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Research Article

Infinite-Dimensional Modular Lie Superalgebra Ω

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All ad-nilpotent elements of the infinite-dimensional Lie superalgebra Ω over a field of positive characteristic are determined. The natural filtration of the Lie superalgebra Ω is proved to be invariant under automorphisms by characterizing ad-nilpotent elements. Then an intrinsic property is obtained by the invariance of the filtration; that is, the integers in the definition of Ω are intrinsic. Therefore, we classify the infinite-dimensional modular Lie superalgebra Ω in the sense of isomorphism.

1. Introduction

The theory of modular Lie superalgebras has obtained many important results during the last twenty years (e.g., see [1–4]). But the complete classification of the simple modular Lie superalgebras remains an open problem. We know that filtration structures play an important role both in the classification of modular Lie algebras and nonmodular Lie superalgebras (see [5–8]). The natural filtrations of finite-dimensional modular Lie algebras of Cartan type were proved to be invariant in [9, 10]. The similar result was obtained for the infinite-dimensional case [11]. In the case of finite-dimensional modular Lie superalgebras of Cartan type, the invariance of the natural filtration was discussed in [12, 13]. The same conclusion was obtained for some infinite-dimensional modular Lie superalgebras of Cartan type (see [14–17]).

In the present paper, we consider the infinite-dimensional modular Lie superalgebra $\Omega(r,m,q)$, which was studied in paper [18]. Denote the natural filtration by $(\Omega(r,m,q)_{[i]})_{i\geq -2}$. We show that the filtration is invariant under automorphisms by determining ad-nilpotent elements and subalgebras generated by certain ad-nilpotent elements. We are thereby able to obtain an intrinsic characterization of Lie superalgebra $\Omega(r,m,q)$.

The paper is organized as follows. In Section 2, we recall some necessary definitions concerning the modular Lie superalgebra Ω. In Section 3, we establish some technical lemmas which will be used to determine the invariance of the filtration. In Section 4, we prove that the natural filtration $(\Omega(r,m,q)_{[i]})_{i\geq -2}$ is invariant. Furthermore, we obtain the sufficient and necessary conditions of $\Omega(r,m,q)\cong\Omega(r',m',q')$; that is, all the Lie superalgebras are classified up to isomorphisms.

2. Preliminaries

Throughout the work $\mathbb{F}$ denotes an algebraically closed field of characteristic $p > 3$ and $\mathbb{F}$ is not equal to its prime field Π. Let $\mathbb{Z}_2 = \{\overline{0},\overline{1}\}$ be the ring of integers module 2. Let $\mathbb{N}$ and $\mathbb{N}_0$ denote the sets of positive integers and non-negative integers, respectively. For $m > 0$, let $\mathbb{E} = \{z_1,\ldots,z_m\}\in\mathbb{F}$ be a subset of $\mathbb{F}$ that is linearly independent over the prime field Π, and let H be the additive subgroup generated by $\mathbb{E}$ that does not contain 1. If $\lambda\in H$, then we let $\lambda=\sum_{i=1}^{m}\lambda_i z_i$ and $y^\lambda = y_1^{\lambda_1}\cdots y_m^{\lambda_m}$, where $0\leq\lambda_i<p$.

Given $n\in\mathbb{N}$ and $r=2n+2$, we put $M=\{1,\ldots,r-1\}$. Let $\mu_1,\ldots,\mu_{r-1}\in\mathbb{F}$ and $\mu_1=0$, $\mu_j+\mu_{n+j}=1$, $j=2,\ldots,n+1$. If $k_i\in\mathbb{N}_0$, then k_i can be uniquely expressed in p-adic form $k_i=\sum_{\nu=0}^{\infty}\varepsilon_\nu(k_i)p^\nu$, where $0\leq\varepsilon_\nu(k_i)<p$. We set $x_i^{k_i}=\prod_{\nu=0}^{\infty}x_{i\nu}^{\varepsilon_\nu(k_i)}$. We define a truncated polynomial algebra

$$A=\mathbb{F}\left[x_{10},x_{11},\ldots,x_{20},x_{21}\ldots,x_{r-1,0},x_{r-1,1},\ldots,y_1,\ldots,y_m\right] \quad (1)$$

such that

$$x_{ij}^{p} = 0, \quad \forall i \in M, \quad j = 0, 1, \ldots,$$
$$y_i^p = 1, \quad i = 1, \ldots, m. \tag{2}$$

For $k_i, k_i' \in \mathbb{N}_0$, it is easy to see that

$$x_i^{k_i} x_i^{k_i'} = x_i^{k_i + k_i'} \neq 0 \Longleftrightarrow \varepsilon_\nu (k_i) + \varepsilon_\nu (k_i') < p, \quad \nu = 0, 1, \ldots. \tag{3}$$

Let $Q = \{(k_1, \ldots, k_{r-1}) \mid k_i \in \mathbb{N}_0, i \in M\}$. If $k = (k_1, \ldots, k_{r-1}) \in Q$, we set $x^k = x_1^{k_1} \cdots x_{r-1}^{k_{r-1}}$.

Let $\Lambda(q)$ be the Grassmann superalgebra over $\mathbb{F}$ in q variables $\xi_{r+1}, \ldots, \xi_{r+q}$, where $q \in \mathbb{N}$ and $q > 1$. Denote by Ω the tensor product $A \bigotimes_{\mathbb{F}} \Lambda(q)$. The trivial $\mathbb{Z}_2$-gradation of A and the natural $\mathbb{Z}_2$-gradation of $\Lambda(q)$ induce a $\mathbb{Z}_2$-gradation of Ω such that Ω is an associative superalgebra:

$$\Omega_{\bar{0}} = A \bigotimes_{\mathbb{F}} \Lambda(q)_{\bar{0}}, \qquad \Omega_{\bar{1}} = A \bigotimes_{\mathbb{F}} \Lambda(q)_{\bar{1}}. \tag{4}$$

For $f \in A$ and $g \in \Lambda(q)$, we abbreviate $f \otimes g$ to fg. Let

$$\mathbb{B}_k = \{\langle i_1, i_2, \ldots, i_k \rangle \mid r+1 \le i_1 < i_2 < \cdots < i_k \le r+q\},$$
$$\mathbb{B}(q) = \bigcup_{k=0}^{q} \mathbb{B}_k, \tag{5}$$

and where $\mathbb{B}_0 = \varnothing$. Given $u = \langle i_1, \ldots, i_k \rangle \in \mathbb{B}_k$, we set $|u| = k, \{u\} = \{i_1, \ldots, i_k\}$ and $\xi^u = \xi_{i_1} \cdots \xi_{i_k} (|\varnothing| = 0, \xi^{\varnothing} = 1)$. Then $\{x^k y^\lambda \xi^u \mid k \in Q, \lambda \in H, u \in \mathbb{B}(q)\}$ is an $\mathbb{F}$-basis of Ω.

If $|f|$ appears in some expression in this paper, we always regard x as a $\mathbb{Z}_2$-homogeneous element and $|f|$ as the $\mathbb{Z}_2$-degree of f.

Let $s = r + q$, $T = \{r+1, \ldots, s\}$ and $R = M \cup T$. Put $M_1 = \{2, \ldots, r-1\}$. Set $e_i = (\delta_{i1}, \ldots, \delta_{ir-1})$ and $x_i = x_i^1 = x_{i0}$ for $i \in M$. Define $\tilde{i} = \bar{0}$, if $i \in M_1$, and $\tilde{i} = \bar{1}$, if $i \in T$. Let

$$i' = \begin{cases} i+n, & 2 \le i \le n+1, \\ i-n, & n+2 \le i \le r-1, \\ i, & r+1 \le i \le s, \end{cases}$$
$$[i] = \begin{cases} 1, & 2 \le i \le n+1, \\ -1, & n+2 \le i \le r-1, \\ 1, & r+1 \le i \le s. \end{cases} \tag{6}$$

Let $D_1, D_2, \ldots, D_s$ be the linear transformations of Ω such that

$$D_i \left(x^k y^\lambda \xi^u \right) = \begin{cases} k_i^* x^{k-e_i} y^\lambda \xi^u, & i \in M, \\ \dfrac{x^k y^\lambda \cdot \partial \xi^u}{\partial \xi_i}, & i \in T, \end{cases} \tag{7}$$

where k_i^* is the first nonzero number of $\varepsilon_0(k_i), \varepsilon_1(k_i), \ldots, \varepsilon_{s_i}(k_i)$. Then $D_1, D_2, \ldots, D_s$ are superderivations of the superalgebra Ω and $|D_i| = \tilde{i}$. Set

$$\bar{\partial} = I - \sum_{j \in M_1} \mu_j x_{j0} \frac{\partial}{\partial x_{j0}} - \sum_{j=1}^{m} z_j y_j \frac{\partial}{\partial y_j} - 2^{-1} \sum_{j \in T} \xi_j \frac{\partial}{\partial \xi_j}, \tag{8}$$

where I is the identity mapping of Ω. Let $f \in \Omega$ be a $\mathbb{Z}_2$-homogeneous element and $g \in \Omega$; we define a bilinear operation in Ω such that

$$[f, g] = D_1 (f) \bar{\partial} (g) - \bar{\partial} (f) D_1 (g) + \sum_{i \in M_1 \cup T} [i] (-1)^{\tilde{i}|f|} D_i (f) D_{i'} (g). \tag{9}$$

Then Ω becomes a simple Lie superalgebra. If $2n + 4 - q \not\equiv 0 \pmod{p}$, we see that $\lambda + 2^{-1} q - n - 2 \neq 0$. In the sequel, we always assume that $2n + 4 - q \not\equiv 0 \pmod{p}$. In some cases, we denote Ω by $\Omega(r, m, q)$ in detail and call $\Omega(r, m, q)$ the Lie superalgebra of Ω-type.

