

Research Article

Controllability of Linear Discrete-Time Systems with Both Delayed States and Delayed Inputs

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The controllability issues for discrete-time linear systems with delay in state and control are addressed. By introducing a new concept, the controllability realization index (CRI), the characteristic of controllability is revealed. An easily testable necessary and sufficient condition for the controllability of discrete-time linear systems with state and control delay is established.

1. Introduction

The concept of controllability, first given by Kalman in the 1960s [1], plays a fundamental role in the modern control theory and has close connections with pole assignment, structure decomposition, quadratic control, and so forth [2, 3]. The various aspects of the controllability of linear systems with delay were considered by several authors [4–11]. The discrete cases have been considered by Klamka [7], Watanabe [8], and Phat [9], but the mathematical conditions given for investigating the controllability are not suitable for real verification and application.

In our recent paper [12], a new concept called *controllability realization index (CRI)* is proposed, which is crucial in determining the controllability of such kind of discrete systems with delays. In that paper, it is proved that the value of CRI is finite for discrete systems with delays, and a general CRI value for planar discrete systems with delays is given. Thus, the judging condition of controllability for planar case is established. In this paper, we will extend our result to the more general case, namely, discrete systems with any order, with time delays both in state and in control.

This paper is organized as follows. In Section 2, some basic definitions and preliminary results are presented. Section 3 is the main results. An easily testable necessary

and sufficient condition for the controllability of discrete-time linear systems with state and control delay is established. A numerical example is given in Section 4. Finally, the conclusion is provided in Section 5.

2. Problem Formulation and Preliminaries

In this paper, we consider the discrete-time case the system model is described as follows:

$$x(k+1) = \sum_{i=0}^p A_i x(k-i) + \sum_{j=0}^q B_j u(k-j), \quad (1)$$

where $x(k) \in \mathbb{R}^n$ is the state, $u(k) \in \mathbb{R}^s$ is the input, $A_i \in \mathbb{R}^{n \times n}$ and $B_j \in \mathbb{R}^{n \times s}$ are constant matrices, and the positive integers i, j are the lengths of the steps of time delays. The initial states $x(-p), x(-p+1), \dots, x(0)$ and the initial input $u(-q), u(-q+1), \dots, u(-1)$ are given arbitrarily.

The controllability discussed here refers to the unconstrained controllability or completely controllability.

Definition 1 (controllability). The system (1) is said to be (completely) controllable if, for any initial input $u(-q), u(-q+1), \dots, u(-1)$, any initial state $x(-p), x(-p+1), \dots, x(0)$, and

any terminal state x_f , there exist a positive integer k and, input $u(0), \dots, u(k-1)$ such that $x(k) = x_f$.

Definition 2 (controllability realization index, CRI). For the system (1), if there exists a positive integer K such that for any initial input $u(-q), u(-q+1), \dots, u(-1)$, initial state $x(-p), x(-p+1), \dots, x(0)$, and any terminal state x_f , there exists an input $u(0), \dots, u(K-1)$ such that $x(K) = x_f$, then one calls K the controllability realization index (CRI) of the system (1). Obviously, if exists, such K is not unique, so the one calls the smallest K among them the minimum controllability realization index (MinCRI).

Denote by \mathbb{N} the nonnegative integer set. The matrices $A_1, \dots, A_N \in \mathbb{R}^{n \times n}$ are said to be linearly dependent on $\mathbb{R}^{n \times n}$, if there exist scalars $c_1, \dots, c_N \in \mathbb{R}$, not all are zero, such that $\sum_{i=1}^N c_i A_i = 0$. In the following statement, $\text{span}\{A_1, \dots, A_N\}$ will be used to denote the space constructed by the linear combinations of matrices $\{A_1, \dots, A_N\}$.

