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Research Article

New Convergence Definitions for Sequences of Sets

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Several notions of *convergence* for subsets of metric space appear in the literature. In this paper, we define *Wijsman I-convergence* and *Wijsman I^*-convergence* for sequences of sets and establish some basic theorems. Furthermore, we introduce the concepts of *Wijsman I-Cauchy* sequence and *Wijsman I^*-Cauchy* sequence and then study their certain properties.

1. Introduction and Background

The concept of convergence of sequences of points has been extended by several authors (see [1–9]) to the concept of convergence of sequences of sets. The one of these such extensions that we will consider in this paper is Wijsman convergence. We will define I-convergence for sequences of sets and establish some basic results regarding these notions.

Let us start with fundamental definitions from the literature. The natural density of a set K of positive integers is defined by

$$\delta(K) := \lim_{n\to\infty}\frac{1}{n}\left|\{k \le n : k \in K\}\right|, \tag{1}$$

where $|k \le n : k \in K|$ denotes the number of elements of K not exceeding n ([10]).

Statistical convergence of sequences of points was introduced by Fast [11]. In [12], Schoenberg established some basic properties of statistical convergence and also studied the concept as a summability method.

A number sequence $x = (x_k)$ is said to be statistically convergent to the number ξ if, for every $\varepsilon > 0$,

$$\lim_{n\to\infty}\frac{1}{n}\left|\{k \le n : |x_k - \xi| \ge \varepsilon\}\right| = 0. \tag{2}$$

In this case, we write $st-\lim x_k = \xi$. Statistical convergence is a natural generalization of ordinary convergence. If $\lim x_k = \xi$, then $st-\lim x_k = \xi$. The converse does not hold in general.

Definition 1 (see [13]). A family of sets $I \subseteq 2^{\mathbb{N}}$ is called an ideal on $\mathbb{N}$ if and only if

(i) $\emptyset \in I$;
(ii) for each $A, B \in I$ one has $A \cup B \in I$;
(iii) for each $A \in I$ and each $B \subseteq A$ one has $B \in I$.

An ideal is called nontrivial if $\mathbb{N} \notin I$, and nontrivial ideal is called admissible if $\{n\} \in I$ for each $n \in \mathbb{N}$.

Definition 2 (see [14]). A family of sets $F \subseteq 2^{\mathbb{N}}$ is a filter in $\mathbb{N}$ if and only if

(i) $\emptyset \notin F$;
(ii) for each $A, B \in F$ one has $A \cap B \in F$;
(iii) for each $A \in F$ and each $B \supseteq A$ one has $B \in F$.

Proposition 3 (see [14]). *I is a nontrivial ideal in $\mathbb{N}$ if and only if*

$$F = F(I) = \{M = \mathbb{N} \setminus A : A \in I\} \tag{3}$$

is a filter in $\mathbb{N}$.

Definition 4 (see [14]). Let I be a nontrivial ideal of subsets of $\mathbb{N}$. A number sequence $(x_n)_{n\in\mathbb{N}}$ is said to be I-convergent to ξ ($\xi = I-\lim_{n\to\infty}x_n$) if and only if for each $\varepsilon > 0$ the set

$$\{k \in \mathbb{N} : |x_k - \xi| \ge \varepsilon\} \tag{4}$$

belongs to I. The element ξ is called the I limit of the number sequence $x = (x_n)_{n\in\mathbb{N}}$.

The concept of I-convergence of real sequences is a generalization of statistical convergence which is based on the structure of the ideal I of subsets of the set of natural numbers. Kostyrko et al. [14] introduced the concept of I-convergence of sequences in a metric space and studied some properties of this convergence. I-convergence of real sequences coincides with the ordinary convergence if I is the ideal of all finite subsets of $\mathbb{N}$ and with the statistical convergence if I is the ideal of subsets of $\mathbb{N}$ of natural density zero.

Definition 5 (see [14]). An admissible ideal $I \subseteq 2^{\mathbb{N}}$ is said to have the property (AP) if for any sequence $\{A_1, A_2, \ldots\}$ of mutually disjoint sets of I, there is sequence $\{B_1, B_2, \ldots\}$ of sets such that each symmetric difference $A_i \Delta B_i$ $(i = 1, 2, \ldots)$ is finite and $\bigcup_{i=1}^{\infty} B_i \in I$.

Definition 5 is similar to the condition (APO) used in [15].

