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*Research Article*

# A Note on Hermite-Hadamard Inequalities for Products of Convex Functions

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We obtain some new Hermite-Hadamard type inequalities for products of convex functions. We conclude that the results obtained in this work are the refinements of the present results.

## 1. Introduction

If $f : I \to R$ is a convex function on the interval $I$, then, for any $a, b \in I$ with $a \neq b$, we have the following double inequality:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(t)\,dt \leq \frac{f(a)+f(b)}{2}. \tag{1}$$

This remarkable result is well known in the literature as the Hermite-Hadamard inequality.

Since then, some refinements of the Hermite-Hadamard inequality on convex functions have been extensively investigated by a number of authors (e.g., [1–7]).

In [8], Pachpatte established two new Hermite-Hadamard type inequalities for products of convex functions as follows.

**Theorem 1.** *Let $f$ and $g$ be real-valued, nonnegative, and convex functions on $[a, b]$. Then,*

$$\frac{1}{b-a}\int_a^b f(x)\,g(x)\,dx \leq \frac{1}{3}M(a,b) + \frac{1}{6}N(a,b), \tag{2}$$

$$\begin{aligned} 2f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,g(x)\,dx \\ + \frac{1}{6}M(a,b) + \frac{1}{3}N(a,b), \end{aligned} \tag{3}$$

*where $M(a,b) = f(a)g(a) + f(b)g(b)$ and $N(a,b) = f(a)g(b) + f(b)g(a)$.*

Some new integral inequalities involving two nonnegative and integrable functions that are related to the Hermite-Hadamard type are also established by many authors. In [9], Pachpatte established some Hermite-Hadamard type inequalities involving two log-convex functions. An analogous result for $s$-convex functions is obtained by Kirmaci et al. in [10]. In [11], Sarikaya presented some integral inequalities for two $h$-convex functions. Some Hermite-Hadamard type inequalities for two operator convex functions are given by Bacak and Türkmen in [12]. For recent results and generalizations concerning Hermite-Hadamard type inequality for product of two functions, see [13] and the references given therein.

In this paper, we obtain new generalizations of Hermite-Hadamard type inequalities for products of convex functions. This result refines the Hermite-Hadamard type inequalities given in Theorem 1.

## 2. Lemma

**Lemma 2.** *Let $f$ and $g$ be real-valued, nonnegative, and convex functions on $[a, b]$. If $[c, d] \subset [a, b]$, then*

$$\begin{aligned} \frac{1}{d-c}\int_c^d f(x)\,g(x)\,dx &\leq \frac{1}{3}M(c,d) + \frac{1}{6}N(c,d), \\ 2f\left(\frac{c+d}{2}\right)g\left(\frac{c+d}{2}\right) &\leq \frac{1}{d-c}\int_c^d f(x)\,g(x)\,dx \\ &\quad + \frac{1}{6}M(c,d) + \frac{1}{3}N(c,d), \end{aligned} \tag{4}$$

*where* $M(c,d) = f(c)g(c) + f(d)g(d)$ *and* $N(c,d) = f(c)g(d) + f(d)g(c)$.

*Proof.* We get the conclusions by $f$ and $g$ are nonnegative and convex functions on $[c,d]$ and Theorem 1. □

## 3. Main Results

**Theorem 3.** *Let $f$ and $g$ be real-valued, nonnegative, and convex functions on $[a,b]$. Then,*

$$\frac{1}{b-a}\int_a^b f(x)\,g(x)\,dx \le L(t) \le \frac{1}{3}M(a,b) + \frac{1}{6}N(a,b), \tag{5}$$

*where* $M(a,b) = f(a)g(a) + f(b)g(b)$, $N(a,b) = f(a)g(b) + f(b)g(a)$, $t \in [0,1]$, *and*

$$\begin{aligned} L(t) = {} & \frac{t}{3}M(a,(1-t)a+tb) + \frac{t}{6}N(a,(1-t)a+tb) \\ & + \frac{1-t}{3}M((1-t)a+tb,b) \\ & + \frac{1-t}{6}N((1-t)a+tb,b). \end{aligned} \tag{6}$$

*Proof.* Since $f$ and $g$ are convex on $[a,b]$ and nonnegative, by (2), we get

$$\frac{1}{b-a}\int_a^b f(x)\,g(x)\,dx \le \frac{1}{3}M(a,b) + \frac{1}{6}N(a,b). \tag{7}$$

Firstly, $f$ and $g$ are convex on $[a,(1-t)a+tb]$ and nonnegative; applying (2) on $[a,(1-t)a+tb]$, we get

$$\begin{aligned} & \frac{1}{t(b-a)}\int_a^{(1-t)a+tb} f(x)\,g(x)\,dx \\ & \quad \le \frac{1}{3}M(a,(1-t)a+tb) \\ & \quad\quad + \frac{1}{6}N(a,(1-t)a+tb). \end{aligned} \tag{8}$$