3. Main Results

3.1. Delay in State. In this section, we first investigate the controllability of the systems only with delay in state

$$x(k+1) = \sum_{i=0}^p A_i x(k-i) + Bu(k). \quad (2)$$

Now, we introduce a matrix sequence $\{G_k\}_{k=0}^\infty \subseteq \mathbb{R}^{n \times n}$ as follows:

$$G_k = \begin{cases} I & \text{if } k = 0 \\ \sum_{i=0}^{k-1} A_i G_{k-1-i} & \text{if } k = 1, \dots, p \\ \sum_{i=0}^p A_i G_{k-1-i} & \text{if } k = p+1, p+2, \dots \end{cases} \quad (3)$$

Lemma 3. The general solution of the system (2) is given by

$$x(k+1) = \Psi(k, x(-p), \dots, x(0)) + \sum_{i=0}^k G_{k-i} Bu(i), \quad k \in \mathbb{N}, \quad (4)$$

where $\Psi(k, x(-p), \dots, x(0))$ is the part of the solution with zero input.

Proof. See Appendix A. □

Lemma 4. The matrix sequence $\{G_k\}_{k=0}^\infty$ given by (3) satisfies

$$\text{span}\{G_0, \dots, G_k, \dots\} = \text{span}\{G_0, \dots, G_{n(p+1)-1}\}. \quad (5)$$

Proof. See Appendix B. □

Theorem 5. The system (2) is controllable if and only if $\text{rank}[G_0 B | \dots | G_{n(p+1)-1} B] = n$.

Proof. (Necessity) If the system is controllable, then we know that $\text{span}\{G_0, \dots, G_k, \dots\} = \mathbb{R}^n$. Thus, by Lemma 4, we have $\text{rank}[G_0 B | \dots | G_{n(p+1)-1} B] = n$.

(Sufficiency) By Lemma 3, we have

$$x(n(p+1)) = \Psi(n(p+1)-1, x(-p), \dots, x(0)) + \sum_{i=0}^{n(p+1)-1} G_{n(p+1)-1-i} Bu(i). \quad (6)$$

Since $\text{rank}[G_0 B | \dots | G_{n(p+1)-1} B] = n$, for any initial state $x(-p), \dots, x(0)$ and any terminal state $x(n(p+1))$, we can select appropriate inputs $u(0), \dots, u(n(p+1)-1)$ such that the equation $x(n(p+1)) = x_f$, where $x(n(p+1))$ is given by the above equation and x_f is arbitrary. Thus, the system is controllable. □

Corollary 6. $n(p+1)$ is a CRI of the system (2).

Proof. It is directly followed from Theorem 5. □

Remark 7. This work has improved the result in [12]. When the system is second order, that is, $n = 2$, we prove $2p+2$ to be a CRI value, which differs from the CRI value $2p+4$ in [12]. The difference lies in that the CRI of a system is not unique. For practical applications, obviously the less, the better.

3.2. Delays in Both State and Input. Now we investigate the controllability of the system (1). We only consider the case when $p = q$, for the case $p \neq q$, the discussion is similar (Without loss of generality, we assume that $p > q$, and let $B_j = 0, j = q+1, q+2, \dots, p$, then we come back to the $p = q$ case.).

Lemma 8. The solution of the system (1) can be expressed as

$$x(k+1) = \Phi(k, x(-p), \dots, x(0), u(-p), \dots, u(-1)) + \sum_{i=0}^k H_{k-i} u(i), \quad k \in \mathbb{N}, \quad (7)$$

where

$$H_k = \begin{cases} G_0 B_0 & \text{if } k = 0 \\ \sum_{i=0}^k G_{k-i} B_i & \text{if } k = 1, \dots, p-1 \\ \sum_{i=0}^p G_{k-i} B_i & \text{if } k = p, p+1, \dots; \end{cases} \quad (8)$$

and $\Phi(k, x(-p), \dots, x(0), u(-p), \dots, u(-1))$ is the part of the solution corresponding only to the initial state and initial input.