In [14], the concept of I^*-convergence which is closely related to I-convergence has been introduced.

Definition 6 (see [14]). A sequence $x = (x_n)$ of elements of X is said to be I^*-convergence to ξ if and only if there exists a set $M \in F(I)$,

$$M = \{m = (m_i) : m_i < m_{i+1},\ i \in \mathbb{N}\} \subset \mathbb{N} \tag{5}$$

such that $\lim_{k\to\infty} x_{m_k} = \xi$.

In [14], it is proved that I-convergence and I^*-convergence are equivalent for admissible ideals with property (AP).

Also, in order to prove that I-convergent sequence coincides with I^*-convergent sequence for admissible ideals with property (AP), we need the following lemma.

Lemma 7 (see [13]). *Let $\{P_i\}_{i=1}^{\infty}$ be a countable collection of subsets of $\mathbb{N}$ such that $P_i \in F(I)$ is a filter which associates with an admissible ideal I with property (AP). Then there exists a set $P \subset \mathbb{N}$ such that $P \in F(I)$ and the set $P \setminus P_i$ is finite for all i.*

Theorem 8 (see [13]). *Let $I \subseteq 2^{\mathbb{N}}$ be an admissible ideals with property (AP) and $x = (x_n)$ be a number sequence. Then $I - \lim_{n\to\infty} x_n = \xi$ if and only if there exists a set $P \in F(I)$, $P = \{p = (p_i) : p_i < p_{i+1},\ i \in \mathbb{N}\}$ such that $\lim_{k\to\infty} x_{p_k} = \xi$.*

Definition 9 (see [9]). Let (X, d) be a metric space. For any nonempty closed subsets $A, A_k \subseteq X$, one says that the sequence $\{A_k\}$ is Wijsman convergent to A:

$$\lim_{k\to\infty} d(x, A_k) = d(x, A) \tag{6}$$

for each $x \in X$. In this case one writes $W - \lim_{k\to\infty} A_k = A$.

As an example, consider the following sequence of circles in the (x, y)-plane: $A_k = \{(x, y) : x^2 + y^2 + 2kx = 0\}$. As $k \to \infty$ the sequence is Wijsman convergent to the y-axis $A = \{(x, y) : x = 0\}$.

Definition 10 (see [16]). Let (X, d) be a metric space. For any nonempty closed subsets $A, A_k \subseteq X$, one says that the sequence $\{A_k\}$ is Wijsman statistical convergent to A if for $\varepsilon > 0$ and for each $x \in X$,

$$\lim_{n\to\infty} \frac{1}{n} \left|\{k \le n : |d(x, A_k) - d(x, A)| \ge \varepsilon\}\right| = 0. \tag{7}$$

In this case one writes $st - \lim_W A_k = A$ or $A_k \to A(\mathrm{WS})$. Consider

$$\mathrm{WS} := \left\{\{A_k\} : st - \lim_W A_k = A\right\}, \tag{8}$$

where WS denotes the set of Wijsman statistical convergence sequences.

Also the concept of bounded sequence for sequences of sets was given by Nuray and Rhoades [16] as follows.

Let (X, ρ) be a metric space. For any nonempty closed subsets A_k of X, one says that the sequence $\{A_k\}$ is bounded if $\sup_k d(x, A_k) < \infty$ for each $x \in X$.

2. Wijsman I-Convergence

In this section, we will define Wijsman I-convergence and Wijsman I^*-convergence of sequences of sets, give the relationship between them, and establish some basic theorems.

Definition 11. Let (X, d) be a metric space and $I \subseteq 2^{\mathbb{N}}$ be a proper ideal in $\mathbb{N}$. For any nonempty closed subsets $A, A_k \subset X$, one says that the sequence $\{A_k\}$ is Wijsman I-convergent to A, if, for each $\varepsilon > 0$ and for each $x \in X$, the set

$$A(x, \varepsilon) = \{k \in \mathbb{N} : |d(x, A_k) - d(x, A)| \ge \varepsilon\} \tag{9}$$

belongs to I. In this case, one writes $I_W - \lim A_k = A$ or $A_k \to A(I_W)$, and the set of Wijsman I-convergent sequences of sets will be denoted by

$$I_W = \{\{A_k\} : \{k \in \mathbb{N} : |d(x, A_k) - d(x, A)| \ge \varepsilon\} \in I\}. \tag{10}$$

