, N *be two integers. Let* Θ_k *be a bifunction from* $C\times C$ *to* $\mathbf{R}$ *satisfying (A1)–(A4) and let* $\varphi_k : C \to \mathbf{R}\cup\{+\infty\}$ *be a proper lower semicontinuous and convex function, where* $k \in \{1,2,\ldots,M\}$*. Let* $R_i : C \to 2^H$ *be a maximal monotone mapping and let* $A_k : H \to H$ *and* $B_i : C \to H$ *be* μ_k*-inverse-strongly monotone and* η_i*-inverse-strongly monotone, respectively, where* $k \in \{1,2,\ldots,M\}$, $i \in \{1,2,\ldots,N\}$*. Let* $F : H \to H$ *be a* κ*-Lipschitzian and* η*-strongly monotone operator with positive constants* $\kappa,\eta > 0$*. Let* $V : H \to H$ *be an* l*-Lipschitzian mapping with constant* $l \geq 0$*. Let* $0 < \mu < 2\eta/\kappa^2$ *and* $0 \leq \gamma l < \tau$*, where* $\tau = 1-\sqrt{1-\mu(2\eta-\mu\kappa^2)}$*. Assume that* $\Omega := \cap_{k=1}^{M}GMEP(\Theta_k,\varphi_k,A_k)\cap\cap_{i=1}^{N}I(B_i,R_i)\cap\Gamma \neq \emptyset$ *and that either (B1) or (B2) holds. For arbitrarily given* $x_1 \in H$*, let* $\{x_n\}$ *be a sequence generated by*

$$\begin{aligned} u_n &= T_{r_{M,n}}^{(\Theta_M,\varphi_M)}(I-r_{M,n}A_M)T_{r_{M-1,n}}^{(\Theta_{M-1},\varphi_{M-1})} \\ &\quad\times(I-r_{M-1,n}A_{M-1})\cdots T_{r_{1,n}}^{(\Theta_1,\varphi_1)}(I-r_{1,n}A_1)x_n, \\ v_n &= J_{R_N,\lambda_{N,n}}(I-\lambda_{N,n}B_N)J_{R_{N-1},\lambda_{N-1,n}} \\ &\quad\times(I-\lambda_{N-1,n}A_{N-1})\cdots J_{R_1,\lambda_{1,n}}(I-\lambda_{1,n}B_1)u_n, \\ x_{n+1} &= s_n\gamma Vx_n+\beta_nx_n+((1-\beta_n)I-s_n\mu F)T_nv_n, \\ &\qquad \forall n \geq 1, \end{aligned} \tag{113}$$

where $P_C(I-\lambda_n\nabla f) = s_nI+(1-s_n)T_n$ *(here* T_n *is nonexpansive and* $s_n = (2-\lambda_nL)/4 \in (0,1/2)$ *for each* $\lambda_n \in (0,2/L)$*). Assume that the following conditions hold:*

(i) $s_n \in (0,1/2)$ *for each* $\lambda_n \in (0,2/L)$*, and* $\lim_{n\to\infty}s_n = 0$ ($\Leftrightarrow \lim_{n\to\infty}\lambda_n = 2/L$);

(ii) $\{\beta_n\} \subset (0,1)$ *and* $0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1$;

(iii) $\{\lambda_{i,n}\} \subset [a_i,b_i] \subset (0,2\eta_i)$ *and* $\lim_{n\to\infty}|\lambda_{i,n+1}-\lambda_{i,n}| = 0$ *for all* $i \in \{1,2,\ldots,N\}$;

(iv) $\{r_{k,n}\} \subset [e_k,f_k] \subset (0,2\mu_k)$ *and* $\lim_{n\to\infty}|r_{k,n+1}-r_{k,n}| = 0$ *for all* $k \in \{1,2,\ldots,M\}$.

Then $\{x_n\}$ *converges strongly as* $\lambda_n \to 2/L$ ($\Leftrightarrow s_n \to 0$) *to a point* $q \in \Omega$*, which is a unique solution of VIP* (50).

Proof. First of all, repeating the same arguments as in Theorem 18, we can write

$$P_C(I-\lambda_n\nabla f) = \frac{2-\lambda_nL}{4}I+\frac{2+\lambda_nL}{4}T_n = s_nI+(1-s_n)T_n, \tag{114}$$

where T_n is nonexpansive and $s_n := s_n(\lambda_n) = (2-\lambda_nL)/4 \in (0,1/2)$ for each $\lambda_n \in (0,2/L)$. It is clear that

$$\lambda_n \longrightarrow \frac{2}{L} \Longleftrightarrow s_n \longrightarrow 0. \tag{115}$$

Put

$$\begin{aligned} \Delta_n^k &= T_{r_{k,n}}^{(\Theta_k,\varphi_k)}(I-r_{k,n}A_k)T_{r_{k-1,n}}^{(\Theta_{k-1},\varphi_{k-1})} \\ &\quad\times(I-r_{k-1,n}A_{k-1})\cdots T_{r_{1,n}}^{(\Theta_1,\varphi_1)}(I-r_{1,n}A_1)x_n \end{aligned} \tag{116}$$

for all $k \in \{1,2,\ldots,M\}$ and $n \geq 1$,

$$\begin{aligned} \Lambda_n^i &= J_{R_i,\lambda_{i,n}}(I-\lambda_{i,n}B_i)J_{R_{i-1},\lambda_{i-1,n}} \\ &\quad\times(I-\lambda_{i-1,n}B_{i-1})\cdots J_{R_1,\lambda_{1,n}}(I-\lambda_{1,n}B_1) \end{aligned} \tag{117}$$

for all $i \in \{1,2,\ldots,N\}$ and $n \geq 1$, and $\Delta_n^0 = \Lambda_n^0 = I$, where I is the identity mapping on H. Then we have that $u_n = \Delta_n^Mx_n$ and $v_n = \Lambda_n^Nu_n$. In addition, taking into consideration conditions (i) and (ii), we may assume, without loss of generality, that $s_n \leq 1-\beta_n$ for all $n \geq 1$.

We divide the remainder of the proof into several steps.

Step 1. Let us show that $\|x_n-p\| \leq \max\{\|x_1-p\|, \|\gamma Vp-\mu Fp\|/(\tau-\gamma l)\}$ for all $n \geq 1$ and $p \in \Omega$. Indeed, take $p \in \Omega$ arbitrarily. Repeating the same arguments as those of (59)–(61) in the proof of Theorem 18, we obtain

$$\begin{aligned} \|u_n-p\| &\leq \|x_n-p\|, \\ \|v_n-p\| &\leq \|u_n-p\|, \\ \|v_n-p\| &\leq \|x_n-p\|. \end{aligned} \tag{118}$$

Then from (118), $T_np = p$, and Lemma 13, we have

$$\begin{aligned} &\|x_{n+1}-p\| \\ &\quad= \|s_n(\gamma Vx_n-\mu Fp)+\beta_n(x_n-p) \\ &\qquad+((1-\beta_n)I-s_n\mu F)T_nv_n \\ &\qquad-((1-\beta_n)I-s_n\mu F)T_np\| \end{aligned}$$

$$\le s_n \|\gamma V x_n - \mu F p\| + \beta_n \|x_n - p\| + (1-\beta_n)\left\|\left(I - \frac{s_n}{1-\beta_n}\mu F\right)T_n v_n - \left(I - \frac{s_n}{1-\beta_n}\mu F\right)T_n p\right\|$$
$$\le s_n (\|\gamma V x_n - \gamma V p\| + \|\gamma V p - \mu F p\|) + \beta_n \|x_n - p\| + (1-\beta_n)\left(1 - \frac{s_n \tau}{1-\beta_n}\right)\|v_n - p\|$$
$$\le s_n (\|\gamma V x_n - \gamma V p\| + \|\gamma V p - \mu F p\|) + \beta_n \|x_n - p\| + (1-\beta_n - s_n\tau)\|x_n - p\|$$
$$\le s_n \gamma l \|x_n - p\| + s_n \|\gamma V p - \mu F p\| + (1 - s_n \tau)\|x_n - p\|$$
$$= (1 - s_n(\tau - \gamma l))\|x_n - p\| + s_n(\tau - \gamma l)\frac{\|\gamma V p - \mu F p\|}{\tau - \gamma l}$$
$$\le \max\left\{\|x_n - p\|, \frac{\|\gamma V p - \mu F p\|}{\tau - \gamma l}\right\}. \tag{119}$$

By induction, we have

$$\|x_n - p\| \le \max\left\{\|x_1 - p\|, \frac{\|\gamma V p - \mu F p\|}{\tau - \gamma l}\right\}, \quad \forall n \ge 1. \tag{120}$$

Hence, $\{x_n\}$ is bounded. According to (118), $\{u_n\}$, $\{v_n\}$, $\{T_n v_n\}$, $\{V x_n\}$, and $\{F T_n v_n\}$ are also bounded.