Proof. The proof is similar to that of Lemma 3. □

Lemma 9. The matrix sequence $\{H_k\}_{k=0}^\infty$ given by (8) satisfies

$$\text{span}\{H_0, \dots, H_k, \dots\} = \text{span}\{H_0, \dots, H_{n(p+1)+p-1}\}. \quad (9)$$

Proof. See Appendix C. □

Theorem 10. *The system (1) is controllable if and only if $\text{rank} [H_0 | \dots | H_{n(p+1)+p-1}] = n$.*

Proof. The proof is similar to that of Theorem 5. □

Corollary 11. *$n(p+1) + p$ is a CRI of the system (1).*

Proof. It is directly followed from Theorem 10. □

Remark 12. This corollary provides a complete and verifiable method to testify the controllability of a general discrete-time system with delay in state or in control or both. Approximately the computation work for each H_k is $O(pn^3)$, and the entire testing work takes $O(p^2n^4)$.

4. Example

In this section, we present a numeric example.

Example 13. Consider the system (1) with $p = q = 2, n = 3$ and

$$\begin{aligned} A_0 &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, & A_1 &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, & A_2 &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \\ B_0 &= \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, & B_1 &= \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, & B_2 &= \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}. \end{aligned} \tag{10}$$

By simple calculation, we get

$$\begin{aligned} G_0 &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, & G_1 &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, & G_2 &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \\ G_3 &= \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, & G_4 &= \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \\ H_0 &= \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, & H_1 &= \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, & H_2 &= \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \\ H_3 &= \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, & H_4 &= \begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix}. \end{aligned} \tag{11}$$

Thus, by Theorem 10, the system is controllable.

5. Conclusion

In this paper, we have investigated the controllability of discrete-time linear systems with time delays. Necessary and sufficient conditions have been established for discrete-time linear systems with state delay or both state and control delays. The proposed conditions are suitable for real verification and can be efficiently computed.

Appendices

A. Proof of Lemma 3

Proof. Now, we prove that (4) holds.

For $k = 0$, we have

$$\begin{aligned} x(1) &= \sum_{i=0}^p A_i x(-i) + Bu(0) \\ &= \Psi(0, x(-p), \dots, x(0)) + G_0 Bu(0). \end{aligned} \tag{A.1}$$

For $k = 1$, we have

$$\begin{aligned} x(2) &= \sum_{i=0}^p A_i x(1-i) + Bu(1) \\ &= \sum_{i=1}^p A_i x(1-i) + A_0 x(1) + Bu(1) \\ &= \sum_{i=1}^p A_i x(1-i) + A_0 \Psi(0, x(-p), \dots, x(0)) \\ &\quad + A_0 G_0 Bu(0) + Bu(1) \\ &= \Psi(1, x(-p), \dots, x(0)) + G_1 Bu(0) + G_0 Bu(1). \end{aligned} \tag{A.2}$$

For $k \leq p$, we have

$$\begin{aligned} x(k+1) &= \sum_{i=0}^p A_i x(k-i) + Bu(k) \\ &= \sum_{i=k}^p A_i x(k-i) + \sum_{i=0}^{k-1} A_i x(k-i) + Bu(k) \\ &= \sum_{i=k}^p A_i x(k-i) \\ &\quad + \sum_{i=0}^{k-1} A_i \left(\Psi(k-i-1, x(-p), \dots, x(0)) \right. \\ &\quad \left. + \sum_{j=0}^{k-i-1} G_{k-i-1-j} Bu(j) \right) + G_0 Bu(k) \\ &= \left(\sum_{i=k}^p A_i x(k-i) \right. \\ &\quad \left. + \sum_{i=0}^{k-1} A_i \Psi(k-i-1, x(-p), \dots, x(0)) \right) \\ &\quad + \sum_{i=0}^{k-1} A_i \left(\sum_{j=0}^{k-i-1} G_{k-i-1-j} Bu(j) \right) + G_0 Bu(k) \end{aligned}$$

$$\begin{aligned}
&= \Psi(k, x(-p), \dots, x(0)) + \sum_{j=0}^{k-1} \sum_{i=0}^{k-1-k-j-1} A_i G_{k-i-1-j} Bu(j) \\
&\quad + G_0 Bu(k) \\
&= \Psi(k, x(-p), \dots, x(0)) + \sum_{j=0}^{k-1} G_{k-j} Bu(j) + G_0 Bu(k).
\end{aligned} \tag{A.3}$$