Example 12. $I \subseteq 2^{\mathbb{N}}$ be a proper ideal in $\mathbb{N}$, (X, d) a metric space, and $A, A_k \subset X$ nonempty closed subsets. Let $X = \mathbb{R}^2$, $\{A_k\}$ be following sequence:

$$A_k = \begin{cases} \left\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 - 2ky = 0\right\} & \text{if, } k \neq n^2 \\ \left\{(x, y) \in \mathbb{R}^2 : y = -1\right\} & \text{if, } k = n^2, \end{cases}$$
$$A = \left\{(x, y) \in \mathbb{R}^2 : y = 0\right\}. \tag{11}$$

For $k = n^2$, $d((x, y), A_{n^2}) = |y+1| \neq d((x, y); A) = |y|$. Let us take a point (x^*, y^*) outside $x^2 + y^2 - 2ky = 0$. For $k \neq n^2$, we write $d((x^*, y^*), A_k) \to d((x^*, y^*), A) = |y^*|$. Since the line equation is

$$\frac{x - 0}{x^*} = \frac{y - k}{y^* - k}, \tag{12}$$

where the line is passing from $(0, k)$ the center point of the circle and (x^*, y^*) the outside of the circle, we write $y = k +$

$((y^* - k)/x^*) \cdot x$. If we write this $y = k + ((y^* - k)/x^*) \cdot x$ value on the circle equation $x^2 + y^2 - 2ky = 0$, we can get

$$x = \frac{|k| \cdot x^*}{\sqrt{(x^*)^2 + (y^* - k)^2}}. \tag{13}$$

For $k \to \infty$, if we take limit, it will be $x \to x^*$. If we write $x = (|k| \cdot x^*)/\sqrt{(x^*)^2 + (y^* - k)^2}$ on the $y = k + ((y^* - k)/x^*) \cdot x$, we get $y \to 0$ $(k \to \infty)$. Thus, for $k \neq n^2$

$$d((x^*, y^*), A_k) = \sqrt{(x - x^*)^2 + (y - y^*)^2} \longrightarrow |y^*|. \tag{14}$$

So we get $d((x^*, y^*), A_k) \to d((x^*, y^*), A) = |y^*|$, for $k \neq n^2$.

For $k = n^2$ and $k \neq n^2$, the set sequence $\{A_k\}$ has two different limits. Thus $\{A_k\}$ is not Wijsman convergent to set A, but

$$\begin{aligned} &\{k \in \mathbb{N} : |d((x, y), A_k) - d((x, y), A)| \geq \varepsilon\} \\ &\quad = \{k \in \mathbb{N} : k = n^2\} \subset I_d. \end{aligned} \tag{15}$$

Thus, suppose that

$$A(x, y, \varepsilon) = \{k \in \mathbb{N} : |d((x, y), A_k) - d((x, y), A)| \geq \varepsilon\} \tag{16}$$

for $\varepsilon > 0$ and for each $(x, y) \in \mathbb{R}^2$.

Since $\lim_{k \to \infty}[|d((x, y), A_k) - d((x, y), A)|] = 0$, for $k \neq n^2$, for each $\varepsilon > 0$,

$$\exists k_\varepsilon \in \mathbb{N} : \forall k > k_\varepsilon : |d((x, y), A_k) - d((x, y), A)| < \varepsilon. \tag{17}$$

Define the set $A_{k_\varepsilon}(x, y)$ as

$$A_{k_\varepsilon}(x, y) := \{k \in \mathbb{N} : |d((x, y), A_k) - d((x, y), A)| > \varepsilon\}. \tag{18}$$

Thus, since $A(x, y, \varepsilon) = A_{k_\varepsilon}(x, y) \cup \{k \in \mathbb{N} : k = n^2\}$ and $A_{k_\varepsilon}(x, y) \in I_d$ and $\{k \in \mathbb{N} : k = n^2\} \in I_d$, we can write

$$\begin{aligned} &A(x, y, \varepsilon) \\ &\quad := \{k \in \mathbb{N} : |d((x, y), A_k) - d((x, y), A)| > \varepsilon\} \in I_d, \end{aligned} \tag{19}$$

where $I_d = \{A : \delta(A) = 0\}$. So the set sequence $\{A_n\}$ is Wijsman I-convergent to set A.

