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Research Article

A Liouville Type Result for Schrödinger Equation on Half-Spaces

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We consider a nonlinear Schrödinger equation with a singular potential on half spaces. Using a Hardy-type inequality and the moving plane method, we obtain a Liouville type result for its nonnegative solutions.

1. Introduction

Recently, properties of nontrivial solutions for nonlinear elliptic equations on half spaces have attracted a great deal of attention from physicians and mathematicians; see, for example, [1–5].

In this paper, we consider nonnegative solutions of the following Schrödinger equation with a singular potential on the half-space:

$$\begin{gathered} -\Delta u - \frac{\beta}{z^2}u - u^{2^*-1} = 0, \quad x \in H, \\ u = 0, \quad x \in \partial H, \end{gathered} \tag{1}$$

where $n \geq 3$, $2^* = 2n/(n-2)$, $\beta > 0$, and

$$H = \mathbb{R}^n_+ = \left\{x = \left(x', z\right) \mid x' \in \mathbb{R}^{n-1}, z > 0\right\}. \tag{2}$$

Equation (1) is related to the Grushin type equation with critical exponent and the Webster scalar curvature equation [6, 7].

We are interested in the Liouville type result for nonnegative solutions of (1). This work is motivated by some monotonicity results and Liouville type results for elliptic equations on half-spaces; see, for example, [2, 3]. In [2], Dancer found some sufficient conditions for nonlinear term $f(u)$ such that the positive bounded solution u of $-\Delta u = f(u)$ with Dirichlet boundary value condition is monotone increasing in z. Guo [3] considered nonnegative solutions for the elliptic system,

$$\begin{gathered} -\Delta u = f(v), \quad \text{in } \mathbb{R}^n_+, \\ -\Delta v = g(u), \quad \text{in } \mathbb{R}^n_+, \\ u = v = 0, \quad \text{on } \partial\mathbb{R}^n_+, \end{gathered} \tag{3}$$

and obtained some sufficient conditions for f and g, under which system (3) admits only trivial solution.

Let $\mathscr{D}^{1,2}_0(H)$ be the space given by the completion of $C^\infty_0(H)$ under the norm $\|u\| = (\int_H |\nabla u|^2 dx)^{1/2}$. We say that u is a weak solution of (1) if $u \in \mathscr{D}^{1,2}_0(H)$ satisfies

$$\int_H \nabla u \cdot \nabla \varphi \, dx = \int_H \frac{\beta}{z^2} u\varphi \, dx + \int_H u^{2^*-1}\varphi \, dx, \tag{4}$$

for all $\varphi \in C^\infty_c(H)$.

Using a Hardy-type inequality and the moving plane method in integral forms [8–10], we obtain the following Liouville type result.

Theorem 1. *Let $u \in \mathscr{D}^{1,2}_0(H)$ be a nonnegative weak solution of (1) with $0 < \beta < 1/16$. Then, $u \equiv 0$.*

Remark 2. For a weak solution $u \in \mathscr{D}^{1,2}_0(H)$, by using a regularity lifting method [8], we know that $u \in C^{2,\alpha}(\Omega)$, for all bounded smooth domain $\Omega \subset H$. Hence, it is a classical solution.

2. Preliminary

In this section, we prepare some lemmas.

Firstly, we recall the Hardy-Sobolev inequality in the half space; see [11–13].

Lemma 3. *Let* $u \in \mathscr{D}_0^{1,2}(\mathbb{R}_+^n)$*; then,*

$$\int_{\mathbb{R}_+^n} \frac{|u|^2}{z^2} dx \le 4 \int_{\mathbb{R}_+^n} |\nabla u|^2 dx. \quad (5)$$

This inequality plays a crucial role in estimating the singular potential term in the following proof.

In the following, we assume that $u \in \mathscr{D}_0^{1,2}(H)$ is a nonnegative weak solution of (1) with $0 < \beta < 1/16$. We are going to use the method of moving plane in the half-space.