For $k > p$, we have

$x(k+1)$

$$\begin{aligned}
&= \sum_{i=0}^p A_i x(k-i) + Bu(k) \\
&= \sum_{i=0}^p A_i \left(\Psi(k-i-1, x(-p), \dots, x(0)) \right. \\
&\quad \left. + \sum_{j=0}^{k-i-1} G_{k-i-1-j} Bu(j) \right) + G_0 Bu(k) \\
&= \left(\sum_{i=0}^p A_i \Psi(k-i-1, x(-p), \dots, x(0)) \right) \\
&\quad + \sum_{i=0}^p A_i \left(\sum_{j=0}^{k-i-1} G_{k-i-1-j} Bu(j) \right) + G_0 Bu(k) \\
&= \Psi(k, x(-p), \dots, x(0)) \\
&\quad + \sum_{j=k-p}^{k-1} \sum_{i=0}^{p-j} A_i G_{k-i-1-j} Bu(j) + G_0 Bu(k) \\
&\quad + \sum_{j=0}^{k-1-p} \sum_{i=0}^p A_i G_{k-i-1-j} Bu(j) + G_0 Bu(k) \\
&= \Psi(k, x(-p), \dots, x(0)) + \sum_{j=0}^{k-1} G_{k-j} Bu(j) + G_0 Bu(k).
\end{aligned} \tag{A.4}$$

□

B. Proof of Lemma 4

Proof. We introduce a new matrix sequence $\{W_k\}_{k=p+1}^{\infty} \in \mathbb{R}^{n(p+1) \times n}$ given by

$$W_k = \begin{bmatrix} G_k \\ G_{k-1} \\ \vdots \\ G_{k-p} \end{bmatrix}. \tag{B.1}$$

It is easy to verify that

$$W_{k+1} = \mathbb{A}W_k, \quad \forall k > p, \tag{B.2}$$

where

$$\mathbb{A} = \begin{bmatrix} A_0 & \cdots & A_{p-1} & A_p \\ I & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & I & 0 \end{bmatrix}. \tag{B.3}$$

By the well-known Hamilton-Caylay Theorem, for any $k \geq n(p+1) + p$, we have

$$W_k \in \text{span} \{W_p, \dots, W_{p+n(p+1)-1}\}. \tag{B.4}$$

It follows that

$$G_{k-p} \in \text{span} \{G_0, \dots, G_{n(p+1)-1}\}. \tag{B.5}$$

Hence, we have

$$\text{span} \{G_0, \dots, G_k, \dots\} = \text{span} \{G_0, \dots, G_{n(p+1)-1}\}. \tag{B.6}$$

□

C. Proof of Lemma 9

Proof. Consider the matrix sequence $\{W_k\}_{k=p+1}^{\infty} \in \mathbb{R}^{n(p+1) \times n}$ given by (B.1), it is easy to verify that

$$H_k = W_k^T \begin{bmatrix} B_0 \\ \vdots \\ B_p \end{bmatrix}, \quad \forall k \geq p. \tag{C.1}$$

By the proof of Lemma 4, we have

$$W_{k+1} = \mathbb{A}W_k, \quad \forall k \geq p. \tag{C.2}$$

This implies that

$$\text{span} \{H_p, \dots, H_k, \dots\} = \text{span} \{H_p, \dots, H_{n(p+1)+p-1}\}. \tag{C.3}$$

Hence, we have

$$\text{span} \{H_0, \dots, H_k, \dots\} = \text{span} \{H_0, \dots, H_{n(p+1)+p-1}\}. \tag{C.4}$$

□

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