Similarly, we can show that

$$\begin{aligned} & \frac{1}{(1-t)(b-a)}\int_{(1-t)a+tb}^{b} f(x)\,g(x)\,dx \\ & \quad \le \frac{1}{3}M((1-t)a+tb,b) \\ & \quad\quad + \frac{1}{6}N((1-t)a+tb,b). \end{aligned} \tag{9}$$

Multiplying (8) by $t$ and (9) by $(1-t)$ and adding the resulting inequalities, we get

$$\begin{aligned} & \frac{1}{b-a}\int_a^b f(x)\,g(x)\,dx \\ & \quad \le \frac{t}{3}M(a,(1-t)a+tb) \\ & \quad\quad + \frac{t}{6}N(a,(1-t)a+tb) \\ & \quad\quad + \frac{1-t}{3}M((1-t)a+tb,b) \\ & \quad\quad + \frac{1-t}{6}N((1-t)a+tb,b) = L(t). \end{aligned} \tag{10}$$

Using the fact that

$$\begin{aligned} & \frac{t}{3}M(a,(1-t)a+tb) + \frac{1-t}{3}M((1-t)a+tb,b) \\ & \quad = \frac{t}{3}\left[f(a)\,g(a) + f((1-t)a+tb)\right. \\ & \quad\quad \left.\times g((1-t)a+tb)\right] \\ & \quad\quad + \frac{1-t}{3}\left[f((1-t)a+tb)\,g((1-t)a+tb)\right. \\ & \quad\quad \left.+ f(b)\,g(b)\right] \\ & \quad = \frac{t}{3}f(a)\,g(a) + \frac{1-t}{3}f(b)\,g(b) \\ & \quad\quad + \frac{1}{3}f((1-t)a+tb)\,g((1-t)a+tb) \\ & \quad \le \frac{t}{3}f(a)\,g(a) + \frac{1-t}{3}f(b)\,g(b) \\ & \quad\quad + \frac{1}{3}\left[(1-t)f(a)+tf(b)\right] \\ & \quad\quad \times\left[(1-t)g(a)+tg(b)\right] \\ & \quad = \frac{t+(1-t)^2}{3}f(a)\,g(a) + \frac{1-t+t^2}{3}f(b)\,g(b) \\ & \quad\quad + \frac{(1-t)t}{3}N(a,b) \\ & \quad = \frac{1-t+t^2}{3}M(a,b) + \frac{(1-t)t}{3}N(a,b), \\ & \frac{t}{6}N(a,(1-t)a+tb) + \frac{1-t}{6}N((1-t)a+tb,b) \\ & \quad = \frac{t}{6}\left[f(a)\,g((1-t)a+tb) + f((1-t)a+tb)\,g(a)\right] \\ & \quad\quad + \frac{1-t}{6}\left[f((1-t)a+tb)\,g(b)\right. \\ & \quad\quad \left.+ f(b)\,g((1-t)a+tb)\right] \end{aligned}$$

$$
\begin{aligned}
&\leq \frac{t}{6}\left[f(a)\left((1-t)g(a)+tg(b)\right)\right.\\
&\qquad \left.+\left((1-t)f(a)+tf(b)\right)g(a)\right]\\
&\quad+\frac{1-t}{6}\left[\left((1-t)f(a)+tf(b)\right)g(b)\right.\\
&\qquad\quad \left.+f(b)\left((1-t)g(a)+tg(b)\right)\right]\\
&=\frac{t}{6}\left[2(1-t)f(a)g(a)+tN(a,b)\right]\\
&\quad+\frac{1-t}{6}\left[2tf(b)g(b)+(1-t)N(a,b)\right]\\
&=\frac{t(1-t)}{3}M(a,b)+\frac{(1-t)^2+t^2}{6}N(a,b),
\end{aligned}
\tag{11}
$$

we have

$$
\begin{aligned}
L(t) &\leq \frac{1-t+t^2}{3}M(a,b)+\frac{(1-t)t}{3}N(a,b)\\
&\quad+\frac{t(1-t)}{3}M(a,b)+\frac{(1-t)^2+t^2}{6}N(a,b)\\
&=\frac{1}{3}M(a,b)+\frac{1}{6}N(a,b),
\end{aligned}
\tag{12}
$$

which completes the proof. □

*Remark 4.* Applying Theorem 3 for $L(0) = L(1) = (1/3)M(a,b) + (1/6)N(a,b)$, we get the result in Theorem 1. So, we conclude that our results give an improvement of Theorem 1.