Step 2. Let us show that $\|x_{n+1} - x_n\| \to 0$ as $n \to \infty$. To this end, define

$$x_{n+1} = \beta_n x_n + (1-\beta_n) z_n, \quad \forall n \ge 1. \tag{121}$$

Observe that, from the definition of z_n,

$$z_{n+1} - z_n = \frac{x_{n+2} - \beta_{n+1}x_{n+1}}{1-\beta_{n+1}} - \frac{x_{n+1} - \beta_n x_n}{1-\beta_n}$$
$$= \frac{s_{n+1}\gamma V x_{n+1} + ((1-\beta_{n+1})I - s_{n+1}\mu F)T_{n+1}v_{n+1}}{1-\beta_{n+1}} - \frac{s_n \gamma V x_n + ((1-\beta_n)I - s_n \mu F)T_n v_n}{1-\beta_n}$$
$$= \frac{s_{n+1}}{1-\beta_{n+1}}\gamma V x_{n+1} - \frac{s_n}{1-\beta_n}\gamma V x_n + T_{n+1}v_{n+1} - T_n v_n + \frac{s_n}{1-\beta_n}\mu F T_n v_n - \frac{s_{n+1}}{1-\beta_{n+1}}\mu F T_{n+1} v_{n+1}$$
$$= \frac{s_{n+1}}{1-\beta_{n+1}}(\gamma V x_{n+1} - \mu F T_{n+1} v_{n+1}) + \frac{s_n}{1-\beta_n}(\mu F T_n v_n - \gamma V x_n) + T_{n+1}v_{n+1} - T_n v_n. \tag{122}$$

Thus, it follows that

$$\|z_{n+1} - z_n\| \le \frac{s_{n+1}}{1-\beta_{n+1}}(\gamma \|V x_{n+1}\| + \mu \|F T_{n+1} v_{n+1}\|) + \frac{s_n}{1-\beta_n}(\mu \|F T_n v_n\| + \gamma \|V x_n\|) + \|T_{n+1}v_{n+1} - T_n v_n\|. \tag{123}$$

On the other hand, since ∇f is $(1/L)$-ism, $P_C(I - \lambda_n \nabla f)$ is nonexpansive for $\lambda_n \in (0, 2/L)$. So, it follows that, for any given $p \in \Omega$,

$$\|P_C(I - \lambda_{n+1}\nabla f)v_n\| \le \|P_C(I - \lambda_{n+1}\nabla f)v_n - p\| + \|p\|$$
$$= \|P_C(I - \lambda_{n+1}\nabla f)v_n - P_C(I - \lambda_{n+1}\nabla f)p\| + \|p\|$$
$$\le \|v_n - p\| + \|p\|$$
$$\le \|v_n\| + 2\|p\|. \tag{124}$$

This together with the boundedness of $\{v_n\}$ implies that $\{P_C(I - \lambda_{n+1}\nabla f)v_n\}$ is bounded. Also, observe that

$$\|T_{n+1}v_n - T_n v_n\|$$
$$= \left\|\frac{4P_C(I - \lambda_{n+1}\nabla f) - (2 - \lambda_{n+1}L)I}{2 + \lambda_{n+1}L}v_n - \frac{4P_C(I - \lambda_n \nabla f) - (2 - \lambda_n L)I}{2 + \lambda_n L}v_n\right\|$$
$$\le \left\|\frac{4P_C(I - \lambda_{n+1}\nabla f)}{2 + \lambda_{n+1}L}v_n - \frac{4P_C(I - \lambda_n \nabla f)}{2 + \lambda_n L}v_n\right\| + \left\|\frac{2 - \lambda_n L}{2 + \lambda_n L}v_n - \frac{2 - \lambda_{n+1}L}{2 + \lambda_{n+1}L}v_n\right\|$$
$$= \|(4(2 + \lambda_n L)P_C(I - \lambda_{n+1}\nabla f)v_n - 4(2 + \lambda_{n+1}L)P_C(I - \lambda_n \nabla f)v_n) \times ((2 + \lambda_{n+1}L)(2 + \lambda_n L))^{-1}\| + \frac{4L|\lambda_{n+1} - \lambda_n|}{(2 + \lambda_{n+1}L)(2 + \lambda_n L)}\|v_n\|$$

$$
\begin{aligned}
&= \Big\| \left(4L\left(\lambda_n-\lambda_{n+1}\right)P_C\left(I-\lambda_{n+1}\nabla f\right)v_n + 4\left(2+\lambda_{n+1}L\right)\right.\\
&\qquad \times \left(P_C\left(I-\lambda_{n+1}\nabla f\right)v_n - P_C\left(I-\lambda_n\nabla f\right)v_n\right)\big)\\
&\qquad \times\left(\left(2+\lambda_{n+1}L\right)\left(2+\lambda_nL\right)\right)^{-1}\Big\|\\
&\quad + \frac{4L\left|\lambda_{n+1}-\lambda_n\right|}{\left(2+\lambda_{n+1}L\right)\left(2+\lambda_nL\right)}\left\|v_n\right\|\\
&\le \frac{4L\left|\lambda_n-\lambda_{n+1}\right|\left\|P_C\left(I-\lambda_{n+1}\nabla f\right)v_n\right\|}{\left(2+\lambda_{n+1}L\right)\left(2+\lambda_nL\right)}\\
&\quad + \left(4\left(2+\lambda_{n+1}L\right)\left\|P_C\left(I-\lambda_{n+1}\nabla f\right)v_n\right.\right.\\
&\qquad\qquad \left.\left. -P_C\left(I-\lambda_n\nabla f\right)v_n\right\|\right)\\
&\quad \times\left(\left(2+\lambda_{n+1}L\right)\left(2+\lambda_nL\right)\right)^{-1}\\
&\quad + \frac{4L\left|\lambda_{n+1}-\lambda_n\right|}{\left(2+\lambda_{n+1}L\right)\left(2+\lambda_nL\right)}\left\|v_n\right\|\\
&\le \left|\lambda_{n+1}-\lambda_n\right|\left[L\left\|P_C\left(I-\lambda_{n+1}\nabla f\right)v_n\right\|\right.\\
&\qquad\qquad \left. +4\left\|\nabla f\left(v_n\right)\right\| + L\left\|v_n\right\|\right]\\
&\le \widetilde{M}\left|\lambda_{n+1}-\lambda_n\right|,
\end{aligned}
\tag{125}
$$

where $\sup_{n\ge1}\{L\|P_C(I-\lambda_{n+1}\nabla f)v_n\|+4\|\nabla f(v_n)\|+L\|v_n\|\}\le\widetilde{M}$ for some $\widetilde{M}>0$. So, by (125), we have that

$$
\begin{aligned}
\left\|T_{n+1}v_{n+1}-T_nv_n\right\| &\le \left\|T_{n+1}v_{n+1}-T_{n+1}v_n\right\|\\
&\quad + \left\|T_{n+1}v_n-T_nv_n\right\|\\
&\le \left\|v_{n+1}-v_n\right\| + \widetilde{M}\left|\lambda_{n+1}-\lambda_n\right|\\
&\le \left\|v_{n+1}-v_n\right\| + \frac{4\widetilde{M}}{L}\left(s_{n+1}+s_n\right).
\end{aligned}
\tag{126}
$$

Note that

$$
\begin{aligned}
\left\|v_{n+1}-v_n\right\| &= \left\|\Lambda_{n+1}^Nu_{n+1}-\Lambda_n^Nu_n\right\|\\
&= \Big\|J_{R_N,\lambda_{N,n+1}}\left(I-\lambda_{N,n+1}B_N\right)\Lambda_{n+1}^{N-1}u_{n+1}\\
&\qquad -J_{R_N,\lambda_{N,n}}\left(I-\lambda_{N,n}B_N\right)\Lambda_n^{N-1}u_n\Big\|\\
&\le \Big\|J_{R_N,\lambda_{N,n+1}}\left(I-\lambda_{N,n+1}B_N\right)\Lambda_{n+1}^{N-1}u_{n+1}\\
&\qquad -J_{R_N,\lambda_{N,n}}\left(I-\lambda_{N,n}B_N\right)\Lambda_{n+1}^{N-1}u_{n+1}\Big\|\\
&\quad + \Big\|J_{R_N,\lambda_{N,n}}\left(I-\lambda_{N,n}B_N\right)\Lambda_{n+1}^{N-1}u_{n+1}\\
&\qquad -J_{R_N,\lambda_{N,n}}\left(I-\lambda_{N,n}B_N\right)\Lambda_n^{N-1}u_n\Big\|\\
&\le \Big\|\left(I-\lambda_{N,n+1}B_N\right)\Lambda_{n+1}^{N-1}u_{n+1}\\
&\qquad -\left(I-\lambda_{N,n}B_N\right)\Lambda_{n+1}^{N-1}u_{n+1}\Big\|\\
&\quad + \Big\|\left(I-\lambda_{N,n}B_N\right)\Lambda_{n+1}^{N-1}u_{n+1}\\
&\qquad -\left(I-\lambda_{N,n}B_N\right)\Lambda_n^{N-1}u_n\Big\|\\
&\le \left|\lambda_{N,n+1}-\lambda_{N,n}\right|\left\|B_N\Lambda_{n+1}^{N-1}u_{n+1}\right\|\\
&\quad + \left\|\Lambda_{n+1}^{N-1}u_{n+1}-\Lambda_n^{N-1}u_n\right\|\\
&\le \left|\lambda_{N,n+1}-\lambda_{N,n}\right|\left\|B_N\Lambda_{n+1}^{N-1}u_{n+1}\right\|\\
&\quad + \left|\lambda_{N-1,n+1}-\lambda_{N-1,n}\right|\left\|B_{N-1}\Lambda_{n+1}^{N-2}u_{n+1}\right\|\\
&\quad + \left\|\Lambda_{n+1}^{N-2}u_{n+1}-\Lambda_n^{N-2}u_n\right\|\\
&\ \ \vdots\\
&\le \left|\lambda_{N,n+1}-\lambda_{N,n}\right|\left\|B_N\Lambda_{n+1}^{N-1}u_{n+1}\right\|\\
&\quad + \left|\lambda_{N-1,n+1}-\lambda_{N-1,n}\right|\left\|B_{N-1}\Lambda_{n+1}^{N-2}u_{n+1}\right\|\\
&\quad + \cdots + \left|\lambda_{1,n+1}-\lambda_{1,n}\right|\left\|B_1\Lambda_{n+1}^{0}u_{n+1}\right\|\\
&\quad + \left\|\Lambda_{n+1}^{0}u_{n+1}-\Lambda_n^{0}u_n\right\|\\
&\le \widetilde{M}_0\sum_{i=1}^{N}\left|\lambda_{i,n+1}-\lambda_{i,n}\right| + \left\|u_{n+1}-u_n\right\|,
\end{aligned}
\tag{127}
$$