For each $\lambda > 0$, let

$$\Sigma_\lambda = \left\{ \left(x', z\right) \mid x' \in \mathbb{R}^{n-1}, z \in (0, \lambda) \right\} = \mathbb{R}^{n-1} \times (0, \lambda). \quad (6)$$

For $x \in \Sigma_\lambda$, we write $x^\lambda = (x_1, \ldots, x_{n-1}, 2\lambda - z)$ which is the reflected point of x with respect to the hyperplane $T_\lambda = \{x = (x', z) \mid z = \lambda\}$ and define

$$u_\lambda(x) = u\left(x^\lambda\right), \qquad w_\lambda(x) = u_\lambda(x) - u(x). \quad (7)$$

Then, direct computation gives

$$\begin{aligned} -\Delta w_\lambda(x) &= -\Delta u_\lambda(x) + \Delta u(x) \\ &= \frac{\beta}{(2\lambda - z)^2} u_\lambda(x) + \left(u_\lambda(x)\right)^{2^*-1} \\ &\quad - \frac{\beta}{z^2} u(x) - (u(x))^{2^*-1} \\ &= \frac{\beta}{(2\lambda - z)^2} u_\lambda(x) - \frac{\beta}{z^2} u_\lambda(x) \\ &\quad + \frac{\beta}{z^2} u_\lambda(x) - \frac{\beta}{z^2} u(x) \\ &\quad + \left(u_\lambda(x)\right)^{2^*-1} - (u(x))^{2^*-1} \\ &= \frac{\beta}{z^2} w_\lambda(x) + \xi(x, \lambda) w_\lambda(x) \\ &\quad + \beta \left(\frac{1}{(2\lambda - z)^2} - \frac{1}{z^2} \right) u_\lambda(x); \end{aligned} \quad (8)$$

here $\xi(x, \lambda) = ((u_\lambda(x))^{2^*-1} - u(x)^{2^*-1})/(u_\lambda(x) - u(x))$.

For $x \in \Sigma_\lambda$, we have $2\lambda - z > z$, $1/(2\lambda - z)^2 < 1/z^2$, $u_\lambda(x) \ge 0$, and $\beta > 0$. Therefore,

$$-\Delta w_\lambda(x) \le \frac{\beta}{z^2} w_\lambda(x) + \xi(x, \lambda) w_\lambda(x). \quad (9)$$

Define $w_\lambda^+(x) = \max\{w_\lambda(x), 0\}$ and $w_\lambda^-(x) = -\min\{w_\lambda(x), 0\}$. Clearly, $w_\lambda^+(x) \ge 0$, $w_\lambda^-(x) \ge 0$ and $w_\lambda(x) = w_\lambda^+(x) - w_\lambda^-(x)$. Define

$$\Sigma_\lambda^- = \left\{ x \in \Sigma_\lambda \mid w_\lambda(x) < 0 \right\}. \quad (10)$$

The heart of our argument is the following lemma.

Lemma 4. *There exists a* $C_0 > 0$*, such that, for* $\lambda > 0$*, if* $\|w_\lambda^-\|_{L^{2^*}(\Sigma_\lambda)} > 0$*, then*

$$\|u(x)\|_{L^{2^*}(\Sigma_\lambda^-)} \ge C_0. \quad (11)$$

Proof. For $0 < \epsilon < \lambda/4$, let $\eta_\epsilon(z) \in C(\mathbb{R}^+)$ be defined by

$$\eta_\epsilon(z) = \begin{cases} 1, & z > \epsilon, \\ \dfrac{\log z - 2\log\epsilon}{-\log\epsilon}, & \epsilon^2 \le z \le \epsilon, \\ 0, & z < \epsilon^2. \end{cases} \quad (12)$$

Testing (9) in Σ_λ with function $\eta_\epsilon^2 w_\lambda^-$, we obtain

$$\begin{aligned} -\int_{\Sigma_\lambda} \Delta w_\lambda(x) \eta_\epsilon^2 w_\lambda^- dx &\le \beta \int_{\Sigma_\lambda} \frac{1}{z^2} w_\lambda(x) \eta_\epsilon^2 w_\lambda^- dx \\ &\quad + \int_{\Sigma_\lambda} \xi(x, \lambda) w_\lambda(x) \eta_\epsilon^2 w_\lambda^- dx. \end{aligned} \quad (13)$$