**Corollary 5.** *With notations above, one has the following inequality:*

$$
\begin{aligned}
\frac{1}{b-a}\int_a^b f(x)g(x)\,dx &\leq \inf_{0\leq t\leq 1} L(t) \leq \sup_{0\leq t\leq 1} L(t)\\
&\leq \frac{1}{3}M(a,b)+\frac{1}{6}N(a,b),
\end{aligned}
\tag{13}
$$

*where $L(t)$ is defined in Theorem 3.*

**Theorem 6.** *Let $f$ and $g$ be real-valued, nonnegative, and convex functions on $[a,b]$. Then,*

$$
\begin{aligned}
2f&\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right)\\
&\leq \frac{1}{b-a}\int_a^b f(x)g(x)\,dx\\
&\quad+\frac{1}{12}\left[N\left(a,\frac{a+b}{2}\right)+N\left(\frac{a+b}{2},a\right)\right]\\
&\quad+\frac{1}{6}\left[N\left(\frac{a+b}{2},\frac{a+b}{2}\right)+N(a,b)\right]\\
&\leq \frac{1}{b-a}\int_a^b f(x)g(x)\,dx\\
&\quad+\frac{1}{6}M(a,b)+\frac{1}{3}N(a,b),
\end{aligned}
\tag{14}
$$

*where $M(a,b) = f(a)g(a) + f(b)g(b)$ and $N(a,b) = f(a)g(b) + f(a)g(b)$.*

*Proof.* Since $f$ and $g$ are convex on $[a,b]$ and nonnegative, for $t \in [0,1]$, we observe that

$$
\begin{aligned}
2f&\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right)\\
&=2f\left(\frac{t((a+b)/2)+(1-t)b}{2}\right.\\
&\qquad\left.+\frac{(1-t)a+t((a+b)/2)}{2}\right)\\
&\quad\times g\left(\frac{t((a+b)/2)+(1-t)b}{2}\right.\\
&\qquad\left.+\frac{(1-t)a+t((a+b)/2)}{2}\right)\\
&\leq \frac{1}{2}\left[f\left(t\frac{a+b}{2}+(1-t)b\right)+f\left((1-t)a+t\frac{a+b}{2}\right)\right]\\
&\quad\times\left[g\left(t\frac{a+b}{2}+(1-t)b\right)+g\left((1-t)a+t\frac{a+b}{2}\right)\right]\\
&\leq \frac{1}{2}\left[f\left(t\frac{a+b}{2}+(1-t)b\right)g\left(t\frac{a+b}{2}+(1-t)b\right)\right.\\
&\qquad\left.+f\left((1-t)a+t\frac{a+b}{2}\right)g\left((1-t)a+t\frac{a+b}{2}\right)\right]\\
&\quad+\frac{1}{2}\left[\left(tf\left(\frac{a+b}{2}\right)+(1-t)f(b)\right)\right.\\
&\qquad\times\left((1-t)g(a)+tg\left(\frac{a+b}{2}\right)\right)\\
&\qquad+\left((1-t)f(a)+tf\left(\frac{a+b}{2}\right)\right)\\
&\qquad\left.\times\left(tg\left(\frac{a+b}{2}\right)+(1-t)g(b)\right)\right]\\
&=\frac{1}{2}\left[f\left(t\frac{a+b}{2}+(1-t)b\right)g\left(t\frac{a+b}{2}+(1-t)b\right)\right.\\
&\qquad\left.+f\left((1-t)a+t\frac{a+b}{2}\right)g\left((1-t)a+t\frac{a+b}{2}\right)\right]\\
&\quad+\frac{1}{2}\left[t(1-t)f\left(\frac{a+b}{2}\right)g(a)\right.\\
&\qquad+t^2f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right)\\
&\qquad+(1-t)^2f(b)g(a)+(1-t)tf(b)g\left(\frac{a+b}{2}\right)
\end{aligned}
$$

$$
\begin{aligned}
&+ (1-t)^2 f(a) g(b) + (1-t) t f(a) g\left(\frac{a+b}{2}\right) \\
&+ t(1-t) f\left(\frac{a+b}{2}\right) g(b) \\
&+ t^2 f\left(\frac{a+b}{2}\right) g\left(\frac{a+b}{2}\right)\Bigg] \\
= &\frac{1}{2}\left[f\left(t\frac{a+b}{2}+(1-t)b\right) g\left(t\frac{a+b}{2}+(1-t)b\right)\right. \\
&\left.+ f\left((1-t)a+t\frac{a+b}{2}\right) g\left((1-t)a+t\frac{a+b}{2}\right)\right] \\
&+ \frac{1}{2}\left[t(1-t)N\left(a,\frac{a+b}{2}\right) + t^2 N\left(\frac{a+b}{2},\frac{a+b}{2}\right)\right. \\
&\left.+ (1-t)^2 N(a,b) + t(1-t) N\left(\frac{a+b}{2},b\right)\right].
\end{aligned} \tag{15}
$$