where $\sup_{n\ge1}\{\sum_{i=1}^N\|B_i\Lambda_{n+1}^{i-1}u_{n+1}\|\}\le\widetilde{M}_0$ for some $\widetilde{M}_0>0$. Also, utilizing Proposition 1(v) we deduce that

$$
\begin{aligned}
&\left\|u_{n+1}-u_n\right\|\\
&\quad = \left\|\Delta_{n+1}^Mx_{n+1}-\Delta_n^Mx_n\right\|\\
&\quad = \Big\|T_{r_{M,n+1}}^{(\Theta_M,\varphi_M)}\left(I-r_{M,n+1}A_M\right)\Delta_{n+1}^{M-1}x_{n+1}\\
&\qquad\quad -T_{r_{M,n}}^{(\Theta_M,\varphi_M)}\left(I-r_{M,n}A_M\right)\Delta_n^{M-1}x_n\Big\|\\
&\quad \le \Big\|T_{r_{M,n+1}}^{(\Theta_M,\varphi_M)}\left(I-r_{M,n+1}A_M\right)\Delta_{n+1}^{M-1}x_{n+1}\\
&\qquad\quad -T_{r_{M,n}}^{(\Theta_M,\varphi_M)}\left(I-r_{M,n}A_M\right)\Delta_{n+1}^{M-1}x_{n+1}\Big\|\\
&\qquad + \Big\|T_{r_{M,n}}^{(\Theta_M,\varphi_M)}\left(I-r_{M,n}A_M\right)\Delta_{n+1}^{M-1}x_{n+1}\\
&\qquad\quad -T_{r_{M,n}}^{(\Theta_M,\varphi_M)}\left(I-r_{M,n}A_M\right)\Delta_n^{M-1}x_n\Big\|\\
&\quad \le \Big\|T_{r_{M,n+1}}^{(\Theta_M,\varphi_M)}\left(I-r_{M,n+1}A_M\right)\Delta_{n+1}^{M-1}x_{n+1}\\
&\qquad\quad -T_{r_{M,n}}^{(\Theta_M,\varphi_M)}\left(I-r_{M,n+1}A_M\right)\Delta_{n+1}^{M-1}x_{n+1}\Big\|
\end{aligned}
$$

$$
\begin{aligned}
&+\left\|T_{r_{M,n}}^{(\Theta_M,\varphi_M)}\left(I-r_{M,n+1}A_M\right)\Delta_{n+1}^{M-1}x_{n+1}\right.\\
&\qquad\left.-T_{r_{M,n}}^{(\Theta_M,\varphi_M)}\left(I-r_{M,n}A_M\right)\Delta_{n+1}^{M-1}x_{n+1}\right\|\\
&+\left\|\left(I-r_{M,n}A_M\right)\Delta_{n+1}^{M-1}x_{n+1}-\left(I-r_{M,n}A_M\right)\Delta_{n}^{M-1}x_{n}\right\|\\
&\leq\frac{\left|r_{M,n+1}-r_{M,n}\right|}{r_{M,n+1}}\left\|T_{r_{M,n+1}}^{(\Theta_M,\varphi_M)}\left(I-r_{M,n+1}A_M\right)\Delta_{n+1}^{M-1}x_{n+1}\right.\\
&\qquad\left.-\left(I-r_{M,n+1}A_M\right)\Delta_{n+1}^{M-1}x_{n+1}\right\|\\
&+\left|r_{M,n+1}-r_{M,n}\right|\left\|A_M\Delta_{n+1}^{M-1}x_{n+1}\right\|\\
&+\left\|\Delta_{n+1}^{M-1}x_{n+1}-\Delta_{n}^{M-1}x_{n}\right\|\\
&=\left|r_{M,n+1}-r_{M,n}\right|\\
&\times\left[\left\|A_M\Delta_{n+1}^{M-1}x_{n+1}\right\|\right.\\
&\qquad+\frac{1}{r_{M,n+1}}\left\|T_{r_{M,n+1}}^{(\Theta_M,\varphi_M)}\left(I-r_{M,n+1}A_M\right)\Delta_{n+1}^{M-1}x_{n+1}\right.\\
&\qquad\left.\left.-\left(I-r_{M,n+1}A_M\right)\Delta_{n+1}^{M-1}x_{n+1}\right\|\right]\\
&+\left\|\Delta_{n+1}^{M-1}x_{n+1}-\Delta_{n}^{M-1}x_{n}\right\|\\
&\vdots\\
&\leq\left|r_{M,n+1}-r_{M,n}\right|\\
&\times\left[\left\|A_M\Delta_{n+1}^{M-1}x_{n+1}\right\|\right.\\
&\qquad+\frac{1}{r_{M,n+1}}\left\|T_{r_{M,n+1}}^{(\Theta_M,\varphi_M)}\left(I-r_{M,n+1}A_M\right)\Delta_{n+1}^{M-1}x_{n+1}\right.\\
&\qquad\left.\left.-\left(I-r_{M,n+1}A_M\right)\Delta_{n+1}^{M-1}x_{n+1}\right\|\right]\\
&+\cdots+\left|r_{1,n+1}-r_{1,n}\right|\\
&\times\left[\left\|A_1\Delta_{n+1}^{0}x_{n+1}\right\|\right.\\
&\qquad+\frac{1}{r_{1,n+1}}\left\|T_{r_{1,n+1}}^{(\Theta_1,\varphi_1)}\left(I-r_{1,n+1}A_1\right)\Delta_{n+1}^{0}x_{n+1}\right.\\
&\qquad\left.\left.-\left(I-r_{1,n+1}A_1\right)\Delta_{n+1}^{0}x_{n+1}\right\|\right]\\
&+\left\|\Delta_{n+1}^{0}x_{n+1}-\Delta_{n}^{0}x_{n}\right\|\\
&\leq\widetilde{M}_1\sum_{k=1}^{M}\left|r_{k,n+1}-r_{k,n}\right|+\left\|x_{n+1}-x_n\right\|,
\end{aligned}
\tag{128}
$$

where $\widetilde{M}_1>0$ is a constant such that for each $n\geq 1$

$$
\begin{aligned}
&\sum_{k=1}^{M}\left[\left\|A_k\Delta_{n+1}^{k-1}x_{n+1}\right\|+\frac{1}{r_{k,n+1}}\right.\\
&\qquad\times\left\|T_{r_{k,n+1}}^{(\Theta_k,\varphi_k)}\left(I-r_{k,n+1}A_k\right)\Delta_{n+1}^{k-1}x_{n+1}\right.\\
&\qquad\left.\left.-\left(I-r_{k,n+1}A_k\right)\Delta_{n+1}^{k-1}x_{n+1}\right\|\right]\leq\widetilde{M}_1.
\end{aligned}
\tag{129}
$$