The left hand side of (13) is

$$\begin{aligned} &-\int_{\Sigma_\lambda} \Delta w_\lambda(x) \eta_\epsilon^2 w_\lambda^- dx \\ &\quad = \int_{\Sigma_\lambda^-} \left|\nabla w_\lambda^-\right|^2 \eta_\epsilon^2 dx + 2 \int_{\Sigma_\lambda^-} \eta_\epsilon w_\lambda^- \nabla w_\lambda(x) \cdot \nabla \eta_\epsilon dx. \end{aligned} \quad (14)$$

Hence, we derive

$$\int_{\Sigma_\lambda^-} \left|\nabla w_\lambda^-\right|^2 \eta_\epsilon^2 dx \le I + II + III, \quad (15)$$

where

$$\begin{aligned} I &= \beta \int_{\Sigma_\lambda^-} \frac{1}{z^2} \eta_\epsilon^2 \left(w_\lambda^-\right)^2 dx, \\ II &= \int_{\Sigma_\lambda^-} \xi(x, \lambda) \eta_\epsilon^2 \left(w_\lambda^-\right)^2 dx, \\ III &= -2 \int_{\Sigma_\lambda^-} \eta_\epsilon w_\lambda^- \nabla w_\lambda(x) \cdot \nabla \eta_\epsilon dx. \end{aligned} \quad (16)$$

Using Lemma 3, we have

$$\begin{aligned} I &= \beta \int_{\Sigma_\lambda^-} \frac{1}{z^2} \eta_\epsilon^2 \left(w_\lambda^-\right)^2 dx \\ &\le 4\beta \int_{\Sigma_\lambda^-} \left|\nabla \left(w_\lambda^- \eta_\epsilon\right)\right|^2 dx \\ &\le 8\beta \int_{\Sigma_\lambda^-} \left(\eta_\epsilon^2 \left|\nabla w_\lambda^-\right|^2 + \left(w_\lambda^-\right)^2 \left|\nabla \eta_\epsilon\right|^2 \right) dx. \end{aligned} \quad (17)$$

For $x \in \Sigma_\lambda^-$, $0 \le u_\lambda(x) < u(x)$, $0 < \xi(x, \lambda) < ((n+2)/(n-2)(u(x))^{4/(n-2)}$, which implies

$$\begin{aligned} II &= \int_{\Sigma_\lambda^-} \xi(x, \lambda) \eta_\epsilon^2 \left(w_\lambda^-\right)^2 dx \\ &\le \frac{n+2}{n-2} \int_{\Sigma_\lambda^-} (u(x))^{4/(n-2)} \eta_\epsilon^2 \left(w_\lambda^-\right)^2 dx. \end{aligned} \quad (18)$$

By using Hölder inequality, we verify that

$$II \le \frac{n+2}{n-2}\left(\int_{\Sigma_\lambda^-} u^{2^*}\eta_\epsilon^{n/2}dx\right)^{2/n} \cdot \left(\int_{\Sigma_\lambda^-} (w_\lambda^-)^{2^*}\eta_\epsilon^{n/(n-2)}dx\right)^{(n-2)/n} \le \frac{n+2}{n-2}\|u\|_{L^{2^*}(\Sigma_\lambda^-)}^{4/(n-2)}\|w_\lambda^-\|_{L^{2^*}(\Sigma_\lambda^-)}^2, \tag{19}$$

$$III = -2\int_{\Sigma_\lambda^-}\eta_\epsilon w_\lambda^-\nabla w_\lambda\cdot\nabla\eta_\epsilon dx \le 2\int_{\Sigma_\lambda^-}\eta_\epsilon w_\lambda^-\left|\nabla w_\lambda^-\right|\left|\nabla\eta_\epsilon\right|dx \le \frac{1}{4}\int_{\Sigma_\lambda^-}\eta_\epsilon^2\left|\nabla w_\lambda^-\right|^2dx + 4\int_{\Sigma_\lambda^-}\left|\nabla\eta_\epsilon\right|^2(w_\lambda^-)^2dx. \tag{20}$$