Example 13. Let $I \subseteq 2^{\mathbb{N}}$ be a proper ideal in $\mathbb{N}$, (X, d) a metric space, and $A, A_n \subset X$ nonempty closed subsets. Let $X = \mathbb{R}^2$, $\{A_n\}$ be following sequence:

$$\begin{aligned} &A_n \\ &= \begin{cases} \left\{(x, y) \in \mathbb{R}^2 : 0 \leq x \leq n, 0 \leq y \leq \frac{1}{n} \cdot x\right\}, & \text{if, } n \neq k^2 \\ \left\{(x, y) \in \mathbb{R}^2 : x \geq 0, y = 1\right\}, & \text{if, } n = k^2, \end{cases} \\ &\qquad A = \left\{(x, y) \in \mathbb{R}^2 : x \geq 0, y = 0\right\}. \end{aligned} \tag{20}$$

Since

$$\lim_{n \to \infty} \frac{1}{n} \left|\{k \leq n : |d((x, y), A_n) - d((x, y), A)| \geq \varepsilon\}\right| = 0, \tag{21}$$

the set sequence $\{A_n\}$ is Wijsman statistical convergent to set A. Thus we can write $st - \lim_W A_n = A$, but this sequence is not Wijsman convergent to set A. Because for $n \neq k^2$, $\lim_{n \to \infty} d((x, y), A_n) = d((x, y), A)$, but for $n = k^2$, $\lim_{n \to \infty} d((x, y), A_n) \neq d((x, y), A)$. Let $I_d \subset 2^{\mathbb{N}}$ be proper ideal. Define set K as

$$K = K(\varepsilon) = \{n \in \mathbb{N} : |d((x, y), A_n) - d((x, y), A)| \geq \varepsilon\}. \tag{22}$$

If we take I_d for I, Wijsman ideal convergent coincides with Wijsman statistical convergent. Really, one has

$$\begin{aligned} &\{n \in \mathbb{N} : |d((x, y), A_n) - d((x, y), A)| \geq \varepsilon\} \\ &\quad = \{n \in \mathbb{N} : n = k^2\} \subset I_d. \end{aligned} \tag{23}$$

Since the Wijsman topology is not first countable in general, if $\{A_k\}$ is convergent to the set A Wijsman sense, every subsequence of $\{A_k\}$ may not be convergent to A. But if X is separable, then every subsequence of a convergent set sequence is convergent to the same limit.

Definition 14. Let $I \subseteq 2^{\mathbb{N}}$ be a proper ideal in $\mathbb{N}$ and (X, d) be a separable metric space. For any nonempty closed subsets $A, A_k \subset X$, one says that the sequence $\{A_k\}$ is Wijsman I^*-convergent to A, if and only if there exists a set $M \in F(I)$, $M = \{m = (m_i) : m_i < m_{i+1}, i \in \mathbb{N}\} \subset \mathbb{N}$ such that for each $x \in X$

$$\lim_{k \to \infty} d(x, A_{m_k}) = d(x, A). \tag{24}$$

In this case, one writes $I_W^* - \lim A_k = A$.

Definition 15. Let $I \subseteq 2^{\mathbb{N}}$ be an admissible ideal in $\mathbb{N}$ and (X, d) be a separable metric space. For any nonempty closed subset A_n in X, one says that the sequence $\{A_n\}$ is Wijsman I-Cauchy sequence if for each $\varepsilon > 0$ and for each $x \in X$, there exists a number $N = N(\varepsilon)$ such that

$$\{n \in \mathbb{N} : |d(x, A_n) - d(x, A_N)| \geq \varepsilon\} \tag{25}$$

belongs to I.

Definition 16. Let $I \subseteq 2^{\mathbb{N}}$ be an admissible ideal in $\mathbb{N}$ and (X, d) be a separable metric space. For any nonempty closed subsets $A_k \subset X$, one says that the sequence $\{A_k\}$ is Wijsman I^*-Cauchy sequences if there exists a set $M = \{m = (m_i) : m_i < m_{i+1}, i \in \mathbb{N}\} \subset \mathbb{N}$, $M \in F(I)$ such that the subsequence $A_M = \{A_{m_k}\}$ is Wijsman Cauchy in X; that is,

$$\lim_{k, p \to \infty} \left|d(x, A_{m_k}) - d(x, A_{m_p})\right| = 0. \tag{26}$$

Now we will prove that Wijsman I-convergence implies the Wijsman I-Cauchy condition.