Combining (123)–(128), we get

$$
\begin{aligned}
&\left\|z_{n+1}-z_n\right\|-\left\|x_{n+1}-x_n\right\|\\
&\quad\leq\frac{s_{n+1}}{1-\beta_{n+1}}\left(\gamma\left\|Vx_{n+1}\right\|+\mu\left\|FT_{n+1}v_{n+1}\right\|\right)\\
&\qquad+\frac{s_n}{1-\beta_n}\left(\mu\left\|FT_nv_n\right\|+\gamma\left\|Vx_n\right\|\right)\\
&\qquad+\left\|T_{n+1}v_{n+1}-T_nv_n\right\|-\left\|x_{n+1}-x_n\right\|\\
&\quad\leq\frac{s_{n+1}}{1-\beta_{n+1}}\left(\gamma\left\|Vx_{n+1}\right\|+\mu\left\|FT_{n+1}v_{n+1}\right\|\right)\\
&\qquad+\frac{s_n}{1-\beta_n}\left(\mu\left\|FT_nv_n\right\|+\gamma\left\|Vx_n\right\|\right)\\
&\qquad+\left\|v_{n+1}-v_n\right\|+\frac{4\widetilde{M}}{L}\left(s_{n+1}+s_n\right)-\left\|x_{n+1}-x_n\right\|\\
&\quad\leq\frac{s_{n+1}}{1-\beta_{n+1}}\left(\gamma\left\|Vx_{n+1}\right\|+\mu\left\|FT_{n+1}v_{n+1}\right\|\right)\\
&\qquad+\frac{s_n}{1-\beta_n}\left(\mu\left\|FT_nv_n\right\|+\gamma\left\|Vx_n\right\|\right)\\
&\qquad+\widetilde{M}_0\sum_{i=1}^{N}\left|\lambda_{i,n+1}-\lambda_{i,n}\right|+\left\|u_{n+1}-u_n\right\|\\
&\qquad+\frac{4\widetilde{M}}{L}\left(s_{n+1}+s_n\right)-\left\|x_{n+1}-x_n\right\|\\
&\quad\leq\frac{s_{n+1}}{1-\beta_{n+1}}\left(\gamma\left\|Vx_{n+1}\right\|+\mu\left\|FT_{n+1}v_{n+1}\right\|\right)\\
&\qquad+\frac{s_n}{1-\beta_n}\left(\mu\left\|FT_nv_n\right\|+\gamma\left\|Vx_n\right\|\right)\\
&\qquad+\widetilde{M}_0\sum_{i=1}^{N}\left|\lambda_{i,n+1}-\lambda_{i,n}\right|+\widetilde{M}_1\sum_{k=1}^{M}\left|r_{k,n+1}-r_{k,n}\right|\\
&\qquad+\left\|x_{n+1}-x_n\right\|+\frac{4\widetilde{M}}{L}\left(s_{n+1}+s_n\right)-\left\|x_{n+1}-x_n\right\|\\
&\quad=\frac{s_{n+1}}{1-\beta_{n+1}}\left(\gamma\left\|Vx_{n+1}\right\|+\mu\left\|FT_{n+1}v_{n+1}\right\|\right)\\
&\qquad+\frac{s_n}{1-\beta_n}\left(\mu\left\|FT_nv_n\right\|+\gamma\left\|Vx_n\right\|\right)
\end{aligned}
$$

$$+ \widetilde{M}_0 \sum_{i=1}^{N} \left|\lambda_{i,n+1} - \lambda_{i,n}\right| + \widetilde{M}_1 \sum_{k=1}^{M} \left|r_{k,n+1} - r_{k,n}\right| + \frac{4\widetilde{M}}{L} \left(s_{n+1} + s_n\right). \tag{130}$$

Thus, it follows from (130) and conditions (i)–(iv) that

$$\limsup_{n\to\infty} \left(\left\|z_{n+1} - z_n\right\| - \left\|x_{n+1} - x_n\right\|\right) \le 0. \tag{131}$$

Hence, by Lemma 11 we have

$$\lim_{n\to\infty} \left\|z_n - x_n\right\| = 0. \tag{132}$$

Consequently,

$$\lim_{n\to\infty} \left\|x_{n+1} - x_n\right\| = \lim_{n\to\infty} \left(1 - \beta_n\right) \left\|z_n - x_n\right\| = 0, \tag{133}$$

and, by (126)–(128),

$$\begin{aligned} &\lim_{n\to\infty} \left\|u_{n+1} - u_n\right\| = 0, \\ &\lim_{n\to\infty} \left\|v_{n+1} - v_n\right\| = 0, \\ &\lim_{n\to\infty} \left\|T_{n+1}v_{n+1} - T_n v_n\right\| = 0. \end{aligned} \tag{134}$$

Step 3. Let us show that $\|A_k\Delta_n^{k-1}x_n - A_k p\| \to 0$ and $\|B_i\Lambda_n^{i-1}u_n - B_i p\| \to 0$ for all $k \in \{1, 2, \ldots, M\}$ and $i \in \{1, 2, \ldots, N\}$.

Indeed, since

$$x_{n+1} = s_n\gamma V x_n + \beta_n x_n + \left(\left(1 - \beta_n\right) I - s_n\mu F\right) T_n v_n, \tag{135}$$

we have

$$\begin{aligned} \left\|x_n - T_n v_n\right\| &\le \left\|x_n - x_{n+1}\right\| + \left\|x_{n+1} - T_n v_n\right\| \\ &\le \left\|x_n - x_{n+1}\right\| + s_n \left\|\gamma V x_n - \mu F T_n v_n\right\| \\ &\quad + \beta_n \left\|x_n - T_n v_n\right\|; \end{aligned} \tag{136}$$

that is,

$$\begin{aligned} \left\|x_n - T_n v_n\right\| \le{}& \frac{1}{1 - \beta_n} \left\|x_n - x_{n+1}\right\| \\ &+ \frac{s_n}{1 - \beta_n} \left(\gamma \left\|V x_n\right\| + \mu \left\|F T_n v_n\right\|\right). \end{aligned} \tag{137}$$

So, from $s_n \to 0$, $\|x_{n+1} - x_n\| \to 0$, and condition (ii), it follows that

$$\lim_{n\to\infty} \left\|x_n - T_n v_n\right\| = 0. \tag{138}$$

Also, from (27) it follows that for all $i \in \{1, 2, \ldots, N\}$ and $k \in \{1, 2, \ldots, M\}$

$$\begin{aligned} \left\|v_n - p\right\|^2 &= \left\|\Lambda_n^N u_n - p\right\|^2 \\ &\le \left\|\Lambda_n^i u_n - p\right\|^2 \\ &= \left\|J_{R_i,\lambda_{i,n}} \left(I - \lambda_{i,n}B_i\right) \Lambda_n^{i-1} u_n - J_{R_i,\lambda_{i,n}} \left(I - \lambda_{i,n}B_i\right) p\right\|^2 \\ &\le \left\|\left(I - \lambda_{i,n}B_i\right) \Lambda_n^{i-1} u_n - \left(I - \lambda_{i,n}B_i\right) p\right\|^2 \\ &\le \left\|\Lambda_n^{i-1} u_n - p\right\|^2 + \lambda_{i,n} \left(\lambda_{i,n} - 2\eta_i\right) \times \left\|B_i\Lambda_n^{i-1} u_n - B_i p\right\|^2 \\ &\le \left\|u_n - p\right\|^2 + \lambda_{i,n} \left(\lambda_{i,n} - 2\eta_i\right) \times \left\|B_i\Lambda_n^{i-1} u_n - B_i p\right\|^2 \\ &\le \left\|x_n - p\right\|^2 + \lambda_{i,n} \left(\lambda_{i,n} - 2\eta_i\right) \times \left\|B_i\Lambda_n^{i-1} u_n - B_i p\right\|^2, \\ \left\|u_n - p\right\|^2 &= \left\|\Delta_n^M x_n - p\right\|^2 \\ &\le \left\|\Delta_n^k x_n - p\right\|^2 \\ &= \left\|T_{r_{k,n}}^{(\Theta_k,\varphi_k)} \left(I - r_{k,n}A_k\right) \Delta_n^{k-1} x_n - T_{r_{k,n}}^{(\Theta_k,\varphi_k)} \left(I - r_{k,n}A_k\right) p\right\|^2 \\ &\le \left\|\left(I - r_{k,n}A_k\right) \Delta_n^{k-1} x_n - \left(I - r_{k,n}A_k\right) p\right\|^2 \\ &\le \left\|\Delta_n^{k-1} x_n - p\right\|^2 + r_{k,n} \left(r_{k,n} - 2\mu_k\right) \times \left\|A_k\Delta_n^{k-1} x_n - A_k p\right\|^2 \\ &\le \left\|x_n - p\right\|^2 + r_{k,n} \left(r_{k,n} - 2\mu_k\right) \times \left\|A_k\Delta_n^{k-1} x_n - A_k p\right\|^2. \end{aligned} \tag{139}$$

Furthermore, utilizing Lemma 7, we deduce from (113) that

$$\begin{aligned} &\left\|x_{n+1} - p\right\|^2 \\ &\quad = \left\|s_n \left(\gamma V x_n - \mu F p\right) + \beta_n \left(x_n - T_n v_n\right) + \left(I - s_n\mu F\right) T_n v_n - \left(I - s_n\mu F\right) p\right\|^2 \\ &\quad \le \left\|\beta_n \left(x_n - T_n v_n\right) + \left(I - s_n\mu F\right) T_n v_n - \left(I - s_n\mu F\right) p\right\|^2 \\ &\qquad + 2s_n \left\langle \gamma V x_n - \mu F p, x_{n+1} - p\right\rangle \end{aligned}$$

$$\begin{aligned}
&\leq [\beta_n \|x_n - T_n v_n\| + \|(I - s_n\mu F) T_n v_n - (I - s_n\mu F) T_n p\|]^2 \\
&\quad + 2s_n \|\gamma V x_n - \mu F p\| \|x_{n+1} - p\| \\
&\leq [\beta_n \|x_n - T_n v_n\| + (1 - s_n\tau) \|v_n - p\|]^2 \\
&\quad + 2s_n \|\gamma V x_n - \mu F p\| \|x_{n+1} - p\| \\
&= (1 - s_n\tau)^2 \|v_n - p\|^2 + \beta_n^2 \|x_n - T_n v_n\|^2 \\
&\quad + 2(1 - s_n\tau) \beta_n \|v_n - p\| \|x_n - T_n v_n\| \\
&\quad + 2s_n \|\gamma V x_n - \mu F p\| \|x_{n+1} - p\| .
\end{aligned} \tag{140}$$