Now we give a $\mathbb{Z}$-gradation of Ω: $\Omega = \bigoplus_{j \in X} \Omega_j$, where

$$\Omega_j = \operatorname{span}_{\mathbb{F}} \left\{ x^k y^\lambda \xi^u \mid \sum_{i \in M_1} k_i + 2k_1 + |u| - 2 = j \right\}. \tag{10}$$

Let $\Omega_{[i]} = \bigoplus_{j \ge i} \Omega_j$ for all $i \ge -2$. Then $\Omega = \Omega_{[-2]} \supset \Omega_{[-1]} \supset \cdots$ are called the natural filtration of Ω.

3. Ad-Nilpotent Elements

Let L be a Lie superalgebra. Recall that an element $y \in L$ is called ad_L-nilpotent if there exists a $t \in \mathbb{N}$ such that $(\mathrm{ad} y)^t (L) = 0$. If $y \in L$ is ad_L-nilpotent, it is also called ad-nilpotent in brief. Let G be a subset of L. Put $\mathrm{nil}(G) = \{x \in G \mid x \text{ is } \mathrm{ad}_L\text{-nilpotent}\}$, and $\mathrm{Nil}(G)$ is the subalgebra of L generated by $\mathrm{nil}(G)$.

Let $a \in \mathbb{N}_0$ and $a = \sum_{\nu=0}^{\infty} \varepsilon_\nu(a) p^\nu$ be the p-adic expression of a, where $0 \le \varepsilon_\nu(a) < p$. Then

$$\mathrm{pad}\,(a) = \left(\mathrm{pad}_0\,(a), \mathrm{pad}_1\,(a), \mathrm{pad}_2\,(a), \ldots\right) \tag{11}$$

is called the p-adic sequence of a, where $\mathrm{pad}_\nu(a) = \varepsilon_\nu(a)$ for all $\nu \in \mathbb{N}_0$. For $k = (k_1, k_2, \ldots, k_{r-1}) \in Q$, we define the p-adic matrix of k to be

$$\mathrm{pad}\,(k) = \begin{pmatrix} \mathrm{pad}\,(k_1) \\ \mathrm{pad}\,(k_2) \\ \vdots \\ \mathrm{pad}\,(k_{r-1}) \end{pmatrix}. \tag{12}$$

Since $\mathrm{pad}(k)$ is a $(r-1) \times \infty$ matrix with only finitely many nonzero elements, we can set

$$ht\,(k) = \max \left\{ j \in \mathbb{N} \mid \exists i \in M, \mathrm{pad}_j\,(k_i) \neq 0 \right\}. \tag{13}$$

If $z = \sum_{k,\lambda,u} \alpha_{k,\lambda,u} x^k y^\lambda \xi^u \in \Omega$ is a nonzero element with $\alpha_{k,\lambda,u} \in \mathbb{F}$, then we may assume

$$ht\,(z) = \max \left\{ ht\,(k) \mid \alpha_{k,\lambda,u} \neq 0 \right\}. \tag{14}$$

For $c, d \in \mathbb{N}_0$, we define $\|k\|_{c,d} := \sum_{i=1}^{r-1} \sum_{j=c}^{d} \mathrm{pad}_j(k_i)$ and $\|k\|_d := \|k\|_{0,d}$. Now for any $t \in \mathbb{N}$ and $x^k y^\lambda \xi^u \in \Omega$, define

$$\mathcal{S}_t \left(x^k y^\lambda \xi^u \right) = \|k\|_t + 2\|k\|_{1,t} + |u| + \mathrm{pad}_0\,(k_1). \tag{15}$$

Lemma 1. *Let $k, k' \in Q$ and $t \in \mathbb{N}_0$. Then the following statements hold.*

(i) $x^k x^{k'} \neq 0 \Leftrightarrow \text{pad}(k) + \text{pad}(k') = \text{pad}(k + k')$.

(ii) *If $k'_i \neq 0$, then $\|k' - e_i\|_t + 2\|k' - e_i\|_{1,t} \geq \|k'\|_t + 2\|k'\|_{1,t} - 1$, where $i \in M_1$.*

(iii) *Let $x^k y^\lambda \xi^u \in \Omega_{[1]}$ and $t \geq ht(x^k y^\lambda \xi^u)$. Then $\mathcal{S}_t(x^k y^\lambda \xi^u) \geq 3$.*

Proof. (i) We see that

$$\text{pad}(k) + \text{pad}(k') = \begin{pmatrix} \text{pad}(k_1) \\ \text{pad}(k_2) \\ \vdots \\ \text{pad}(k_n) \end{pmatrix} + \begin{pmatrix} \text{pad}(k'_1) \\ \text{pad}(k'_2) \\ \vdots \\ \text{pad}(k'_n) \end{pmatrix} = \begin{pmatrix} \text{pad}(k_1) + \text{pad}(k'_1) \\ \text{pad}(k_2) + \text{pad}(k'_2) \\ \vdots \\ \text{pad}(k_n) + \text{pad}(k'_n) \end{pmatrix}. \tag{16}$$

Note that $x^k x^{k'} \neq 0 \Leftrightarrow \varepsilon_v(k_i) + \varepsilon_v(k'_i) < p \Leftrightarrow \text{pad}_v(k_i) + \text{pad}_v(k'_i) < p \Leftrightarrow \text{pad}_v(k_i) + \text{pad}_v(k'_i) \not\equiv 0 \pmod p$, for all $i \in M, v \in \mathbb{N}_0$. By the uniqueness of p-adic expression, we have $x^k x^{k'} \neq 0 \Leftrightarrow \text{pad}_v(k_i) + \text{pad}_v(k'_i) = \text{pad}_v(k_i + k'_i)$, for all $i \in M, v \in \mathbb{N}_0$. Thus

$$x^k x^{k'} \neq 0 \Longleftrightarrow \text{pad}(k) + \text{pad}(k') = \begin{pmatrix} \text{pad}(k_1) + \text{pad}(k'_1) \\ \text{pad}(k_2) + \text{pad}(k'_2) \\ \vdots \\ \text{pad}(k_n) + \text{pad}(k'_n) \end{pmatrix} = \begin{pmatrix} \text{pad}(k_1 + k'_1) \\ \text{pad}(k_2 + k'_2) \\ \vdots \\ \text{pad}(k_n + k'_n) \end{pmatrix} = \text{pad}(k + k'), \tag{17}$$

as desired.

(ii) If $\text{pad}_0(k'_i) \neq 0$, then $\text{pad}(k'_i - 1) = (\text{pad}_0(k'_i) - 1, \text{pad}_1(k'_i), \ldots)$. We see that $\|k' - e_i\|_t = \|k'\|_t - 1$, $\|k' - e_i\|_{1,t} = \|k'\|_{1,t}$. So (ii) holds.

If $\text{pad}_0(k'_i) = 0$ and $\text{pad}_b(k'_i) \neq 0$ for $b \geq 1$, then we can assume that

$$\text{pad}(k'_i) = (0, \ldots, 0, \text{pad}_b(k'_i), \text{pad}_{b+1}(k'_i), \ldots). \tag{18}$$

Hence $\text{pad}(k'_i - 1) = (p-1, \ldots, p-1, \text{pad}_b(k'_i), \text{pad}_{b+1}(k'_i), \ldots)$.

(a) If $t < b$, then we get

$$\begin{aligned} \|k' - e_i\|_t + 2\|k' - e_i\|_{1,t} &= (t+1)(p-1) + 2t(p-1) \\ &> -1, \\ \|k'\|_t + 2\|k'\|_{1,t} - 1 &= -1. \end{aligned} \tag{19}$$

(b) If $t = b$ and $p > 3$, then

$$\begin{aligned} \|k' - e_i\|_t + 2\|k' - e_i\|_{1,t} &= t(p-1) + 2(b+1)(p-1) \\ &\quad + 3\left(3\text{pad}_t(k'_i) - 1\right) \\ &> 3\text{pad}_t(k'_i) - 1, \\ \|k'\|_t + 2\|k'\|_{1,t} - 1 &= 3\text{pad}_b(k'_i) - 1. \end{aligned} \tag{20}$$

(c) If $t > b$, then

$$\begin{aligned} &\|k' - e_i\|_t + 2\|k' - e_i\|_{1,t} \\ &\quad = \|k' - e_i\|_b + 2\|k' - e_i\|_{1,b} + 3\|k' - e_i\|_{b+1,t} \\ &\quad \geq \|k'\|_b + 2\|k'\|_{1,b} - 1 + 3\|k' - e_i\|_{b+1,t} \\ &\quad = \|k'\|_t + 2\|k'\|_{1,t} - 1. \end{aligned} \tag{21}$$

Thus (ii) holds.