Theorem 17. *Let I be an arbitrary admissible ideal and let X be a separable metric space. Then $I_W - \lim A_n = A$ implies that $\{A_n\}$ is Wijsman I-Cauchy sequence.*

Proof. Let I be an arbitrary admissible ideal and $I_W - \lim A_n = A$. Then for each $\varepsilon > 0$ and for each $x \in X$, we have

$$A(x,\varepsilon) = \{n \in \mathbb{N} : |d(x, A_n) - d(x, A)| \geq \varepsilon\} \quad (27)$$

that belongs to I. Since I is an admissible ideal, there exists an $n_0 \in \mathbb{N}$ such that $n_0 \notin A(x,\varepsilon)$.

Let $B(x,\varepsilon) = \{n \in \mathbb{N} : |d(x, A_n) - d(x, A_{n_0})| \geq 2\varepsilon\}$. Taking into account the inequality

$$\begin{aligned} &\left|d(x, A_n) - d\left(x, A_{n_0}\right)\right| \\ &\quad \leq |d(x, A_n) - d(x, A)| + \left|d\left(x, A_{n_0}\right) - d(x, A)\right|, \end{aligned} \quad (28)$$

we observe that if $n \in B(x,\varepsilon)$, then

$$|d(x, A_n) - d(x, A)| + \left|d\left(x, A_{n_0}\right) - d(x, A)\right| \geq 2\varepsilon. \quad (29)$$

On the other hand, since $n_0 \notin A(x,\varepsilon)$, we have $|d(x, A_{n_0}) - d(x, A)| < \varepsilon$. Here we conclude that $|d(x, A_n) - d(x, A)| \geq \varepsilon$; hence $n \in A(x,\varepsilon)$. Observe that $B(x,\varepsilon) \subset A(x,\varepsilon) \in I$ for each $\varepsilon > 0$ and for each $x \in X$. This gives that $B(x,\varepsilon) \in I$; that is $\{A_n\}$ is Wijsman I-Cauchy sequence. □

Theorem 18. *Let I be an admissible ideal and let X be a separable metric space. If $\{A_n\}$ is Wijsman I^*-Cauchy sequence, then it is Wijsman I-Cauchy sequence.*

Proof. Let $\{A_n\}$ be Wijsman I^*-Cauchy sequence; then by the definition, there exists a set $M = \{m = (m_i) : m_i < m_{i+1},\ i \in \mathbb{N}\} \subset \mathbb{N}$, $M \in F(I)$ such that

$$\left|d\left(x, A_{m_k}\right) - d\left(x, A_{m_p}\right)\right| < \varepsilon \quad (30)$$

for each $\varepsilon > 0$, for each $x \in X$, and for all $k, p > k_0 = k_0(\varepsilon)$.

Let $N = N(\varepsilon) = m_{k_0+1}$. Then for every $\varepsilon > 0$, we have

$$\left|d\left(x, A_{m_k}\right) - d(x, A_N)\right| < \varepsilon, \quad k > k_0. \quad (31)$$

Now let $H = \mathbb{N} \setminus M$. It is clear that $H \in I$ and that

$$\begin{aligned} A(x,\varepsilon) = &\ \{n \in \mathbb{N} : |d(x, A_n) - d(x, A_N)| \geq \varepsilon\} \\ &\subset H \cup \left\{m_1, m_2, \ldots, m_{k_0}\right\} \end{aligned} \quad (32)$$

belongs to I. Therefore, for every $\varepsilon > 0$, we can find a $N = N(\varepsilon)$ such that $A(x,\varepsilon) \in I$; that is, $\{A_n\}$ is Wijsman I-Cauchy sequence. Hence the proof is complete. □

In order to prove that Wijsman I-convergent sequence coincides with Wijsman I^*-convergent sequence for admissible ideals with property (AP), we need the following lemma.