From (139)–(140), it follows that

$$\begin{aligned}
\|x_{n+1} - p\|^2 &\leq \|v_n - p\|^2 + \beta_n^2 \|x_n - T_n v_n\|^2 \\
&\quad + 2(1 - s_n\tau) \beta_n \|v_n - p\| \|x_n - T_n v_n\| \\
&\quad + 2s_n \|\gamma V x_n - \mu F p\| \|x_{n+1} - p\| \\
&\leq \|u_n - p\|^2 + \lambda_{i,n} (\lambda_{i,n} - 2\eta_i) \\
&\quad \times \left\|B_i \Lambda_n^{i-1} u_n - B_i p\right\|^2 + \beta_n^2 \|x_n - T_n v_n\|^2 \\
&\quad + 2(1 - s_n\tau) \beta_n \|v_n - p\| \|x_n - T_n v_n\| \\
&\quad + 2s_n \|\gamma V x_n - \mu F p\| \|x_{n+1} - p\| \\
&\leq \|x_n - p\|^2 + r_{k,n} (r_{k,n} - 2\mu_k) \\
&\quad \times \left\|A_k \Delta_n^{k-1} x_n - A_k p\right\|^2 \\
&\quad + \lambda_{i,n} (\lambda_{i,n} - 2\eta_i) \left\|B_i \Lambda_n^{i-1} u_n - B_i p\right\|^2 \\
&\quad + \beta_n^2 \|x_n - T_n v_n\|^2 \\
&\quad + 2(1 - s_n\tau) \beta_n \|v_n - p\| \|x_n - T_n v_n\| \\
&\quad + 2s_n \|\gamma V x_n - \mu F p\| \|x_{n+1} - p\| ,
\end{aligned} \tag{141}$$

and so

$$\begin{aligned}
&r_{k,n} (2\mu_k - r_{k,n}) \left\|A_k \Delta_n^{k-1} x_n - A_k p\right\|^2 \\
&\quad + \lambda_{i,n} (2\eta_i - \lambda_{i,n}) \left\|B_i \Lambda_n^{i-1} u_n - B_i p\right\|^2 \\
&\leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + \beta_n^2 \|x_n - T_n v_n\|^2 \\
&\quad + 2(1 - s_n\tau) \beta_n \|v_n - p\| \|x_n - T_n v_n\| \\
&\quad + 2s_n \|\gamma V x_n - \mu F p\| \|x_{n+1} - p\| \\
&\leq (\|x_n - p\| + \|x_{n+1} - p\|) \\
&\quad \times \|x_n - x_{n+1}\| + \beta_n^2 \|x_n - T_n v_n\|^2 \\
&\quad + 2(1 - s_n\tau) \beta_n \|v_n - p\| \|x_n - T_n v_n\| \\
&\quad + 2s_n \|\gamma V x_n - \mu F p\| \|x_{n+1} - p\| .
\end{aligned} \tag{142}$$

Since $\{\lambda_{i,n}\} \subset [a_i, b_i] \subset (0, 2\eta_i)$ and $\{r_{k,n}\} \subset [e_k, f_k] \subset (0, 2\mu_k)$ for all $i \in \{1, 2, \ldots, N\}$ and $k \in \{1, 2, \ldots, M\}$, by (133), (138), and (142) we conclude immediately that

$$\begin{aligned}
&\lim_{n\to\infty} \left\|A_k \Delta_n^{k-1} x_n - A_k p\right\| = 0, \\
&\lim_{n\to\infty} \left\|B_i \Lambda_n^{i-1} u_n - B_i p\right\| = 0,
\end{aligned} \tag{143}$$

for all $k \in \{1, 2, \ldots, M\}$ and $i \in \{1, 2, \ldots, N\}$.

Step 4. Let us show that $\|x_n - u_n\| \to 0$, $\|u_n - v_n\| \to 0$, and $\|v_n - T_n v_n\| \to 0$ as $n \to \infty$.

Indeed, by Proposition 1(iii) we obtain that for each $k \in \{1, 2, \ldots, M\}$

$$\begin{aligned}
&\left\|\Delta_n^k x_n - p\right\|^2 \\
&= \left\|T_{r_{k,n}}^{(\Theta_k,\varphi_k)} (I - r_{k,n} A_k) \Delta_n^{k-1} x_n - T_{r_{k,n}}^{(\Theta_k,\varphi_k)} (I - r_{k,n} A_k) p\right\|^2 \\
&\leq \left\langle (I - r_{k,n} A_k) \Delta_n^{k-1} x_n - (I - r_{k,n} A_k) p, \Delta_n^k x_n - p \right\rangle \\
&= \frac{1}{2} \left(\left\|(I - r_{k,n} A_k) \Delta_n^{k-1} x_n - (I - r_{k,n} A_k) p\right\|^2 \right. \\
&\quad + \left\|\Delta_n^k x_n - p\right\|^2 - \left\|(I - r_{k,n} A_k) \Delta_n^{k-1} x_n \right. \\
&\qquad - (I - r_{k,n} A_k) p \\
&\qquad \left.\left. - \left(\Delta_n^k x_n - p\right)\right\|^2 \right) \\
&\leq \frac{1}{2} \left(\left\|\Delta_n^{k-1} x_n - p\right\|^2 + \left\|\Delta_n^k x_n - p\right\|^2 \right. \\
&\quad \left. - \left\|\Delta_n^{k-1} x_n - \Delta_n^k x_n - r_{k,n} \left(A_k \Delta_n^{k-1} x_n - A_k p\right)\right\|^2 \right),
\end{aligned} \tag{144}$$

which implies that

$$\begin{aligned}
\left\|\Delta_n^k x_n - p\right\|^2 &\leq \left\|\Delta_n^{k-1} x_n - p\right\|^2 \\
&\quad - \left\|\Delta_n^{k-1} x_n - \Delta_n^k x_n \right. \\
&\qquad \left. - r_{k,n} \left(A_k \Delta_n^{k-1} x_n - A_k p\right)\right\|^2 \\
&= \left\|\Delta_n^{k-1} x_n - p\right\|^2 - \left\|\Delta_n^{k-1} x_n - \Delta_n^k x_n\right\|^2 \\
&\quad - r_{k,n}^2 \left\|A_k \Delta_n^{k-1} x_n - A_k p\right\|^2 \\
&\quad + 2r_{k,n} \left\langle \Delta_n^{k-1} x_n - \Delta_n^k x_n, A_k \Delta_n^{k-1} x_n - A_k p \right\rangle \\
&\leq \left\|\Delta_n^{k-1} x_n - p\right\|^2 - \left\|\Delta_n^{k-1} x_n - \Delta_n^k x_n\right\|^2 \\
&\quad + 2r_{k,n} \left\|\Delta_n^{k-1} x_n - \Delta_n^k x_n\right\| \left\|A_k \Delta_n^{k-1} x_n - A_k p\right\| \\
&\leq \|x_n - p\|^2 - \left\|\Delta_n^{k-1} x_n - \Delta_n^k x_n\right\|^2 \\
&\quad + 2r_{k,n} \left\|\Delta_n^{k-1} x_n - \Delta_n^k x_n\right\| \left\|A_k \Delta_n^{k-1} x_n - A_k p\right\| .
\end{aligned} \tag{145}$$