(iii) By $|k| + 2k_1 + |u| = i + 2 \geq 3$, we have $\sum(\text{pad}_0(k_i)p^0 + \text{pad}_1(k_i)p^1 + \cdots) + 2\text{pad}_0(k_1) + |u| \geq 3$; that is, $\sum \text{pad}_0(k_i) + 2\text{pad}_0(k_1) + |u| \geq 3$. □

Lemma 2. *Let $x^k y^\lambda \xi^u \in \Omega$, $x^{k'} y^\eta \xi^v \in \Omega_{[1]}, t \geq \max\{1, ht(x^{k'} y^\eta \xi^v)\}, i \in M_1 \cup T$. Then the following statements hold.*

(i) *If $x^{k'} y^\eta \xi^v D_1(x^k y^\lambda \xi^u) \neq 0$, then $\mathcal{S}_t(x^{k'} y^\eta \xi^v D_1(x^k y^\lambda \xi^u)) \geq \mathcal{S}_t(x^k y^\lambda \xi^u) + 1$.*

(ii) *If $x^k y^\lambda \xi^u D_1(x^{k'} y^\eta \xi^v) \neq 0$, then $\mathcal{S}_t(x^k y^\lambda \xi^u D_1(x^{k'} y^\eta \xi^v)) \geq \mathcal{S}_t(x^k y^\lambda \xi^u) + 1$.*

(iii) *If $D_i(x^{k'} y^\eta \xi^v) D_{i'}(x^k y^\lambda \xi^u) \neq 0$, then $\mathcal{S}_t(D_i(x^{k'} y^\eta \xi^v) D_{i'}(x^k y^\lambda \xi^u)) \geq \mathcal{S}_t(x^k y^\lambda \xi^u) + 1$.*

Proof. (i) As $x^{k'} y^\eta \xi^v k_1^* x^{k-e_1} y^\lambda \xi u = k_1^* x^{k'} x^{k-e_1} y^{\eta+\lambda} \xi^u \xi^v \neq 0$, $x^{k'} x^{k-e_1} = x^{k'+k-e_1} \neq 0$. Then we have

$$\begin{aligned} \mathcal{S}_t\left(x^{k'} y^\eta \xi^v D_1\left(x^k y^\lambda \xi^u\right)\right) &= \left\|k' + (k - e_1)\right\|_t \\ &\quad + 2\left\|k' + (k - e_1)\right\|_{1,t} + |u| \\ &\quad + |v| + \text{pad}_0\left(k'_1 + (k_1 - 1)\right). \end{aligned} \tag{22}$$

By the equality above and Lemma 1, we get $\mathrm{pad}(k'+(k-e_1)) = \mathrm{pad}(k') + \mathrm{pad}(k-e_1)$. Thus

$$\begin{aligned} \mathrm{pad}_0\left(k_1' + (k_1 - 1)\right) &= \mathrm{pad}_0\left(k_1'\right) + \mathrm{pad}_0\left(k_1 - 1\right), \\ \left\|k' + (k - e_1)\right\|_t &= \left\|k'\right\|_t + \left\|k - e_1\right\|_t, \\ \left\|k' + (k - e_1)\right\|_{1,t} &= \left\|k'\right\|_{1,t} + \left\|k - e_1\right\|_{1,t}. \end{aligned} \tag{23}$$

Hence

$$\begin{aligned} &\mathcal{S}_t\left(x^{k'}y^{\eta}\xi^{v}D_1\left(x^k y^{\lambda}\xi^u\right)\right) \\ &\quad = \left\|k'\right\|_t + 2\left\|k'\right\|_{1,t} + |v| + \mathrm{pad}_0\left(k_1'\right) \\ &\qquad + \left\|k - e_1\right\|_t + 2\left\|k - e_1\right\|_{1,t} + |u| + \mathrm{pad}_0\left(k_1 - 1\right) \\ &\quad \geq \mathcal{S}_t\left(x^{k'}y^{\eta}\xi^{v}\right) + \|k\|_t + 2\|k\|_{1,t} + |u| \\ &\qquad + \mathrm{pad}_0\left(k_1\right) - 1 - 1 \\ &\quad \geq \mathcal{S}_t\left(x^k y^{\lambda}\xi^u\right) + 1. \end{aligned} \tag{24}$$

(ii) The proof is completely analogous to (i).

(iii) For $i \in M_1$, by assumption of this lemma, we have $x^{k'-e_i}x^{k-e_{i'}} \neq 0$, which combined with Lemma 1 yield

$$\begin{aligned} &\mathcal{S}_t\left(D_i\left(x^{k'}y^{\eta}\xi^{v}\right)D_{i'}\left(x^k y^{\lambda}\xi^u\right)\right) \\ &\quad = \mathcal{S}_t\left(k_i^* k_{i'}^* x^{(k'+e_i)+(k-e_{i'})}y^{\lambda+\eta}\xi^u\xi^v\right) \\ &\quad = \left\|\left(k' - e_i\right) + \left(k - e_{i'}\right)\right\|_t + 2\left\|\left(k' - e_i\right) + \left(k - e_{i'}\right)\right\|_{1,t} \\ &\qquad + |u| + |v| + \mathrm{pad}_0\left(k_1 + k_1'\right) \\ &\quad \geq \left\|k'\right\|_t + 2\left\|k'\right\|_{1,t} + |v| - 1 + \|k\|_t + 2\|k\|_{1,t} \\ &\qquad + |u| - 1 + \mathrm{pad}_0\left(k_1\right) + \mathrm{pad}_0\left(k_1'\right) \\ &\quad \geq \mathcal{S}_t\left(x^k y^{\lambda}\xi^u\right) + 1. \end{aligned} \tag{25}$$

For $i \in T$, it is easily seen that $\varepsilon_v(k' - e_i) + \varepsilon_v(k - e_{i'}) = \varepsilon_v(k') + \varepsilon_v(k) < p$. Also by Lemma 1, we obtain

$$\begin{aligned} &\mathcal{S}_t\left(D_i\left(x^{k'}y^{\eta}\xi^{v}\right)D_{i'}\left(x^k y^{\lambda}\xi^u\right)\right) \\ &\quad = \mathcal{S}_t\left(x^{k'+k}y^{\lambda+\eta}\xi^{u-\langle i\rangle}\xi^{v-\langle i'\rangle}\right) \\ &\quad = \left\|k' + k\right\|_t + 2\left\|k' + k\right\|_{1,t} + |u - 1| + |v - 1| \\ &\qquad + \mathrm{pad}_0\left(k_1' + k_1\right) \\ &\quad \geq \mathcal{S}_t\left(x^k y^{\lambda}\xi^u\right) + 3 - 2 \\ &\quad = \mathcal{S}_t\left(x^k y^{\lambda}\xi^u\right) + 1. \end{aligned} \tag{26}$$

Hence Lemma 2 holds. □

Lemma 3. *Let $x^k y^{\lambda}\xi^u \in \Omega, x^{k'}y^{\eta}\xi^v \in \Omega_{[1]}$ and $t \geq \max\{1, ht(x^{k'}y^{\eta}\xi^v)\}$. Let $x^{k''}y^{\lambda'}\xi^{u'}$ be a nonzero summand of $[x^{k'}y^{\eta}\xi^v, x^k y^{\lambda}\xi^u]$. Then $\mathcal{S}_t(x^{k''}y^{\lambda'}\xi^{u'}) \geq \mathcal{S}_t(x^k y^{\lambda}\xi^u) + 1$.*

Proof. By a direct computation, we obtain that

$$\begin{aligned} &\left[x^{k'}y^{\eta}\xi^{v}, x^k y^{\lambda}\xi^u\right] \\ &\quad = D_1\left(x^{k'}y^{\eta}\xi^{v}\right)\partial\left(x^k y^{\lambda}\xi^u\right) - D_1\left(x^k y^{\lambda}\xi^u\right)\partial\left(x^{k'}y^{\eta}\xi^{v}\right) \\ &\qquad + \sum_{i\in M_1\cup T}[i](-1)^{\bar{\tilde{i}}|f|}D_i\left(x^{k'}y^{\eta}\xi^{v}\right)D_{i'}\left(x^k y^{\lambda}\xi^u\right), \end{aligned} \tag{27}$$

which satisfies the conditions of Lemma 2. □

Lemma 4. $\Omega_{[1]} \subseteq \mathrm{nil}(\Omega)$.

Proof. Given $t \in \mathbb{N}$, put

$$\begin{aligned} l_t = {} & (r-1)(t+1)(p-1) + 2(r-1)t(p-1) \\ & + (p-1) + n + 1. \end{aligned} \tag{28}$$

Clearly, we have $\mathcal{S}_t(x^k y^{\lambda}\xi^u) < l_t$ for all standard basis element $x^k y^{\lambda}\xi^u$ of Ω.

Let $t \in \mathbb{N}$ such that $t \geq ht(z)$. For any $z = \sum_{k',v}c_{k',v}x^{k'}y^{\eta}\xi^v \in \Gamma_{[1]}$ with $0 \neq c_{k',v} \in \mathbb{F}$, we have

$$\begin{aligned} \mathrm{ad}z\left(x^k y^{\lambda}\xi^u\right) = \left[z, x^k y^{\lambda}\xi^u\right] &= D_1(z)\,\partial\left(x^k y^{\lambda}\xi^u\right) \\ &\quad - \partial(z)\,D_1\left(x^k y^{\lambda}\xi^u\right) \\ &\quad + \sum_{i\in M_1\cup T}[i]\,D_i(z)\,D_{i'}\left(x^k y^{\lambda}\xi^u\right). \end{aligned} \tag{29}$$

Note that $t \geq \max\{1, ht(x^{k'}y^{\eta}\xi^v)\}$. By using Lemma 3 repeatedly we see that $(\mathrm{ad}z)^{p^{l_t}}(x^k y^{\lambda}\xi^u) = 0$. Hence $\Omega_{[1]} \subseteq \mathrm{nil}(\Omega)$. □

Lemma 5. *(i) If $f = \sum_{t=k}^{l} f_t \in \mathrm{nil}(\Omega)$, where $f_t \in \Omega_t$, then $f_t \in \mathrm{nil}(\Omega)$.*

(ii) If $f = \sum_{t=-2}^{l} f_t \in \mathrm{nil}(\Omega_{\bar{0}})$, then $f_{-2} = 0$.

(iii) If $f = \sum_{t=-1}^{l} f_t \in \mathrm{nil}(\Omega_{\bar{0}})$, then $f_{-1} = 0$.

(iv) $\mathrm{Nil}(\Omega_{[0]} \cap \Omega_{\bar{0}}) = \mathrm{Nil}(\Omega_0 \cap \Omega_{\bar{0}}) + \Omega_{[1]} \cap \Omega_{\bar{0}}$.

(v) $\mathrm{Nil}(\Omega_{\bar{0}}) = \mathrm{Nil}(\Omega_{[0]} \cap \Omega_{\bar{0}})$.

Proof. (i) See Lemma 5 in [14].