Lemma 19. *Let $I \subseteq 2^{\mathbb{N}}$ be an admissible ideal with property (AP) and (X, d) a separable metric space. If $I_W - \lim_{n\to\infty} d(x, A_n) = d(x, A)$, then there exists a set $P \in F(I)$ $P = \{p = (p_i) : p_i < p_{i+1},\ i \in \mathbb{N}\}$ such that $I_W - \lim_{k\to\infty} d(x, A_{p_k}) = d(x, A)$.*

Theorem 20. *Let $I \subseteq 2^{\mathbb{N}}$ be an admissible ideal with property (AP), let (X, d) be an arbitrary separable metric space and $x = (x_n) \in X$. Then, $I_W - \lim_{n\to\infty} d(x, A_n) = d(x, A)$, if and only if there exists a set $P \in F(I)$, $P = \{p = (p_i) : p_i < p_{i+1},\ i \in \mathbb{N}\}$ such that $I_W - \lim_{k\to\infty} d(x, A_{p_k}) = d(x, A)$.*

Now we prove that, a Wijsman I-Cauchy sequence coincides with a Wijsman I^*-Cauchy sequence for admissible ideals with property (AP).

Theorem 21. *If $I \subseteq 2^{\mathbb{N}}$ is an admissible ideal with property (AP) and if (X, d) is a separable metric space, then the concepts Wijsman I-Cauchy sequence and Wijsman I^*-Cauchy sequence coincide.*

Proof. If a sequence is Wijsman I^*-Cauchy, then it is Wijsman I-Cauchy by Theorem 18 where I does not need to have the (AP) property. Now it is sufficient to prove that $\{A_n\}$ is Wijsman I^*-Cauchy sequence in X under assumption that $\{A_n\}$ is a Wijsman I-Cauchy sequence. Let $\{A_n\}$ be a Wijsman I-Cauchy sequence. Then by definition, there exists a $N = N(\varepsilon)$ such that

$$A(x,\varepsilon) = \{n \in \mathbb{N} : |d(x, A_n) - d(x, A_N)| \geq \varepsilon\} \in I \quad (33)$$

for each $\varepsilon > 0$ and for each $x \in X$.

Let $P_i = \{n \in \mathbb{N} : |d(x, A_n) - d(x, A_{m_i})| < 1/i\}$, $i = 1, 2, \ldots$ where $m_i = N(1/i)$. It is clear that $P_i \in F(I)$ for $i = 1, 2, \ldots$. Since I has (AP) property, then by Lemma 7 there exists a set $P \subset \mathbb{N}$ such that $P \in F(I)$ and $P \setminus P_i$ is finite for all i. Now we show that

$$\lim_{n,m\to\infty} |d(x, A_n) - d(x, A_m)| = 0. \quad (34)$$

To prove this, let $\varepsilon > 0$, $x \in X$, and $j \in \mathbb{N}$ such that $j > 2/\varepsilon$. If $m, n \in P$ then $P \setminus P_i$ is finite set, therefore there exists $k = k(j)$ such that

$$\begin{aligned} &\left|d(x, A_n) - d\left(x, A_{m_j}\right)\right| < \frac{1}{j}, \\ &\left|d(x, A_m) - d\left(x, A_{m_j}\right)\right| < \frac{1}{j} \end{aligned} \quad (35)$$

for all $m, n > k(j)$. Hence it follows that

$$\begin{aligned} |d(x, A_n) - d(x, A_m)| < &\left|d(x, A_n) - d\left(x, A_{m_j}\right)\right| \\ &+ \left|d(x, A_m) - d\left(x, A_{m_j}\right)\right| < \varepsilon \end{aligned} \quad (36)$$

for $m, n > k(j)$.

Thus, for any $\varepsilon > 0$, there exists $k = k(\varepsilon)$ and $n, m \in P \in F(I)$:

$$|d(x, A_n) - d(x, A_m)| < \varepsilon. \quad (37)$$

This shows that the sequences $\{A_n\}$ is a Wijsman I^*-Cauchy sequence. □

Theorem 22. *Let I be an admissible ideal and (X, d) a separable metric space. Then $I_W^* - \lim A_k = A$ implies that $\{A_n\}$ is a Wijsman I-Cauchy sequence.*

Proof. Let $I_W^* - \lim A_k = A$. Then by definition there exists a set $M \in F(I)$, $M = \{m = (m_i) : m_i < m_{i+1},\ i \in \mathbb{N}\} \subset \mathbb{N}$ such that

$$\lim_{k\to\infty} d\left(x, A_{m_k}\right) = d(x, A) \tag{38}$$

for each $\varepsilon > 0$ and for each $x \in X$, and $k, p > k_0$,

$$\begin{aligned} &\left|d\left(x, A_{m_k}\right) - d\left(x, A_{m_p}\right)\right| \\ &\quad < \left|d\left(x, A_{m_k}\right) - d(x, A)\right| + \left|d\left(x, A_{m_p}\right) - d(x, A)\right| \\ &\quad < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned} \tag{39}$$