Also, by Lemma 14, we obtain that for each $i \in \{1, 2, \dots, N\}$

$$\begin{aligned}
&\left\|\Lambda_n^i u_n - p\right\|^2 \\
&\quad = \left\|J_{R_i,\lambda_{i,n}} (I - \lambda_{i,n} B_i) \Lambda_n^{i-1} u_n - J_{R_i,\lambda_{i,n}} (I - \lambda_{i,n} B_i) p\right\|^2 \\
&\quad \le \left\langle (I - \lambda_{i,n} B_i) \Lambda_n^{i-1} u_n - (I - \lambda_{i,n} B_i) p, \Lambda_n^i u_n - p \right\rangle \\
&\quad = \frac{1}{2} \left(\left\|(I - \lambda_{i,n} B_i) \Lambda_n^{i-1} u_n - (I - \lambda_{i,n} B_i) p\right\|^2 \right. \\
&\qquad + \left\|\Lambda_n^i u_n - p\right\|^2 - \left\|(I - \lambda_{i,n} B_i) \Lambda_n^{i-1} u_n \right. \\
&\qquad\qquad - (I - \lambda_{i,n} B_i) p \\
&\qquad\qquad \left.\left. - \left(\Lambda_n^i u_n - p\right)\right\|^2 \right) \\
&\quad \le \frac{1}{2} \left(\left\|\Lambda_n^{i-1} u_n - p\right\|^2 + \left\|\Lambda_n^i u_n - p\right\|^2 \right. \\
&\qquad \left. - \left\|\Lambda_n^{i-1} u_n - \Lambda_n^i u_n - \lambda_{i,n} \left(B_i \Lambda_n^{i-1} u_n - B_i p\right)\right\|^2 \right) \\
&\quad \le \frac{1}{2} \left(\left\|u_n - p\right\|^2 + \left\|\Lambda_n^i u_n - p\right\|^2 \right. \\
&\qquad \left. - \left\|\Lambda_n^{i-1} u_n - \Lambda_n^i u_n - \lambda_{i,n} \left(B_i \Lambda_n^{i-1} u_n - B_i p\right)\right\|^2 \right),
\end{aligned} \tag{146}$$

which implies

$$\begin{aligned}
\left\|\Lambda_n^i u_n - p\right\|^2 &\le \left\|u_n - p\right\|^2 \\
&\quad - \left\|\Lambda_n^{i-1} u_n - \Lambda_n^i u_n - \lambda_{i,n} \left(B_i \Lambda_n^{i-1} u_n - B_i p\right)\right\|^2 \\
&= \left\|u_n - p\right\|^2 - \left\|\Lambda_n^{i-1} u_n - \Lambda_n^i u_n\right\|^2 \\
&\quad - \lambda_{i,n}^2 \left\|B_i \Lambda_n^{i-1} u_n - B_i p\right\|^2 \\
&\quad + 2\lambda_{i,n} \left\langle \Lambda_n^{i-1} u_n - \Lambda_n^i u_n, B_i \Lambda_n^{i-1} u_n - B_i p \right\rangle \\
&\le \left\|u_n - p\right\|^2 - \left\|\Lambda_n^{i-1} u_n - \Lambda_n^i u_n\right\|^2 \\
&\quad + 2\lambda_{i,n} \left\|\Lambda_n^{i-1} u_n - \Lambda_n^i u_n\right\| \left\|B_i \Lambda_n^{i-1} u_n - B_i p\right\|.
\end{aligned} \tag{147}$$

Thus, from (140), (145), and (147), we have

$$\begin{aligned}
&\left\|x_{n+1} - p\right\|^2 \\
&\quad \le (1 - s_n \tau)^2 \left\|v_n - p\right\|^2 + \beta_n^2 \left\|x_n - T_n v_n\right\|^2 \\
&\qquad + 2 (1 - s_n \tau) \beta_n \left\|v_n - p\right\| \left\|x_n - T_n v_n\right\| \\
&\qquad + 2s_n \left\|\gamma V x_n - \mu F p\right\| \left\|x_{n+1} - p\right\| \\
&\quad = (1 - s_n \tau)^2 \left\|\Lambda_n^N u_n - p\right\|^2 + \beta_n^2 \left\|x_n - T_n v_n\right\|^2 \\
&\qquad + 2 (1 - s_n \tau) \beta_n \left\|v_n - p\right\| \left\|x_n - T_n v_n\right\| \\
&\qquad + 2s_n \left\|\gamma V x_n - \mu F p\right\| \left\|x_{n+1} - p\right\| \\
&\quad \le (1 - s_n \tau)^2 \left\|\Lambda_n^i u_n - p\right\|^2 + \beta_n^2 \left\|x_n - T_n v_n\right\|^2 \\
&\qquad + 2 (1 - s_n \tau) \beta_n \left\|v_n - p\right\| \left\|x_n - T_n v_n\right\| \\
&\qquad + 2s_n \left\|\gamma V x_n - \mu F p\right\| \left\|x_{n+1} - p\right\| \\
&\quad \le (1 - s_n \tau)^2 \left[\left\|u_n - p\right\|^2 - \left\|\Lambda_n^{i-1} u_n - \Lambda_n^i u_n\right\|^2 \right. \\
&\qquad\qquad + 2\lambda_{i,n} \left\|\Lambda_n^{i-1} u_n - \Lambda_n^i u_n\right\| \\
&\qquad\qquad \left. \times \left\|B_i \Lambda_n^{i-1} u_n - B_i p\right\| \right] \\
&\qquad + \beta_n^2 \left\|x_n - T_n v_n\right\|^2 + 2 (1 - s_n \tau) \beta_n \left\|v_n - p\right\| \\
&\qquad \times \left\|x_n - T_n v_n\right\| + 2s_n \left\|\gamma V x_n - \mu F p\right\| \left\|x_{n+1} - p\right\| \\
&\quad = (1 - s_n \tau)^2 \left[\left\|\Delta_n^M x_n - p\right\|^2 - \left\|\Lambda_n^{i-1} u_n - \Lambda_n^i u_n\right\|^2 \right. \\
&\qquad\qquad + 2\lambda_{i,n} \left\|\Lambda_n^{i-1} u_n - \Lambda_n^i u_n\right\| \\
&\qquad\qquad \left. \times \left\|B_i \Lambda_n^{i-1} u_n - B_i p\right\| \right] \\
&\qquad + \beta_n^2 \left\|x_n - T_n v_n\right\|^2 + 2 (1 - s_n \tau) \beta_n \left\|v_n - p\right\| \\
&\qquad \times \left\|x_n - T_n v_n\right\| + 2s_n \left\|\gamma V x_n - \mu F p\right\| \left\|x_{n+1} - p\right\| \\
&\quad \le (1 - s_n \tau)^2 \left[\left\|\Delta_n^k x_n - p\right\|^2 - \left\|\Lambda_n^{i-1} u_n - \Lambda_n^i u_n\right\|^2 \right. \\
&\qquad\qquad + 2\lambda_{i,n} \left\|\Lambda_n^{i-1} u_n - \Lambda_n^i u_n\right\| \\
&\qquad\qquad \left. \times \left\|B_i \Lambda_n^{i-1} u_n - B_i p\right\| \right] \\
&\qquad + \beta_n^2 \left\|x_n - T_n v_n\right\|^2 + 2 (1 - s_n \tau) \beta_n \left\|v_n - p\right\| \\
&\qquad \times \left\|x_n - T_n v_n\right\| + 2s_n \left\|\gamma V x_n - \mu F p\right\| \left\|x_{n+1} - p\right\| \\
&\quad \le (1 - s_n \tau)^2 \left[\left\|x_n - p\right\|^2 - \left\|\Delta_n^{k-1} x_n - \Delta_n^k x_n\right\|^2 \right. \\
&\qquad\qquad + 2r_{k,n} \left\|\Delta_n^{k-1} x_n - \Delta_n^k x_n\right\| \\
&\qquad\qquad \times \left\|A_k \Delta_n^{k-1} x_n - A_k p\right\| \\
&\qquad\qquad - \left\|\Lambda_n^{i-1} u_n - \Lambda_n^i u_n\right\|^2 \\
&\qquad\qquad + 2\lambda_{i,n} \left\|\Lambda_n^{i-1} u_n - \Lambda_n^i u_n\right\| \\
&\qquad\qquad \left. \times \left\|B_i \Lambda_n^{i-1} u_n - B_i p\right\| \right] \\
&\qquad + \beta_n^2 \left\|x_n - T_n v_n\right\|^2 \\
&\qquad + 2 (1 - s_n \tau) \beta_n \left\|v_n - p\right\| \left\|x_n - T_n v_n\right\| \\
&\qquad + 2s_n \left\|\gamma V x_n - \mu F p\right\| \left\|x_{n+1} - p\right\| \\
&\quad \le \left\|x_n - p\right\|^2 - (1 - s_n \tau)^2 \left\|\Delta_n^{k-1} x_n - \Delta_n^k x_n\right\|^2 \\
&\qquad + 2r_{k,n} \left\|\Delta_n^{k-1} x_n - \Delta_n^k x_n\right\| \left\|A_k \Delta_n^{k-1} x_n - A_k p\right\|
\end{aligned}$$

$$\begin{aligned}&-(1-s_n\tau)^2\left\|\Lambda_n^{i-1}u_n-\Lambda_n^iu_n\right\|^2\\&+2\lambda_{i,n}\left\|\Lambda_n^{i-1}u_n-\Lambda_n^iu_n\right\|\left\|B_i\Lambda_n^{i-1}u_n-B_ip\right\|\\&+\beta_n^2\left\|x_n-T_nv_n\right\|^2+2(1-s_n\tau)\beta_n\left\|v_n-p\right\|\\&\times\left\|x_n-T_nv_n\right\|+2s_n\left\|\gamma Vx_n-\mu Fp\right\|\left\|x_{n+1}-p\right\|;\end{aligned}\tag{148}$$