(ii) By (i), we see that f_{-2} is ad-nilpotent. If $f_{-2} \neq 0$, then $[y^{\lambda}, x^{me_1}] = -D_1(x^{(me_1)}\partial(y^{\lambda})) = (\lambda-1)k_1^* x^{(m-1)e_1}y^{\lambda}$ for all $m > 0$ and $\lambda \neq 1$. By a direct computation, we obtain $(\mathrm{ad}y^{\lambda})^m(x^{me_1}) = (-1)^m(1-\lambda)^m y^{m\lambda} \neq 0$, contradicting the nilpotency of f_{-2}. Hence $f_{-2} = 0$, as desired.

(iii) Clearly, f_{-1} is ad-nilpotent by virtue of (i). If $f_{-1} \neq 0$, then we can suppose that $f_{-1} = \sum_{i \in M_1} \gamma_i x_i \neq 0$, where $\gamma_i \in \mathbb{F}$. Thus there exists some $\gamma_j \neq 0$. By computation, we have

$$\begin{aligned} \left[\Sigma \gamma_i x_i y^\lambda, x^{me_j}\right] &= \Sigma [i] (-1)^{\bar{i}|f|} D_i \left(\Sigma \gamma_i x_i y^\lambda\right) D_{i'} \left(x^{me_j}\right) \\ &= [j] \gamma_j k_{j'}^* x^{(m-1)e_{j'}} y^\lambda. \end{aligned} \tag{30}$$

Similarly, we get $(\mathrm{ad} f_{-1})^m (x^{me_{j'}}) = [j]^m (\gamma_j)^m (k_{j'}^*)^m y^{m\lambda} \neq 0$, a contradiction. Consequently, $f_{-1} = 0$.

(iv) Suppose that $f = f_0 + f_{[1]}$ is an arbitrary element of $\mathrm{nil}(\Omega_{[0]} \cap \Omega_{\bar{0}})$, where $f_0 \in \Omega_0 \cap \Omega_{\bar{0}}$, $f_{[1]} \in \Omega_{[1]} \cap \Omega_{\bar{0}}$. By (1) of this lemma, we see that $f_0 \in \mathrm{nil}(\Omega_0 \cap \Omega_{\bar{0}}) \subseteq \mathrm{Nil}(\Omega_0 \cap \Omega_{\bar{0}})$. Hence $\mathrm{nil}(\Omega_{[0]} \cap \Omega_{\bar{0}}) \subseteq \mathrm{Nil}(\Omega_0 \cap \Omega_{\bar{0}}) + \Omega_{[1]} \cap \Omega_{\bar{0}}$ and $\mathrm{Nil}(\Omega_{[0]} \cap \Omega_{\bar{0}}) \subseteq \mathrm{Nil}(\Omega_0 \cap \Omega_{\bar{0}}) + \Omega_{[1]} \cap \Omega_{\bar{0}}$.

Conversely, we have $\Omega_{[1]} \cap \Omega_{\bar{0}} \subseteq \mathrm{Nil}(\Omega_{[0]} \cap \Omega_{\bar{0}})$ by means of Lemma 4. Clearly, $\mathrm{Nil}(\Omega_0 \cap \Omega_{\bar{0}}) \subseteq \mathrm{Nil}(\Omega_{[0]} \cap \Omega_{\bar{0}})$. Thus $\mathrm{Nil}(\Omega_0 \cap \Omega_{\bar{0}}) + \Omega_{[1]} \cap \Omega_{\bar{0}} \subseteq \mathrm{Nil}(\Omega_{[0]} \cap \Omega_{\bar{0}})$. This shows that $\mathrm{Nil}(\Omega_{[0]} \cap \Omega_{\bar{0}}) = \mathrm{Nil}(\Omega_0 \cap \Omega_{\bar{0}}) + \Omega_{[1]} \cap \Omega_{\bar{0}}$, as desired.

(v) It is obvious that $\mathrm{Nil}(\Omega_{[0]} \cap \Omega_{\bar{0}}) \subseteq \mathrm{Nil}(\Omega_{\bar{0}})$. Conversely, we assume that $f = \sum_{t=-2}^{l} f_t \in \mathrm{nil}(\Omega_{\bar{0}})$.

Then by (ii) of this lemma, $f_{-2} = 0$. Hence $f = f_{-1} + f_{[0]} \in \mathrm{nil}(\Omega_{\bar{0}})$, where $f_{-1} \in \Omega_{-1} \cap \Omega_{\bar{0}}$ and $f_0 \in \Omega_{[0]} \cap \Omega_{\bar{0}}$. It follows from (iii) that $f_{-1} = 0$. Now $f = f_{[0]} \in \mathrm{nil}(\Omega_{\bar{0}})$. Noting that $\Omega_{[1]} \subseteq \mathrm{nil}(\Omega)$, we have $f = f_{[0]} \subseteq \mathrm{nil}(\Omega_{[0]} \cap \Omega_{\bar{0}}) \subseteq \mathrm{Nil}(\Omega_{[0]} \cap \Omega_{\bar{0}})$. Thus $\mathrm{nil}(\Omega_{\bar{0}}) \subseteq \mathrm{Nil}(\Omega_{[0]} \cap \Omega_{\bar{0}})$, and the assertion holds. □

Lemma 6. *Let $i, j \in M_1$. Suppose that $x^k y^\lambda \xi^u$ is an arbitrary standard element of Ω. Then the following statements hold.*

(i) $x_i^2 y^\lambda \in \mathrm{nil}(\Omega_0 \cap \Omega_{\bar{0}})$.

(ii) *If $[i] = [j]$ and $i \neq j$, then* $x_i x_j y^\lambda \in \mathrm{nil}(\Omega_0 \cap \Omega_{\bar{0}})$.

(iii) *If $[i] \neq [j]$ and $j \neq i'$, then* $x_i x_j y^\lambda \in \mathrm{nil}(\Omega_0 \cap \Omega_{\bar{0}})$.

(iv) $x_i x_{i'} y^\lambda \in \mathrm{nil}(\Omega_0 \cap \Omega_{\bar{0}})$.

Proof. (i) By a direct computation, we get

$$\begin{aligned} \left(\mathrm{ad} x_i^2 y^\lambda\right)\left(x^k y^\lambda \xi^u\right) &= \left[x_i^2 y^\lambda, x^k y^\lambda \xi^u\right] \\ &= -\left(1 - 2\mu_i - \lambda\right) x_i^2 y^\lambda D_1 \left(x^k y^\lambda \xi^u\right) \\ &\quad + 2 [i] x_i y^\lambda D_{i'} \left(x^k y^\lambda \xi^u\right) \end{aligned} \tag{31}$$

and $x_i^2 y^\lambda D_1 \circ x_i y^\lambda D_{i'} = x_i y^\lambda D_{i'} \circ x_i^2 y^\lambda D_1$. It follows from the binomial theorem that

$$\begin{aligned} \left(\mathrm{ad} x_i^2 y^\lambda\right)^{p^n} \left(x^k y^\lambda \xi^u\right) &= \Big(\left(-\left(1 - 2\mu_i - \lambda\right) x_i^2 y^\lambda D_1\right)^{p^n} \\ &\quad + \left(2 [i] x_i y^\lambda D_{i'}\right)^{p^n}\Big) \left(x^k y^\lambda \xi^u\right) = 0. \end{aligned} \tag{32}$$

(ii) Also by a direct calculation, we have

$$\begin{aligned} &\left(\mathrm{ad} x_i x_j y^\lambda\right)\left(x^k y^\lambda \xi^u\right) \\ &\quad = \left[x_i x_j y^\lambda, x^k y^\lambda \xi^u\right] \\ &\quad = -\left(1 - \mu_i - \mu_j - \lambda\right) x_i x_j y^\lambda D_1 \left(x^k y^\lambda \xi^u\right) \\ &\qquad + [i] x_j y^\lambda D_{i'} \left(x^k y^\lambda \xi^u\right) + [j] x_i y^\lambda D_{j'} \left(x^k y^\lambda \xi^u\right). \end{aligned} \tag{33}$$

Put $A = -(1 - \mu_i - \mu_j - \lambda) x_i x_j y^\lambda D_1$, $B = [i] x_j y^\lambda D_{i'}$, and $C = [j] x_i y^\lambda D_{j'}$. Obviously,

$$\begin{aligned} &x_i x_j y^\lambda D_1 \left(x_j y^\lambda D_{i'} + x_i y^\lambda D_{j'}\right) \\ &\quad = \left(x_j y^\lambda D_{i'} + x_i y^\lambda D_{j'}\right) x_i x_j y^\lambda D_1, \\ &x_j y^\lambda D_{i'} x_i y^\lambda D_{j'} = x_i y^\lambda D_{j'} \left(x_j y^\lambda D_{i'}\right). \end{aligned} \tag{34}$$

Hence $(\mathrm{ad} x_i x_j y^\lambda)^{p^n} (x^k y^\lambda \xi^u) = A^{p^n} + B^{p^n} + C^{p^n} = 0$.

(iii) The proof is completely analogous to (ii).

(iv) According to (i), we see that $x_i x_{i'} y^\lambda = (1/4)[i, x_i x_i y^\lambda, x_{i'} x_{i'}] \in \mathrm{nil}(\Omega_0 \cap \Omega_{\bar{0}})$. □

Lemma 7. *Suppose that $i, j, k \in T$ are different from each other, and $\gamma, \chi \in \mathbb{F}$. Then*

(i) $f = \gamma \xi_i \xi_j y^\lambda + \chi \xi_i \xi_k y^\lambda \in \mathrm{nil}(\Omega)$ for $\gamma^2 + \chi^2 = 0$.

(ii) $\xi_i \xi_j y^\lambda \in \mathrm{nil}(\Omega_0 \cap \Omega_{\bar{0}})$.