Putting (17), (19), and (20) into (15) and using the assumption $1<\beta<1/16$, we then deduce that

$$\int_{\Sigma_\lambda^-}\left|\nabla w_\lambda^-\right|^2\eta_\epsilon^2dx \le 9\int_{\Sigma_\lambda^-}\left|\nabla\eta_\epsilon\right|^2(w_\lambda^-)^2dx + 4\cdot\frac{n+2}{n-2}\|u\|_{L^{2^*}(\Sigma_\lambda^-)}^{4/(n-2)}\|w_\lambda^-\|_{L^{2^*}(\Sigma_\lambda^-)}^2. \tag{21}$$

Moreover, by the Sobolev inequality, we know that

$$\|w_\lambda^-\eta_\epsilon\|_{L^{2^*}(\Sigma_\lambda^-)}^2 \le C^2\|\nabla(w_\lambda^-\eta_\epsilon)\|_{L^2(\Sigma_\lambda^-)} \le C^2\int_{\Sigma_\lambda^-}\left|\eta_\epsilon\nabla w_\lambda^- + w_\lambda^-\nabla\eta_\epsilon\right|^2dx \le 2C^2\int_{\Sigma_\lambda^-}\left(\eta_\epsilon^2\left|\nabla w_\lambda^-\right|^2 + (w_\lambda^-)^2\left|\nabla\eta_\epsilon\right|^2\right)dx. \tag{22}$$

Combine the above inequality with (21) to get

$$\|w_\lambda^-\eta_\epsilon\|_{L^{2^*}(\Sigma_\lambda^-)}^2 \le 20C^2\int_{\Sigma_\lambda^-}\left|\nabla\eta_\epsilon\right|^2(w_\lambda^-)^2dx + 8C^2\frac{n+2}{n-2}\|u\|_{L^{2^*}(\Sigma_\lambda^-)}^{(n-2)/4}\|w_\lambda^-\|_{L^{2^*}(\Sigma_\lambda^-)}^2. \tag{23}$$

Now we claim that

$$\int_{\Sigma_\lambda^-}\left|\nabla\eta_\epsilon\right|^2(w_\lambda^-)^2dx \longrightarrow 0, \quad \text{as } \epsilon\longrightarrow 0. \tag{24}$$

Notice that, for $x\in\Sigma_\lambda^-$, $0<w_\lambda^-(x)=u(x)-u_\lambda(x)\le u(x)$. Hence,

$$0 \le \int_{\Sigma_\lambda^-}\left|\nabla\eta_\epsilon\right|^2(w_\lambda^-)^2dx \le \int_{\Sigma_\lambda^-}\left|\eta_\epsilon'(z)\right|^2u^2dx = \int_{\epsilon^2\le z\le\epsilon}\frac{(u(x))^2}{z^2(\log\epsilon)^2}dx \le \frac{1}{(\log\epsilon)^2}\int_H\frac{(u(x))^2}{z^2}dx \le 4\frac{1}{(\log\epsilon)^2}\int_H|\nabla u|^2dx. \tag{25}$$

Since $u\in\mathcal{D}_0^{1,2}(H)$, $\int_H|\nabla u|^2dx<+\infty$. Thus (24) is valid.

Now, letting $\epsilon \to 0$ in (23), by using dominated convergence theorem, we obtain

$$1 \le 8C^2\cdot\frac{n+2}{n-2}\|u\|_{L^{2^*}(\Sigma_\lambda^-)}^{4/(n-2)} \tag{26}$$

if $\|w_\lambda^-\|_{L^{2^*}(\Sigma_\lambda^-)}\ne 0$.

One can choose $C_0 = ((n-2)/8C^2(n+2))^{(n-2)/4}$, where C is the best constant in the Sobolev inequality. □

Using Lemma 4, we now can start the moving plane process as the following Lemma.