that is,

$$\begin{aligned}&(1-s_n\tau)^2\left[\left\|\Delta_n^{k-1}x_n-\Delta_n^kx_n\right\|^2+\left\|\Lambda_n^{i-1}u_n-\Lambda_n^iu_n\right\|^2\right]\\&\le\left\|x_n-p\right\|^2-\left\|x_{n+1}-p\right\|^2\\&\quad+2r_{k,n}\left\|\Delta_n^{k-1}x_n-\Delta_n^kx_n\right\|\left\|A_k\Delta_n^{k-1}x_n-A_kp\right\|\\&\quad+2\lambda_{i,n}\left\|\Lambda_n^{i-1}u_n-\Lambda_n^iu_n\right\|\left\|B_i\Lambda_n^{i-1}u_n-B_ip\right\|\\&\quad+\beta_n^2\left\|x_n-T_nv_n\right\|^2\\&\quad+2(1-s_n\tau)\beta_n\left\|v_n-p\right\|\left\|x_n-T_nv_n\right\|\\&\quad+2s_n\left\|\gamma Vx_n-\mu Fp\right\|\left\|x_{n+1}-p\right\|\\&\le\left(\left\|x_n-p\right\|+\left\|x_{n+1}-p\right\|\right)\left\|x_n-x_{n+1}\right\|\\&\quad+2r_{k,n}\left\|\Delta_n^{k-1}x_n-\Delta_n^kx_n\right\|\left\|A_k\Delta_n^{k-1}x_n-A_kp\right\|\\&\quad+2\lambda_{i,n}\left\|\Lambda_n^{i-1}u_n-\Lambda_n^iu_n\right\|\left\|B_i\Lambda_n^{i-1}u_n-B_ip\right\|\\&\quad+\beta_n^2\left\|x_n-T_nv_n\right\|^2+2(1-s_n\tau)\beta_n\left\|v_n-p\right\|\\&\quad\times\left\|x_n-T_nv_n\right\|+2s_n\left\|\gamma Vx_n-\mu Fp\right\|\left\|x_{n+1}-p\right\|.\end{aligned}\tag{149}$$

So, from $s_n\to 0$, (133), (138), and (143), we immediately get

$$\lim_{n\to\infty}\left\|\Lambda_n^{i-1}u_n-\Lambda_n^iu_n\right\|=0,\qquad\lim_{n\to\infty}\left\|\Delta_n^{k-1}x_n-\Delta_n^kx_n\right\|=0,\tag{150}$$

for all $i\in\{1,2,\dots,N\}$ and $k\in\{1,2,\dots,M\}$. Note that

$$\begin{aligned}\left\|x_n-u_n\right\|&=\left\|\Delta_n^0x_n-\Delta_n^Mx_n\right\|\\&\le\left\|\Delta_n^0x_n-\Delta_n^1x_n\right\|+\left\|\Delta_n^1x_n-\Delta_n^2x_n\right\|\\&\quad+\cdots+\left\|\Delta_n^{M-1}x_n-\Delta_n^Mx_n\right\|,\\\left\|u_n-v_n\right\|&=\left\|\Lambda_n^0u_n-\Lambda_n^Nu_n\right\|\\&\le\left\|\Lambda_n^0u_n-\Lambda_n^1u_n\right\|+\left\|\Lambda_n^1u_n-\Lambda_n^2u_n\right\|\\&\quad+\cdots+\left\|\Lambda_n^{N-1}u_n-\Lambda_n^Nu_n\right\|.\end{aligned}\tag{151}$$

Thus, from (150) we have

$$\lim_{n\to\infty}\left\|x_n-u_n\right\|=0,\qquad\lim_{n\to\infty}\left\|u_n-v_n\right\|=0.\tag{152}$$

It is easy to see that as $n\to\infty$

$$\left\|x_n-v_n\right\|\le\left\|x_n-u_n\right\|+\left\|u_n-v_n\right\|\longrightarrow 0.\tag{153}$$

Also, observe that

$$\left\|T_nv_n-v_n\right\|\le\left\|T_nv_n-x_n\right\|+\left\|x_n-v_n\right\|.\tag{154}$$

Hence, we have from (138)

$$\lim_{n\to\infty}\left\|T_nv_n-v_n\right\|=0.\tag{155}$$

Step 5. Let us show that $\limsup_{n\to\infty}\langle(\mu F-\gamma V)q,q-x_n\rangle\le 0$, where $q\in\Omega$ is the same as in Theorem 18; that is, $q\in\Omega$ is a unique solution of VIP (50). To show this inequality, we choose a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\begin{aligned}&\limsup_{n\to\infty}\langle(\mu F-\gamma V)q,q-x_n\rangle\\&\quad=\lim_{i\to\infty}\left\langle(\mu F-\gamma V)q,q-x_{n_i}\right\rangle.\end{aligned}\tag{156}$$

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_{i_j}}\}$ of $\{x_{n_i}\}$ which converges weakly to w. Without loss of generality, we may assume that $x_{n_i}\rightharpoonup w$. From Step 4, we have that $\Delta_{n_i}^kx_{n_i}\rightharpoonup w$, $\Lambda_{n_i}^mu_{n_i}\rightharpoonup w$, $u_{n_i}\rightharpoonup w$, and $v_{n_i}\rightharpoonup w$, where $k\in\{1,2,\dots,M\}$, $m\in\{1,2,\dots,N\}$. Since $v_n-T_nv_n\to 0$ by Step 4, by the same arguments as in the proof of Theorem 18, we get $w\in\Omega$. Since $q=P_\Omega(I-\mu F+\gamma V)q$, it follows that

$$\begin{aligned}\limsup_{n\to\infty}\langle(\mu F-\gamma V)q,q-x_n\rangle&=\lim_{i\to\infty}\left\langle(\mu F-\gamma V)q,q-x_{n_i}\right\rangle\\&=\langle(\mu F-\gamma V)q,q-w\rangle\le 0.\end{aligned}\tag{157}$$

Step 6. Let us show that $\lim_{n\to\infty}\|x_n-q\|=0$, where $q\in\Omega$ is the same as in Theorem 18; that is, $q\in\Omega$ is a unique solution of VIP (50). From (113), we know that

$$\begin{aligned}x_{n+1}-q&=s_n(\gamma Vx_n-\mu Fq)+\beta_n(T_nv_n-q)\\&\quad+((1-\beta_n)I-s_n\mu F)T_nv_n\\&\quad-((1-\beta_n)I-s_n\mu F)q.\end{aligned}\tag{158}$$

Applying Lemmas 7 and 13 and noticing $T_nq=q$ and $\|v_n-q\|\le\|x_n-q\|$ for all $n\ge 1$, we have

$$\begin{aligned}&\left\|x_{n+1}-q\right\|^2\\&\le\left\|\beta_n(T_nv_n-q)+((1-\beta_n)I-s_n\mu F)T_nv_n\right.\\&\quad\left.-((1-\beta_n)I-s_n\mu F)q\right\|^2\\&\quad+2s_n\langle\gamma Vx_n-\mu Fq,x_{n+1}-q\rangle\end{aligned}$$

$$\begin{aligned}
&\leq [\beta_n \|T_n v_n - q\| + \|((1-\beta_n) I - s_n \mu F) T_n v_n \\
&\qquad - ((1-\beta_n) I - s_n \mu F) q\|]^2 \\
&\quad + 2s_n \langle \gamma V x_n - \gamma V q, x_{n+1} - q \rangle \\
&\quad + 2s_n \langle \gamma V q - \mu F q, x_{n+1} - q \rangle \\
&= \Big[\beta_n \|T_n v_n - q\| + (1-\beta_n) \\
&\qquad \times \Big\| \Big(I - \frac{s_n}{1-\beta_n}\mu F\Big) T_n v_n - \Big(I - \frac{s_n}{1-\beta_n}\mu F\Big) T_n q \Big\| \Big]^2 \\
&\quad + 2s_n \langle \gamma V x_n - \gamma V q, x_{n+1} - q \rangle \\
&\quad + 2s_n \langle \gamma V q - \mu F q, x_{n+1} - q \rangle \\
&\leq \Big[\beta_n \|v_n - q\| + (1-\beta_n)\Big(1 - \frac{s_n \tau}{1-\beta_n}\Big) \|v_n - q\|^2 \Big]^2 \\
&\quad + 2s_n \langle \gamma V x_n - \gamma V q, x_{n+1} - q \rangle \\
&\quad + 2s_n \langle \gamma V q - \mu F q, x_{n+1} - q \rangle \\
&= [\beta_n \|v_n - q\| + (1 - \beta_n - s_n \tau) \|v_n - q\|]^2 \\
&\quad + 2s_n \langle \gamma V x_n - \gamma V q, x_{n+1} - q \rangle \\
&\quad + 2s_n \langle \gamma V q - \mu F q, x_{n+1} - q \rangle \\
&\leq [\beta_n \|x_n - q\| + (1 - \beta_n - s_n \tau) \|x_n - q\|]^2 \\
&\quad + 2s_n \langle \gamma V x_n - \gamma V q, x_{n+1} - q \rangle \\
&\quad + 2s_n \langle \gamma V q - \mu F q, x_{n+1} - q \rangle \\
&\leq (1 - s_n \tau)^2 \|x_n - q\|^2 + 2 s_n \gamma l \|x_n - q\| \\
&\quad \times \|x_{n+1} - q\| + 2s_n \langle \gamma V q - \mu F q, x_{n+1} - q \rangle \\
&\leq (1 - s_n \tau)^2 \|x_n - q\|^2 + s_n \gamma l \left(\|x_n - q\|^2 + \|x_{n+1} - q\|^2\right) \\
&\quad + 2s_n \langle \gamma V q - \mu F q, x_{n+1} - q \rangle .
\end{aligned} \tag{159}$$