Proof. (i) Set $x^k y^\lambda \xi^u$ be an arbitrary standard element of Ω. Then

$$\begin{aligned} &(\mathrm{ad} f)\left(x^k y^\lambda \xi^u\right) \\ &= \left[f, x^k y^\lambda \xi^u\right] \\ &= \left[\gamma \xi_i \xi_j y^\lambda, x^k y^\lambda \xi^u\right] \\ &\quad + \left[\chi \xi_i \xi_k y^\lambda, x^k y^\lambda \xi^u\right] \\ &= \gamma \lambda \xi_i \xi_j y^\lambda D_1 \left(x^k y^\lambda \xi^u\right) + \gamma \xi_j y^\lambda D_i \left(x^k y^\lambda \xi^u\right) \\ &\quad + \gamma \xi_i y^\lambda D_j \left(x^k y^\lambda \xi^u\right) + \chi \lambda \xi_i \xi_j y^\lambda D_1 \left(x^k y^\lambda \xi^u\right) \\ &\quad + \chi \xi_k y^\lambda D_j \left(x^k y^\lambda \xi^u\right) + \chi \xi_i y^\lambda D_k \left(x^k y^\lambda \xi^u\right). \end{aligned} \tag{35}$$

Put

$$\begin{aligned} A &= \gamma \lambda \xi_i \xi_j y^\lambda D_1, \qquad B = \gamma \xi_j y^\lambda D_i, \\ C &= \gamma \xi_i y^\lambda D_j, \qquad D = \chi \lambda \xi_i \xi_j y^\lambda D_1, \\ E &= \chi \xi_k y^\lambda D_i, \qquad F = \chi \xi_i y^\lambda D_k. \end{aligned} \tag{36}$$

Obviously,

$$\begin{aligned} A^2 &= B^2 = C^2 = D^2 = E^2 = F^2 = 0, \\ AB &= BA = AC = CA = AD = DA = AF \\ &= FA = BD = DB = BE = EB = CD \\ &= DC = CF = FC = DF = FD = 0. \end{aligned} \tag{37}$$

Hence

$$\begin{aligned} (\operatorname{ad} f)^2 = {} & AE + EA + BC + CB + BF + FB \\ & + CE + EC + DE + ED + EF + FE. \end{aligned} \tag{38}$$

Noting that $\gamma^2 + \chi^2 = 0$, we obtain

$$(\operatorname{ad} f)^3 = BCB + CBC + EFE + FEF. \tag{39}$$

Similarly, $(\operatorname{ad} f)^4 = 0$, and then $f = \gamma\xi_i\xi_j y^\lambda + \chi\xi_i\xi_k y^\lambda \in \operatorname{nil}(\Omega)$.

(ii) Let $k \in T \setminus \{i, j\}$. It follows from (i) that

$$\begin{aligned} f_1 &= \gamma\xi_i\xi_j y^\lambda + \xi_i\xi_k y^\lambda \in \operatorname{Nil}(\Omega_0 \cap \Omega_{\bar 0}), \\ f_2 &= \xi_i\xi_j + \gamma\xi_i\xi_k \in \operatorname{Nil}(\Omega_0 \cap \Omega_{\bar 0}), \quad \text{where } \gamma^2 = -1. \end{aligned} \tag{40}$$

Hence $\xi_i\xi_j y^\lambda = -(1/2)(\gamma f_1 - f_2) \in \operatorname{Nil}(\Omega_0 \cap \Omega_{\bar 0})$. □

Lemma 8. $x_i\xi_j y^\lambda \in \operatorname{nil}(\Omega_0 \cap \Omega_{\bar 1})$ *for* $i \in M_1$ *and* $j \in T$.

Proof. By a direct computation, we obtain

$$\begin{aligned} & (\operatorname{ad} x_i\xi_j y^\lambda)(x^k y^\lambda \xi^u) \\ & = [x_i\xi_j y^\lambda, x^k y^\lambda \xi^u] \\ & = \left(-\left(\frac{1}{2} - \mu_i - \lambda\right) x_i\xi_j y^\lambda D_1 + [i]\,\xi_j y^\lambda D_{i'} - x_i y^\lambda D_j\right) \\ & \quad \times (x^k y^\lambda \xi^u). \end{aligned} \tag{41}$$

Put $A = -(1/2 - \mu_i - \lambda)x_i\xi_j y^\lambda D_1, B = [i]\xi_j y^\lambda D_{i'}$ and $C = -x_i y^\lambda D_j$. Observing $A^2 = B^2 = C^2 = 0$ and $AB = BA = 0$, we see that

$$\begin{aligned} (\operatorname{ad} x_i\xi_j y^\lambda)^2 &= AC + BC + CA + CB, \\ (\operatorname{ad} x_i\xi_j y^\lambda)^{p^n} &= 0, \quad n \in \mathbb{N}. \end{aligned} \tag{42}$$

Thus $x_i\xi_j y^\lambda \in \operatorname{nil}(\Omega_0 \cap \Omega_{\bar 1})$, as required. □

Lemma 9. *The following statements hold*

(i) $\operatorname{nil}(\Omega_0) = \operatorname{span}_{\mathbb{F}}\{x_i x_j y^\lambda, x_i\xi_j y^\lambda, \xi_i\xi_j y^\lambda \mid i, j \in M_1 \cup T\}$.

(ii) *For* $q \geq 3$, $\operatorname{nil}(\Omega_0 \cap \Omega_{\bar 0}) = \operatorname{span}_{\mathbb{F}}\{x_i x_j y^\lambda, \xi_i\xi_j y^\lambda \mid i, j \in M_1 \cup T[i], = [j]\}$.

(iii) *For* $q = 2$, $\operatorname{nil}(\Omega_0 \cap \Omega_{\bar 0}) = \operatorname{span}_{\mathbb{F}}\{x_i x_j y^\lambda \mid i, j \in M_1\}$.

Proof. (i) Suppose that $f = \gamma_1 x_1 y^\lambda + \sum_{i,j\in M_1\cup T}(x_i x_j y^\lambda + x_i\xi_j y^\lambda + \xi_i\xi_j y^\lambda)$ is an arbitrary element of $\operatorname{nil}(\Omega_0)$, where $\gamma_1 \in \mathbb{F}$. If $\gamma_1 \neq 0$, then $[\gamma_1 x_1 y^\lambda, y^\lambda] = \gamma_1(1-\lambda)y^{2\lambda}$. A direct calculation shows that $(\operatorname{ad} f)^m(y^\lambda) = \gamma_1^m(1-\lambda)(1-2\lambda)\cdots(1-m\lambda)y^{m\lambda} \neq 0$. Thus f is not ad-nilpotent, contradicting the nilpotency of f. Hence $\gamma_1 = 0$ and $\operatorname{nil}(\Omega_0) \subseteq \operatorname{span}_{\mathbb{F}}\{x_i x_j y^\lambda, x_i\xi_j y^\lambda, \xi_i\xi_j y^\lambda \mid i, j \in M_1 \cup T\}$. Obviously, $\operatorname{span}_{\mathbb{F}}\{x_i x_j y^\lambda, x_i\xi_j y^\lambda, \xi_i\xi_j y^\lambda \mid i, j \in M_1 \cup T\}$ is a subalgebra of Ω, which yields $\operatorname{Nil}(\Omega_0) \subseteq \operatorname{span}_{\mathbb{F}}\{x_i x_j y^\lambda, x_i\xi_j y^\lambda, \xi_i\xi_j y^\lambda \mid i, j \in M_1 \cup T\}$.

Conversely, we have $\operatorname{span}_{\mathbb{F}}\{x_i x_j y^\lambda, x_i\xi_j y^\lambda, \xi_i\xi_j y^\lambda \mid i, j \in M_1 \cup T\} \subseteq \operatorname{Nil}(\Omega_0)$ by virtue of Lemmas 6, 7, and 8. It follows that $\operatorname{Nil}(\Omega_0) = \operatorname{span}_{\mathbb{F}}\{x_i x_j y^\lambda, x_i\xi_j y^\lambda, \xi_i\xi_j y^\lambda \mid i, j \in M_1 \cup T\}$.

(ii) By (i) of the lemma, we have

$$\begin{aligned} & \operatorname{Nil}(\Omega_0 \cap \Omega_{\bar 0}) \\ & \subseteq \operatorname{Nil}(\Omega_0) \cap \Omega_{\bar 0} \\ & = \operatorname{span}_{\mathbb{F}}\{x_i x_j y^\lambda, \xi_i\xi_j y^\lambda \mid i, j \in M_1 \cup T, [i] = [j]\}. \end{aligned} \tag{43}$$

Conversely, the assertion $\operatorname{span}_{\mathbb{F}}\{x_i x_j y^\lambda, \xi_i\xi_j y^\lambda \mid i, j \in M_1 \cup T, [i] = [j]\} \subseteq \operatorname{Nil}(\Omega_0 \cap \Omega_{\bar 0})$ follows from Lemmas 6 and 7.