Integrating both sides of the previous inequality with respect to $t$ over $[0,1]$, we obtain

$$
\begin{aligned}
&2f\left(\frac{a+b}{2}\right) g\left(\frac{a+b}{2}\right) \\
&\le \frac{1}{2}\left[\int_0^1 f\left(t\frac{a+b}{2}+(1-t)b\right)\right. \\
&\qquad \times g\left(t\frac{a+b}{2}+(1-t)b\right) dt \\
&\qquad + \int_0^1 f\left((1-t)a+t\frac{a+b}{2}\right) \\
&\qquad \left.\times g\left((1-t)a+t\frac{a+b}{2}\right) dt\right] \\
&\quad + \frac{1}{2}\left[\frac{1}{6}N\left(a,\frac{a+b}{2}\right) + \frac{1}{3}N\left(\frac{a+b}{2},\frac{a+b}{2}\right)\right. \\
&\qquad \left.+ \frac{1}{3}N(a,b) + \frac{1}{6}N\left(\frac{a+b}{2},b\right)\right] \\
&= \frac{1}{2}\left[\frac{2}{b-a}\int_{(a+b)/2}^{b} f(x) g(x)\, dx\right. \\
&\qquad \left.+ \frac{2}{b-a}\int_a^{(a+b)/2} f(x) g(x)\, dx\right] \\
&\quad + \frac{1}{2}\left[\frac{1}{6}N\left(a,\frac{a+b}{2}\right) + \frac{1}{3}N\left(\frac{a+b}{2},\frac{a+b}{2}\right)\right. \\
&\qquad \left.+ \frac{1}{3}N(a,b) + \frac{1}{6}N\left(\frac{a+b}{2},b\right)\right] \\
&= \frac{1}{b-a}\int_a^b f(x) g(x)\, dx \\
&\quad + \frac{1}{12}N\left(a,\frac{a+b}{2}\right) + \frac{1}{6}N\left(\frac{a+b}{2},\frac{a+b}{2}\right) \\
&\quad + \frac{1}{6}N(a,b) + \frac{1}{12}N\left(\frac{a+b}{2},b\right).
\end{aligned} \tag{16}
$$

On the other hand, we have

$$
\begin{aligned}
&N\left(a,\frac{a+b}{2}\right) \\
&\quad = f(a) g\left(\frac{a+b}{2}\right) + f\left(\frac{a+b}{2}\right) g(a) \\
&\quad \le \frac{1}{2}\left[f(a)(g(a)+g(b)) + (f(a)+f(b)) g(a)\right] \\
&\quad = f(a) g(a) + \frac{1}{2}N(a,b).
\end{aligned} \tag{17}
$$

By a similar way, we obtain

$$
\begin{aligned}
&N\left(\frac{a+b}{2},b\right) \\
&\quad = f(b) g\left(\frac{a+b}{2}\right) + f\left(\frac{a+b}{2}\right) g(b) \\
&\quad \le \frac{1}{2}\left[f(b)(g(a)+g(b)) + (f(a)+f(b)) g(b)\right] \\
&\quad = f(b) g(b) + \frac{1}{2}N(a,b),
\end{aligned} \tag{18}
$$

$$
\begin{aligned}
N\left(\frac{a+b}{2},\frac{a+b}{2}\right) &= 2f\left(\frac{a+b}{2}\right) g\left(\frac{a+b}{2}\right) \\
&\le \frac{1}{2}\left[(g(a)+g(b))(f(a)+f(b))\right] \\
&= \frac{1}{2}M(a,b) + \frac{1}{2}N(a,b).
\end{aligned}
$$

So,

$$
\begin{aligned}
&\frac{1}{12}N\left(a,\frac{a+b}{2}\right) + \frac{1}{6}N\left(\frac{a+b}{2},\frac{a+b}{2}\right) \\
&\qquad + \frac{1}{6}N(a,b) + \frac{1}{12}N\left(\frac{a+b}{2},b\right) \\
&\quad \le \frac{1}{12}\left[f(a) g(a) + \frac{1}{2}N(a,b)\right. \\
&\qquad \left.+ f(b) g(b) + \frac{1}{2}N(a,b)\right] \\
&\qquad + \frac{1}{6}\left[\frac{1}{2}M(a,b) + \frac{1}{2}N(a,b)\right] + \frac{1}{6}N(a,b) \\
&\quad = \frac{1}{6}M(a,b) + \frac{1}{3}N(a,b),
\end{aligned} \tag{19}
$$

which yields the desired result. □

## 4. Conclusion

In this paper, we establish some new Hermite-Hadamard type inequalities for products of convex functions. The results obtained in this work are the refinements of the present results. An interesting topic is whether we can use the methods in this paper to refine the Hermite-Hadamard inequality for coordinated convex functions.

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