This implies that

$$\begin{aligned}
&\|x_{n+1} - q\|^2 \\
&\quad \leq \frac{1 - 2\tau s_n + \tau^2 s_n^2 + s_n \gamma l}{1 - s_n \gamma l} \|x_n - q\|^2 \\
&\qquad + \frac{2s_n}{1 - s_n \gamma l} \langle \gamma V q - \mu F q, x_{n+1} - q \rangle \\
&\quad = \Big(1 - \frac{2(\tau - \gamma l) s_n}{1 - s_n \gamma l}\Big) \|x_n - q\|^2 + \frac{\tau^2 s_n^2}{1 - s_n \gamma l} \|x_n - q\|^2 \\
&\qquad + \frac{2s_n}{1 - s_n \gamma l} \langle \gamma V q - \mu F q, x_{n+1} - q \rangle \\
&\quad \leq \Big(1 - \frac{2(\tau - \gamma l)}{1 - s_n \gamma l} s_n\Big) \|x_n - q\|^2 + \frac{2(\tau - \gamma l) s_n}{1 - s_n \gamma l} \\
&\qquad \times \Big(\frac{\tau^2 s_n}{2(\tau - \gamma l)} \widetilde{M}_2 \\
&\qquad\qquad + \frac{1}{\tau - \gamma l} \langle \mu F q - \gamma V q, q - x_{n+1} \rangle \Big) \\
&\quad = (1 - \sigma_n) \|x_n - q\|^2 + \sigma_n \delta_n,
\end{aligned} \tag{160}$$

where $\widetilde{M}_2 = \sup_{n\geq 1} \|x_n - q\|^2$, $\sigma_n = (2(\tau - \gamma l)/(1 - s_n \gamma l)) s_n$, and

$$\delta_n = \frac{\tau^2 s_n}{2(\tau - \gamma l)} \widetilde{M}_2 + \frac{1}{\tau - \gamma l} \langle \mu F q - \gamma V q, q - x_{n+1} \rangle . \tag{161}$$

From condition (i) and Step 5, it is easy to see that $\sigma_n \to 0$, $\sum_{n=0}^{\infty} \sigma_n = \infty$ and $\limsup_{n\to\infty} \delta_n \leq 0$. Hence, by Lemma 10, we conclude that $x_n \to q$ as $n \to \infty$. This completes the proof. □

Corollary 22. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $f : C \to \mathbf{R}$ be a convex functional with L-Lipschitz continuous gradient ∇f. Let Θ be a bifunction from $C \times C$ to $\mathbf{R}$ satisfying (A1)–(A4) and let $\varphi : C \to \mathbf{R} \cup \{+\infty\}$ be a proper lower semicontinuous and convex function. Let $R_i : C \to 2^H$ be a maximal monotone mapping and let $A : H \to H$ and $B_i : C \to H$ be ζ-inverse-strongly monotone and η_i-inverse-strongly monotone, respectively, for $i = 1, 2$. Let $F : H \to H$ be a κ-Lipschitzian and η-strongly monotone operator with positive constants $\kappa, \eta > 0$. Let $V : H \to H$ be an l-Lipschitzian mapping with constant $l \geq 0$. Let $0 < \mu < 2\eta/\kappa^2$ and $0 \leq \gamma l < \tau$, where $\tau = 1 - \sqrt{1 - \mu(2\eta - \mu\kappa^2)}$. Assume that $\Omega := GMEP(\Theta, \varphi, A) \cap I(B_1, R_1) \cap I(B_2, R_2) \cap \Gamma \neq \emptyset$ and that either (B1) or (B2) holds. For arbitrarily given $x_1 \in H$, let $\{x_n\}$ be a sequence generated by*

$$\begin{aligned}
&\Theta(u_n, y) + \varphi(y) - \varphi(u_n) + \langle A x_n, y - u_n \rangle \\
&\qquad + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq 0, \quad \forall y \in C, \\
&v_n = J_{R_2,\lambda_{2,n}} (I - \lambda_{2,n} B_2) J_{R_1,\lambda_{1,n}} (I - \lambda_{1,n} B_1) u_n, \\
&x_{n+1} = s_n \gamma V x_n + \beta_n x_n + ((1 - \beta_n) I - s_n \mu F) T_n v_n, \\
&\qquad\qquad \forall n \geq 1,
\end{aligned} \tag{162}$$

where $P_C(I - \lambda_n \nabla f) = s_n I + (1 - s_n) T_n$ (here T_n is nonexpansive and $s_n = (2 - \lambda_n L)/4 \in (0, 1/2)$ for each $\lambda_n \in (0, 2/L)$). Assume that the following conditions hold:

(i) *$s_n \in (0, 1/2)$ for each $\lambda_n \in (0, 2/L)$, and $\lim_{n\to\infty} s_n = 0$ ($\Leftrightarrow \lim_{n\to\infty} \lambda_n = 2/L$);*

(ii) *$\{\beta_n\} \subset (0, 1)$ and $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;*

(iii) *$\{\lambda_{i,n}\} \subset [a_i, b_i] \subset (0, 2\eta_i)$ and $\lim_{n\to\infty} |\lambda_{i,n+1} - \lambda_{i,n}| = 0$ for $i = 1, 2$;*

(iv) *$\{r_n\} \subset [e, f] \subset (0, 2\zeta)$ and $\lim_{n\to\infty} |r_{n+1} - r_n| = 0$.*

Then $\{x_n\}$ *converges strongly as* $\lambda_n \to 2/L$ ($\Leftrightarrow s_n \to 0$) *to a point* $q \in \Omega$*, which is a unique solution of VIP* (110).

Corollary 23. *Let* C *be a nonempty closed convex subset of a real Hilbert space* H*. Let* $f : C \to \mathbf{R}$ *be a convex functional with L-Lipschitz continuous gradient* ∇f*. Let* Θ *be a bifunction from* $C\times C$ *to* $\mathbf{R}$ *satisfying (A1)–(A4) and let* $\varphi : C \to \mathbf{R}\cup\{+\infty\}$ *be a proper lower semicontinuous and convex function. Let* $R : C \to 2^H$ *be a maximal monotone mapping and let* $A : H \to H$ *and* $B : C \to H$ *be* ζ*-inverse-strongly monotone and* ξ*-inverse-strongly monotone, respectively. Let* $F : H \to H$ *be a* κ*-Lipschitzian and* η*-strongly monotone operator with positive constants* $\kappa, \eta > 0$*. Let* $V : H \to H$ *be an* l*-Lipschitzian mapping with constant* $l \geq 0$*. Let* $0 < \mu < 2\eta/\kappa^2$ *and* $0 \leq \gamma l < \tau$*, where* $\tau = 1 - \sqrt{1 - \mu(2\eta - \mu\kappa^2)}$*. Assume that* $\Omega := GMEP(\Theta, \varphi, A) \cap I(B, R) \cap \Gamma \neq \emptyset$ *and that either (B1) or (B2) holds. For arbitrarily given* $x_1 \in H$*, let* $\{x_n\}$ *be a sequence generated by*

$$\begin{aligned} &\Theta(u_n, y) + \varphi(y) - \varphi(u_n) + \langle Ax_n, y - u_n\rangle \\ &\qquad + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ &x_{n+1} = s_n\gamma V x_n + \beta_n x_n \\ &\qquad + ((1-\beta_n) I - s_n\mu F) T_n J_{R,\rho_n}(u_n - \rho_n B u_n), \quad \forall n \geq 1, \end{aligned} \tag{163}$$

where $P_C(I - \lambda_n\nabla f) = s_n I + (1 - s_n)T_n$ *(here* T_n *is nonexpansive and* $s_n = (2-\lambda_n L)/4 \in (0, 1/2)$ *for each* $\lambda_n \in (0, 2/L)$*). Assume that the following conditions hold:*

(i) $s_n \in (0, 1/2)$ *for each* $\lambda_n \in (0, 2/L)$*, and* $\lim_{n\to\infty} s_n = 0$ ($\Leftrightarrow \lim_{n\to\infty}\lambda_n = 2/L$);

(ii) $\{\beta_n\} \subset (0, 1)$ *and* $0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1$;

(iii) $\{\rho_n\} \subset [a, b] \subset (0, 2\xi)$ *and* $\lim_{n\to\infty}|\rho_{n+1} - \rho_n| = 0$;

(iv) $\{r_n\} \subset [e, f] \subset (0, 2\zeta)$ *and* $\lim_{n\to\infty}|r_{n+1} - r_n| = 0$.

Then $\{x_n\}$ *converges strongly as* $\lambda_n \to 2/L$ ($\Leftrightarrow s_n \to 0$) *to a point* $q \in \Omega$*, which is a unique solution of VIP* (112).

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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