(iii) Suppose $f = \gamma\xi_{r+1}\xi_s y^\lambda + \sum_{i,j\in M_1}\gamma_{ij}x_i x_j y^\lambda \in \operatorname{nil}(\Omega_0 \cap \Omega_{\bar 0})$, where $\gamma, \gamma_{ij} \in \mathbb{F}$. If $\gamma \neq 0$, then $\operatorname{ad} f(\xi_s y^\lambda) = [f, \xi_s y^\lambda] = \gamma\xi_{r+1}y^{2\lambda}$ and $(\operatorname{ad} f)^2(\xi_s y^\lambda) = \gamma^2\xi_s y^{3\lambda}$. A direct calculation shows that $(\operatorname{ad} f)^{2m}(\xi_s y^\lambda) = \gamma^{2m}\xi_s y^{2m\lambda} \neq 0$. It follows that f is not ad-nilpotent. Thus $\gamma = 0$. Then $f = \sum_{i,j\in M_1}\gamma_{ij}x_i x_j y^\lambda \in \operatorname{nil}(\Omega_0 \cap \Omega_{\bar 0})$ and $\operatorname{nil}(\Omega_0 \cap \Omega_{\bar 0}) \subseteq \operatorname{span}_{\mathbb{F}}\{x_i x_j y^\lambda \mid i, j \in M_1\}$. Note that $\operatorname{span}_{\mathbb{F}}\{x_i x_j y^\lambda \mid i, j \in M_1\}$ is a subalgebra of Ω. Hence $\operatorname{Nil}(\Omega_0 \cap \Omega_{\bar 0}) \subseteq \operatorname{span}_{\mathbb{F}}\{x_i x_j y^\lambda \mid i, j \in M_1\}$ and (iii) holds. □

Let ρ be the corresponding representation with respect to $\Omega_{\bar 0}$ module Ω_{-1}; that is, $\rho(f) = \operatorname{ad} f|_{\Omega_{-1}}$ for all $f \in \Omega_0$. It is easily seen that ρ is faithful. For $f \in \Omega_0$, we denote by $\rho(f)$ the matrix of $\rho(f)$ relative to the fixed ordered $\mathbb{F}$-basis:

$$\{x_2 y^\lambda, x_3 y^\lambda, \ldots, x_{r-1}y^\lambda, \xi_{r+1}y^\lambda, \ldots, \xi_s y^\lambda\}. \tag{44}$$

Denote by $\mathfrak{gl}(2n, q)$ the general liner Lie superalgebra of $(2n+q)\times(2n+q)$ matrices over $\mathbb{F}$. Let e_{ij} denote the $(s-2)\times(s-2)$ matrix whose (i, j)-entry is 1 and 0 elsewhere. Let E_n denote the identity matrix of size $n \times n$. Put $G = \begin{pmatrix} 0 & E_n \\ -E_n & 0 \end{pmatrix}$. Let $\mathfrak{sp}(2n, \mathbb{F})$ be all the $(2n)\times(2n)$ matrices set filled with $A^T G + GA = 0$. Put

$$\begin{aligned} \mathscr{W} = \Bigg\{ & \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathfrak{gl}(2n, q) \mid A \in \mathfrak{sp}(2n, \mathbb{F}), \\ & B^T G + C = 0, D = -D^T \Bigg\}. \end{aligned} \tag{45}$$

Set

$$\mathscr{L} = \mathscr{W} \oplus \mathbb{F}E_{s-2}. \tag{46}$$

Lemma 10. *(i) $\rho(\Omega_0) = \mathscr{L}$.*

(ii) If $f \in \operatorname{nil}(\Omega_0)$, then $\rho(f)$ is a nilpotent matrix.

Proof. (i) Let $i, j, k \in M_1 \cup T$. By computation, we have

$$\begin{aligned}\left[x_i x_j, x_k y^{\lambda}\right] &= [i]\,\delta_{i'k} x_j y^{\lambda} + [j]\,\delta_{j'k} x_i y^{\lambda},\\ \left[x_1, x_k y^{\lambda}\right] &= (1-\mu_k-\lambda)\, x_k y^{\lambda}.\end{aligned} \tag{47}$$

Hence $\rho(x_i x_j) = [i]e_{j,i'} + [j]e_{ij'}$ and $\rho(x_1) = (1-\mu_k-\lambda)E_{s-2}$. The other cases are treated similarly. Thus (i) holds.

(ii) As f is a nilpotent elements, $\rho(f)$ is a nilpotent liner transformation. Then by the definition of ρ, we see that $\rho(f)$ is a nilpotent matrix. □

Lemma 11. *If $f \in \operatorname{nil}(\Omega_0 \cap \Omega_{\bar{0}})$, $f \neq 0$, then there exists a $z \in \Omega_0 \cap \Omega_{\bar{0}}$ such that $[f, z] \notin \operatorname{nil}(\Omega_0 \cap \Omega_{\bar{0}})$.*

Proof. By Lemma 9, we can assume that $f = \sum_{l\in M_1} \gamma_l x_l^2 y^{\lambda} + \sum_{l,t\in M_1, l<t} \beta_{lt} x_l x_t y^{\lambda} + \sum_{l,t\in T, l<t} \chi_{lt}\xi_l\xi_t y^{\lambda}$, where $\gamma_l, \beta_{lt}, \chi_{lt} \in \mathbb{F}$.

Suppose $\gamma_i \neq 0$ for some $i \in M_1$ and $z = x_{i'}^2$. A direct calculation shows that

$$[f, z] = [i]\, 4\gamma_i x_i x_{i'} y^{\lambda} + h, \tag{48}$$

where every item of h does not contain x_i. Then $(\operatorname{ad}[f,z])(x_{i'}) = 4\gamma_i y^{\lambda} x_{i'}$. Thus $(\operatorname{ad}[f,z])^n(x_{i'}) = 4^n \gamma_i^n y^{n\lambda} x_{i'} \neq 0$, for all $n \in \mathbb{N}$, which implies that $[f,z]$ is not a nilpotent element.

If $\gamma_i = 0$ for all $i \in M_1$, then we let $\beta_{ij} \neq 0$ for some $i, j \in M_1$ and $z = x_{i'} x_{j'} y^{\lambda}$. Also by computation, we have

$$[f, z] = [j]\,\beta_{ij} x_i x_{i'} y^{\lambda} + h, \tag{49}$$

where every item of h does not contain x_i. Similarly, $(\operatorname{ad}[f,z])^n(x_{i'}) = [j]^n \beta_{ij}^n y^{n\lambda} x_{i'} \neq 0$, for all $n \in \mathbb{N}$, and then $[f,z]$ is not nilpotent.

Hence our assertion follows.

If $\gamma_i = 0$ and $\beta_{ij} = 0$ for all $i, j \in M_1$, then $f = \sum_{l,t\in T, l<t} \chi_{lt}\xi_l\xi_t y^{\lambda} \in \operatorname{span}_{\mathbb{F}}\{\xi_l\xi_t y^{\lambda} \mid l, t \in T, \lambda \in H\}$. We see that $\rho(f)$ is a antisymmetric nilpotent matrix. By Lemmas 9(ii) and 10, it is easy to see that there is $z \in \operatorname{span}_{\mathbb{F}}\{\xi_i\xi_j y^{\lambda} \mid i, j \in T\}$ such that $[\rho(f), \rho(z)]$ is not a nilpotent matrix; that is, $[f, z]$ is not an ad-nilpotent element. Hence our assertion holds. □

Proposition 12. $\operatorname{nil}\Omega = \Omega_{[1]} \oplus \operatorname{span}_{\mathbb{F}}\{x_i x_j y^{\lambda}, x_i\xi_j y^{\lambda}, \xi_i\xi_j y^{\lambda} \mid \lambda \in H, i, j \in M_1 \cup T\}$.

Proof. According to Lemma 4, we need only to determine all ad-nilpotent elements in $\Omega_{-2} \oplus \Omega_{-1} \oplus \Omega_0$. For any $n \in \mathbb{N}$, a direct computation shows that

$$\begin{aligned}\left(\operatorname{ad} y^{\lambda}\right)^n &= \left((1-\lambda) D_1\right)^n \neq 0,\\ \left(\operatorname{ad} x_i y^{\lambda}\right)^n &= \left(-\left(2^{-1}-\mu_i-\lambda\right) x_i y^{\lambda} D_1 + [i]\, y^{\lambda} D_{i'}\right)^n \neq 0,\\ \left(\operatorname{ad}\xi_j y^{\lambda}\right)^n &= \left(-\left(2^{-1}-\lambda\right)\xi_j y^{\lambda} D_1 + y^{\lambda} D_j\right)^n \neq 0,\\ \left(\operatorname{ad} x_1 y^{\lambda}\right)^n &= \left(y^{\lambda}\bar{\partial} - (1-\lambda)\, x_1 y^{\lambda} D_1\right)^n \neq 0.\end{aligned} \tag{50}$$

Lemmas 6, 7, and 8 imply that $x_i x_j y^{\lambda}$, $x_i\xi_j y^{\lambda}$, and $\xi_i\xi_j y^{\lambda}$ are ad-nilpotent for $i, j \in M_1 \cup T$. Hence our assertion holds. □

4. Filtration and Intrinsic Property

Lemma 13. *$\Omega_{[0]} \cap \Omega_{\bar{0}} = \operatorname{Nor}_{\Omega_{\bar{0}}}(\operatorname{Nil}(\Omega_{\bar{0}}))$ and $\Omega_{[0]}$ is invariant.*

Proof. Firstly, we prove the inclusion $\Omega_{[0]} \cap \Omega_{\bar{0}} \subseteq \operatorname{Nor}_{\Omega_{\bar{0}}}(\operatorname{Nil}(\Omega_{\bar{0}}))$. Lemmas 6–9 show that $\operatorname{Nil}(\Omega_0 \cap \Omega_{\bar{0}}) \lhd \Omega_0 \cap \Omega_{\bar{0}}$, which combined with (iv) and (v) of Lemma 5, yield

$$\begin{aligned}\operatorname{Nil}\left(\Omega_{\bar{0}}\right) &= \operatorname{Nil}\left(\Omega_{[0]} \cap \Omega_{\bar{0}}\right) = \operatorname{Nil}\left(\Omega_0 \cap \Omega_{\bar{0}}\right) + \Omega_{[1]} \cap \Omega_{\bar{0}},\\ \left[\Omega_0 \cap \Omega_{\bar{0}}, \operatorname{Nil}\left(\Omega_{[0]} \cap \Omega_{\bar{0}}\right)\right] &\subseteq \operatorname{Nil}\left(\Omega_0 \cap \Omega_{\bar{0}}\right),\\ \left[\Omega_0 \cap \Omega_{\bar{0}}, \Omega_{[1]} \cap \Omega_{\bar{0}}\right] &\subseteq \Omega_{[1]} \cap \Omega_{\bar{0}}.\end{aligned} \tag{51}$$

Then

$$\begin{aligned}\left[\Omega_0 \cap \Omega_{\bar{0}}, \operatorname{Nil}\left(\Omega_{\bar{0}}\right)\right] &= \left[\Omega_0 \cap \Omega_{\bar{0}}, \operatorname{Nil}\left(\Omega_0 \cap \Omega_{\bar{0}}\right)\right]\\ &\quad + \left[\Omega_0 \cap \Omega_{\bar{0}}, \Omega_{[1]} \cap \Omega_{\bar{0}}\right]\\ &\subseteq \operatorname{Nil}\left(\Omega_0 \cap \Omega_{\bar{0}}\right)\\ &\quad + \Omega_{[1]} \cap \Omega_{\bar{0}} = \operatorname{Nil}\left(\Omega_{\bar{0}}\right),\end{aligned} \tag{52}$$

that is, $\Omega_{[0]} \cap \Omega_{\bar{0}} \subseteq \operatorname{Nor}_{\Omega_{\bar{0}}}(\operatorname{Nil}(\Omega_{\bar{0}}))$.