Lemma 5. *There is a $\lambda_0>0$, such that, for all $0<\lambda<\lambda_0$,*

$$w_\lambda(x)\ge 0, \quad \forall x\in\Sigma_\lambda. \tag{27}$$

Proof. Since $u\in\mathcal{D}_0^{1,2}(H)$, using Sobolev inequality, we have $u(x)\in L^{2^*}(H)$. Choose $\lambda_0>0$ small enough such that

$$\|u\|_{L^{2^*}(\Sigma_{\lambda_0})} < C_0, \tag{28}$$

where C_0 is the same as in Lemma 4.

Hence, for all $0<\lambda<\lambda_0$,

$$\|u\|_{L^{2^*}(\Sigma_\lambda^-)} \le \|u\|_{L^{2^*}(\Sigma_\lambda)} \le \|u\|_{L^{2^*}(\Sigma_{\lambda_0})} < C_0, \tag{29}$$

which is a contradiction to Lemma 4, if $\|w_\lambda^-\|_{L^{2^*}(\Sigma_\lambda^-)}\ne 0$. That is to say,

$$\|w_\lambda^-\|_{L^{2^*}(\Sigma_\lambda^-)} = 0, \tag{30}$$

which implies that $w_\lambda\ge 0$, for $x\in\Sigma_\lambda$. □

Now we move the hyperplane T_λ upwards by increasing the value of λ continuously as long as (27) holds. We will show that the hyperplane will be moved to the infinity. Precisely, define

$$\Lambda = \sup\left\{\lambda>0 \mid w_\mu(x)\ge 0, \forall x\in\Sigma_\mu, \forall 0<\mu\le\lambda\right\}. \tag{31}$$

By the result of Lemma 5, $\Lambda\ge\lambda_0>0$.

Lemma 6. *We have* $\Lambda = +\infty$.

Proof. Suppose $\Lambda < +\infty$.

On one hand, by continuity we know that $w_\Lambda(x) \geq 0$, for all $x \in \Sigma_\Lambda$, which means

$$\Sigma_\Lambda^- = \emptyset. \tag{32}$$

On the other hand, by the definition of Λ, there is $\{\delta_i\}_{i=1}^{\infty}$ that satisfy (i) $\delta_i \to 0$, as $i \to \infty$, and (ii) $\|w_{\Lambda+\delta_i}^-(x)\|_{L^{2^*}(\Sigma_{\Lambda+\delta_i})} > 0$, for all i. By Lemma 4, we get $\|u(x)\|_{L^{2^*}(\Sigma_{\Lambda+\delta_i}^-)} \geq C_0 > 0$. By using the dominated convergence theorem, we obtain

$$\|u(x)\|_{L^{2^*}(\Sigma_\Lambda^-)} \geq C_0 > 0, \tag{33}$$

which is a contradiction to (32). □

3. Proof of Theorem 1

In this section, we prove Theorem 1.

Since u is a superharmonic continuous function in H (see Remark 2), we have either $u \equiv 0$ in H or $u > 0$ in H.

If $u > 0$ in H, then there is some $(x_0', z_0) \in H$ satisfying $u(x_0', z_0) = c > 0$. Moreover, by continuity, there is a $\delta > 0$, such that $u(x', z_0) > a/2$, for all $|x' - x_0'| < \delta$. By using Lemma 6, we know that $u(x)$ is increasing with respect to z in H. Thus, $u(x', z) \geq a/2$ for all $|x' - x_0'| < \delta$ and $z \geq z_0$. Hence,

$$\begin{aligned} &\int_{z_0}^{+\infty} \int_{|x'-x_0'|<\delta} \left|u\left(x', z\right)\right|^{2^*} dx' dz \\ &\quad \geq \int_{z_0}^{+\infty} \int_{|x'-x_0'|<\delta} \left(\frac{a}{2}\right)^{2^*} dx' dz = +\infty, \end{aligned} \tag{34}$$

which contradicts the fact that $u \in L^{2^*}(H)$.

Therefore, $u \equiv 0$ in H.

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