Let us consider the converse inclusion. Suppose that $f = f_{-2} + f_{[-1]} \in \operatorname{Nor}_{\Omega_{\bar{0}}}(\operatorname{Nil}(\Omega_{\bar{0}}))$, where $f_{-2} \in \Omega_{-2}$ and $f_{[-1]} \in \Omega_{[-1]}$. If $f_{-2} \neq 0$, then $[f, x_1\xi_s] = [f_{-2}, x_1\xi_s] + [f_{[-1]}, x_1\xi_s] = \gamma f_{-2}\xi_s + h \notin \operatorname{Nil}(\Omega_{\bar{0}})$, where $h \in \Omega_{[0]}$ and $\gamma \in \mathbb{F}$, a contradiction. Consequently, $f_{-2} = 0$.

Now suppose $f = f_{-1} + f_{[0]}$, where $f_{-1} = \sum_{i\in M_1} \gamma_i x_i y^{\lambda} \in \Omega_{-1}$, $\gamma_i \in \mathbb{F}$, $f_{[0]} \in \Omega_{[0]}$. If $y_{-1} \neq 0$ and $\gamma_j \neq 0$ for some $j \in M_1$, then we have $[f, x_{j'}^2] = 2[j]\gamma_j x_{j'} y^{\lambda} + h \notin \operatorname{Nil}(\Omega_{\bar{0}})$, where $h \in \Omega_{[0]}$, a contradiction. Thus $f_{-1} = 0$ and $f = f_{[0]} \in \Omega_{[0]} \cap \Omega_{\bar{0}}$. This proves the asserted inclusion.

By the proof above, we know that $\Omega_{[0]}$ is invariant. □

Lemma 14. *$\Omega_{[1]} \cap \Omega_{\bar{0}} = \{f \in \operatorname{nil}(\Omega_{\bar{0}}) \mid [f, \Omega_{[0]} \cap \Omega_{\bar{0}}] \subseteq \operatorname{nil}(\Omega_{\bar{0}})\}$ and $\Omega_{[1]}$ is invariant.*

Proof. Let $\mathscr{M} = \{f \in \operatorname{nil}(\Omega_{\bar{0}}) \mid [f, \Omega_{[0]} \cap \Omega_{\bar{0}}] \subseteq \operatorname{nil}(\Omega_{\bar{0}})\}$. Suppose $f \in \operatorname{nil}(\Omega_{\bar{0}})$. By Lemma 5, $f_{-2} = 0$ and

$f_{-1}=0$. Then we can assume that $f=f_0+f_{[1]}\in\mathscr{M}$, where $f_0\in\Omega_0\cap\Omega_{\bar{0}}$, $f_{[1]}\in\Omega_{[1]}\cap\Omega_{\bar{0}}$. Let $f_0\neq 0$. Clearly, $f_0\in\mathrm{nil}(\Omega_0)$. Lemma 5(1) implies that $f_0\in\mathrm{nil}(\Omega_0\cap\Omega_{\bar{0}})$. According to Lemma 11, there exists $z\in\Omega_0\cap\Omega_{\bar{0}}$ such that $[f_0,z]\notin\mathrm{nil}(\Omega_0\cap\Omega_{\bar{0}})$. Thus $[f_{[1]},z]\notin\mathrm{nil}(\Omega_0\cap\Omega_{\bar{0}})$ and $f_{[1]}$ is not nilpotent, contradicting the result of Lemma 5. Hence $f_0=0$ and $f\in\Omega_{[1]}\cap\Omega_{\bar{0}}$; that is, $\mathscr{M}\subseteq\Omega_{[1]}\cap\Omega_{\bar{0}}$.

Conversely, $[\Omega_{[1]}\cap\Omega_{\bar{0}},\Omega_{[0]}\cap\Omega_{\bar{0}}]\subseteq\Omega_{[1]}\cap\Omega_{\bar{0}}\subseteq\mathrm{nil}(\Omega_0\cap\Omega_{\bar{0}})\subseteq\mathrm{Nil}(\Omega_{\bar{0}})=\mathrm{Nil}(\Omega_{[0]}\cap\Omega_{\bar{0}})$. It is obvious that $\Omega_{[1]}\cap\Omega_{\bar{0}}\subseteq\mathscr{M}$ and the proof is complete. □

Lemma 15. *(i)* $\Omega_{[1]}\cap\Omega_{\bar{1}}=\{f\in\Omega_{\bar{1}}\mid[f,\Omega_{\bar{1}}]\subseteq\Omega_{[0]}\cap\Omega_{\bar{0}}\}$.

(ii) $\Omega_{[0]}\cap\Omega_{\bar{1}}=[\Omega_{\bar{1}},\Omega_{[1]}\cap\Omega_{\bar{0}}]$.

(iii) $\Omega_{[-1]}=\{f\in\Omega\mid[f,\Omega_{[1]}]\subseteq\Omega_{[0]}\}$.

Proof. (i) Put $\mathscr{A}=\{f\in\Omega_{\bar{1}}\mid[f,\Omega_{\bar{1}}]\subseteq\Omega_{[0]}\cap\Omega_{\bar{0}}\}$. Suppose $f=f_{-1}+f_{[0]}$, where $f_{-1}=\sum_{i\in T}\gamma_i\xi_i y^\lambda\in\Omega_{-1}\cap\Omega_{\bar{1}}$, $f_{[0]}\in\Omega_{[0]}\cap\Omega_{\bar{1}}$ and $\gamma_i\in\mathbb{F}$. Let $\gamma_j\neq 0$ for some one $j\in T$. Then $[f,\xi_j]=[f_{-1},\xi_j]+[f_{[0]},\xi_j]=\gamma_j y^\lambda+[f_{[0]},\xi_j]\notin\Omega_{[-1]}\cap\Omega_{\bar{0}}$; that is, $[f,\xi_j]\notin\Omega_{[0]}\cap\Omega_{\bar{0}}$. This contradicts $f\in\mathscr{A}$. Thus $f_{-1}=0$.

Let $f=f_0+f_{[1]}$ be an arbitrary element of $\mathscr{A}$, where $f_0=\sum_{i\in M_1,j\in T}\gamma_{i,j}x_i\xi_j y^\lambda\in\Omega_0\cap\Omega_{\bar{1}}$, $f_{[1]}\in\Omega_{[1]}\cap\Omega_{\bar{1}}$ and $\gamma_{i,j}\in\mathbb{F}$. If there is a $\gamma_{it}\neq 0$, then $[f,\xi_t]=[f_0,\xi_t]+[f_{[1]},\xi_t]=\gamma_{it}x_i y^\lambda+[f_{[1]},\xi_t]\notin\Omega_{[0]}\cap\Omega_{\bar{0}}$, contradicting $f\in\mathscr{A}$. Hence $f_0=0$ and $f=f_{[1]}\in\Omega_{[1]}\cap\Omega_{\bar{1}}$. Consequently, $\mathscr{A}\subseteq\Omega_{[1]}\cap\Omega_{\bar{1}}$.

Conversely, let $f\in\Omega_{[1]}\cap\Omega_{\bar{1}}$. Then $[f,\Omega_{\bar{1}}]\subseteq[f,\Omega_{[-1]}]\subseteq\Omega_{[0]}\cap\Omega_{\bar{0}}$, which shows that $\Omega_{[1]}\cap\Omega_{\bar{1}}\subseteq\mathscr{A}$.

(ii) We first prove the inclusion $\Omega_{[0]}\cap\Omega_{\bar{1}}\subseteq[\Omega_{\bar{1}},\Omega_{[1]}\cap\Omega_{\bar{0}}]$. Suppose $x^k y^\lambda\xi^u\in\Omega_{[0]}\cap\Omega_{\bar{1}}$, where $\xi^u=\xi_i\xi^v$. Noting that $x_1x^k y^\lambda\xi^v\in\Omega_{[1]}\cap\Omega_{\bar{0}}$ and $x^k y^\lambda\xi^u=-2[\xi_i,x_1x^k y^\lambda\xi^v]$, we obtain $x^k y^\lambda\xi^u\in[\Omega_{\bar{1}},\Omega_{[1]}\cap\Omega_{\bar{0}}]$. The converse inclusion is obvious.

(iii) The proof is completely analogous to (ii). □

Theorem 16. Ω *is transitive.*

Proof. Assume the contrary. Suppose that there exists a nonzero $f\in\Omega_l$ such that $[f,\Omega_{-1}]=0$, where $l\in\mathbb{N}_0$. Let t be the maximal exponent of x_1 of all monomial expressions occurring in f. Then we may assume that

$$f=\sum_{k_1=t}\chi x^k y^\lambda\xi^u+\sum_{k_1<t}\omega x^{k'}y^\eta\xi^v, \tag{53}$$

where $\chi,\omega\in\mathbb{F}$. For any $j\in M_1$, we have

$$0=\left[f,x_j\right]=\sum_{k_1=t}\chi\left[j'\right]k^*_{j'}x^{k-e_{j'}}y^\lambda\xi^u+h, \tag{54}$$

where h is the sum of summand that the exponent of x_1 is less than t. Since $\sum_{k_1=t}\chi[j']k^*_{j'}x^{k-e_{j'}}y^\lambda\xi^u$ is linear independence, $x^{k-e_{j'}}y^\lambda\xi^u=0$; that is, $k_{j'}=0$. For any $j\in T$, we get

$$0=\left[f,\xi_j\right]=\sum_{k_1=t}\chi x^k y^\lambda\xi^{u-\langle j\rangle}+h, \tag{55}$$

where h is the sum of summand that the exponent of x_1 is less than t. Thus $\sum_{k_1=t}\chi x^k y^\lambda\xi^{u-\langle j\rangle}=0$; that is, $x^k y^\lambda\xi^u$ does not contain ξ_j and $f=x^{te_1}y^\lambda$.

If $t=0$, then $f\in\Omega_{-2}$, contradicting $l\geq 0$. Let $t>0$. Then we can suppose

$$f=\alpha x^{te_1}y^\lambda+\sum_{k_1=t-1}\chi_{k,\lambda,u}x^k y^\lambda\xi^u+\sum_{k_1<t-1}\omega_{k',\eta,v}x^{k'}y^\eta\xi^v, \tag{56}$$

where $\alpha,\chi_{k,\lambda,u},\omega_{k',\eta,v}\in\mathbb{F}$. For $j\in M_1$, we have

$$\begin{aligned}0=\left[f,x_j\right]&=\alpha\left(1-\mu_j\right)x^{(t-1)e_1}y^\lambda x_j\\&\quad+\sum_{k_1=t-1}\chi_{k,\lambda,u}\left[j'\right]k^*_{j'}x^{k-e_{j'}}y^\lambda\xi^u+h,\end{aligned} \tag{57}$$

where h is the sum of summand that the exponent of x_1 is less than $t-1$. Thus

$$\alpha\left(1-\mu_j\right)x^{(t-1)e_1}y^\lambda x_j+\sum_{k_1=t-1}\chi_{k,\lambda,u}\left[j'\right]k^*_{j'}x^{k-e_{j'}}y^\lambda\xi^u=0. \tag{58}$$

By $\alpha_t(1-\mu_j)x^{(t-1)e_1}y^\lambda x_j$ contains x_j and $\sum_{k_1=t-1}\chi_{k,\lambda,u}[j']k^*_{j'}x^{k-e_{j'}}y^\lambda\xi^u$ dose not contain x_j, we have $t=0$, contradicting $t>0$. Hence Ω is transitive, as required. □

Theorem 17. *Suppose that* Ω *and* Ω' *are the Lie superalgebras of* Ω*-type. If* φ *is an isomorphism of* Ω *onto* Ω'*, then* $\varphi(\Omega_{[i]})=\Omega'_{[i]}$ *for all* $i\geq -2$.

Proof. As the isomorphism φ is an even mapping, we have $\varphi(\Omega_{\bar{0}})=\Omega'_{\bar{0}}$ and $\varphi(\mathrm{Nil}(\Omega_{\bar{0}}))=\mathrm{Nil}(\Omega'_{\bar{0}})$. Hence $\varphi(\mathrm{Nor}_{\Omega_{\bar{0}}}(\mathrm{Nil}(\Omega_{\bar{0}}))=\mathrm{Nor}_{\Omega'_{\bar{0}}}(\mathrm{Nil}(\Omega'_{\bar{0}}))$. By Lemmas 13 and 14, we obtain

$$\begin{aligned}&\varphi\left(\Omega_{[0]}\cap\Omega_{\bar{0}}\right)=\Omega'_{[0]}\cap\Omega'_{\bar{0}},\\&\varphi\left(\Omega_{[1]}\cap\Omega_{\bar{0}}\right)\\&\quad=\varphi\left(\{f\in\mathrm{nil}\left(\Omega_{[0]}\right)\mid\left[f,\Omega_{[0]}\cap\Omega_{\bar{0}}\right]\subseteq\mathrm{nil}\left(\Omega_{[0]}\right)\}\right)\\&\quad=\left\{f\in\mathrm{nil}\left(\Omega'_{[0]}\right)\mid\left[f,\Omega'_{[0]}\cap\Omega'_{\bar{0}}\right]\subseteq\mathrm{nil}\left(\Omega'_{\bar{0}}\right)\right\}\\&\quad=\Omega'_{[1]}\cap\Omega'_{\bar{0}}.\end{aligned} \tag{59}$$

Now by virtue of Lemma 15, we get

$$\begin{aligned}&\varphi\left(\Omega_{[0]}\cap\Omega_{\bar{1}}\right)\\&\quad=\varphi\left(\{f\in\Omega_{[-1]}\cap\Omega_{\bar{1}}\mid\left[f,\Omega_{\bar{0}}\right]\subseteq\Omega_{[-1]}\cap\Omega_{\bar{1}}\}\right)\\&\quad=\left\{f\in\Omega'_{[-1]}\cap\Omega'_{\bar{1}}\mid\left[f,\Omega'_{\bar{0}}\right]\subseteq\Omega'_{[-1]}\cap\Omega'_{\bar{1}}\right\}\\&\quad=\Omega'_{[0]}\cap\Omega'_{\bar{1}},\\&\varphi\left(\Omega_{[-1]}\cap\Omega_{\bar{1}}\right)\\&\quad=\varphi\left(\left[\Omega_{\bar{1}},\Omega_{[1]}\cap\Omega_{\bar{0}}\right]\right)\\&\quad=\left[\Omega'_{\bar{1}},\Omega_{[1]}\cap\Omega'_{\bar{0}}\right]=\Omega'_{[-1]}\cap\Omega'_{\bar{1}}.\end{aligned} \tag{60}$$

It follows that $\varphi(\Omega_{[0]}) = \Omega'_{[0]}$. Because $\Omega_{\bar{0}} \subseteq \Omega_{[-1]}$, we have

$$\begin{aligned}\varphi\left(\Omega_{[-1]}\right) &= \varphi\left(\Omega_{[-1]} \cap \Omega_{\bar{0}} + \Omega_{[-1]} \cap \Omega_{\bar{1}}\right)\\ &= \varphi\left(\Omega_{\bar{0}} + \Omega_{[-1]} \cap \Omega_{\bar{1}}\right) = \Omega'_{\bar{0}} + \Omega'_{[-1]} \cap \Omega'_{\bar{1}}\\ &= \Omega'_{[-1]}.\end{aligned} \tag{61}$$

Since Ω is transitive by the lemma above, we have $\Omega_{[i+1]} = \{f \in \Omega_{[i]} \mid [f, \Omega_{[-1]}] \subseteq \Omega_{[i]}\}$ for all $i \geq 0$. It is easy to show that $\varphi(\Omega_{[i]}) = \Omega'_{[i]}$ by induction on i. □

Theorem 18. *Suppose that ϕ is an automorphism of Ω. Then $\phi(\Omega_{[i]}) = \Omega_{[i]}$ for all $i \geq -2$, that is; the filtration of Ω is invariant under the automorphism group of Ω.*

Proof. This is a direct consequence of Theorem 17. □

Theorem 19. *Let $\Omega(r, m, q)$ and $\Omega(r', m', q')$ be the Lie superalgebras of Ω-type. Then $\Omega(r, m, q) \cong \Omega(r', m', q')$ if and only if $r = r', m = m', q = q'$.*

Proof. The sufficient condition is obvious. We will prove the necessary condition. Assume that $\varphi : \Omega(r, m, q) \to \Omega(r', m', q')$ is an isomorphism of Lie superalgebra. According to Theorem 17, we have

$$\begin{aligned}\varphi\left(\Omega(r, m, q)_{[-2]}\right) &= \Omega\left(r', m', q'\right)_{[-2]},\\ \varphi\left(\Omega(r, m, q)_{[-1]}\right) &= \Omega\left(r', m', q'\right)_{[-1]}.\end{aligned} \tag{62}$$

Then φ induces an isomorphism of $\mathbb{Z}_2$-graded spaces

$$\frac{\Omega(r, m, q)_{[-2]}}{\Omega(r, m, q)_{[-1]}} \longrightarrow \frac{\Omega\left(r', m', q'\right)_{[-2]}}{\Omega(r', m', q')_{[-1]}}. \tag{63}$$

It is easy to see that

$$\begin{aligned}\frac{\Omega(r, m, q)_{[-2]}}{\Omega(r, m, q)_{[-1]}} &\cong \Omega(r, m, q)_{-2},\\ \frac{\Omega\left(r', m', q'\right)_{[-2]}}{\Omega(r', m', q')_{[-1]}} &\cong \Omega\left(r', m', q'\right)_{-2}.\end{aligned} \tag{64}$$

We conclude that $m = m'$ by the dimension comparison.

Similarly, we also obtain an isomorphism of $\mathbb{Z}_2$-graded spaces:

$$\frac{\Omega(r, m, q)_{[-1]}}{\Omega(r, m, q)_{[0]}} \cong \frac{\Omega\left(r', m', q'\right)_{[-1]}}{\Omega(r', m', q')_{[0]}}, \tag{65}$$

and the quotient space $\Omega_{[-1]}/\Omega_{[0]}$ is isomorphic to Ω_{-1}. A comparison of dimensions shows that $r + q = r' + q'$. Note that $\varphi(\Omega(r, m, q)_{\alpha}) = \Omega(r', m', q')_{\alpha}$, where $\alpha \in \mathbb{Z}_2$. It follows that $r = r'$ and $q = q'$. □

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