Therefore,

$$\lim_{k,p\to\infty} \left|d\left(x, A_{m_k}\right) - d\left(x, A_{m_p}\right)\right| = 0. \tag{40}$$

Hence, $\{A_n\}$ is a Wijsman I-Cauchy sequence. □

Theorem 23. *Let I be an admissible ideal and (X, d) a separable metric space. If the ideal I has property (AP) and if (X, d) is an arbitrary metric space, then for arbitrary sequence $\{A_n\}_{n\in\mathbb{N}}$ of elements of X $I_W - \lim A_n = A$ implies $I_W^* - \lim A_n = A$.*

Proof. Suppose that I satisfies condition (AP). Let $I_W - \lim A_n = A$. Then

$$T(\varepsilon, x) = \left\{n \in \mathbb{N} : \left|d\left(x, A_n\right) - d(x, A)\right| \geq \varepsilon\right\} \in I \tag{41}$$

for each $\varepsilon > 0$ and for each $x \in X$. Put

$$\begin{aligned} T_1 &= \left\{n \in \mathbb{N} : \left|d\left(x, A_n\right) - d(x, A)\right| \geq 1\right\}, \\ T_n &= \left\{n \in \mathbb{N} : \frac{1}{n} \leq \left|d\left(x, A_n\right) - d(x, A)\right| < \frac{1}{n-1}\right\} \end{aligned} \tag{42}$$

for $n \geq 2$, and $n \in \mathbb{N}$. Obviously $T_i \cap T_j = \emptyset$ for $i \neq j$. By condition (AP) there exists a sequence of sets $\{V_n\}_{n\in\mathbb{N}}$ such that $T_j \Delta V_j$ are finite sets for $j \in \mathbb{N}$ and $V = \bigcup_{j=1}^{\infty} V_j \in I$. It is sufficient to prove that for $M = \mathbb{N} \setminus V$, $M = \{m = (m_i) : m_i < m_{i+1},\ i \in \mathbb{N}\} \in F(I)$, we have $\lim_{k\to\infty} d(x, A_{m_k}) = d(x, A)$.

Let $\gamma > 0$. Choose $k \in \mathbb{N}$ such that $1/(k+1) < \gamma$. Then

$$\left\{n \in \mathbb{N} : \left|d\left(x, A_n\right) - d(x, A)\right| \geq \gamma\right\} \subset \bigcup_{j=1}^{k+1} T_j. \tag{43}$$

Since $T_j \Delta V_j$, $j = 1, 2, \ldots$ are finite sets, there exists $n_0 \in \mathbb{N}$ such that

$$\begin{aligned} &\left(\bigcup_{j=1}^{k+1} V_j\right) \cap \left\{n \in \mathbb{N} : n > n_0\right\} \\ &\quad = \left(\bigcup_{j=1}^{k+1} T_j\right) \cap \left\{n \in \mathbb{N} : n > n_0\right\}. \end{aligned} \tag{44}$$

If $n > n_0$ and $n \notin V$, so $n \notin \bigcup_{j=1}^{k+1} V_j$ and by (44) $n \notin \bigcup_{j=1}^{k+1} T_j$. But then $|d(x, A_n) - d(x, A)| < 1/(n+1) < \gamma$ for each $x \in X$, so we have $\lim_{k\to\infty} d(x, A_{m_k}) = d(x, A)$. □

3. Wijsman I-Limit Points and Wijsman I-Cluster Points Sequences of Sets

In this section, we introduce Wijsman I-limit points of sequences of sets and Wijsman I-cluster points of sequences of sets, prove some basic properties of these concepts, and establish some basic theorems.

Definition 24. Let $I \subseteq 2^{\mathbb{N}}$ a proper ideal in $\mathbb{N}$ and (X, d) a separable metric space. For any nonempty closed subsets A_n, $B_n \subset X$, one says that the sequences $\{A_n\}$ and $\{B_n\}$ are almost equal with respect to I if

$$\left\{n \in \mathbb{N} : A_n \neq B_n\right\} \in I, \tag{45}$$

and we write I-a.a.n $A_n = B_n$.

Definition 25. Let $I \subseteq 2^{\mathbb{N}}$ be a proper ideal in $\mathbb{N}$ and let (X, d) be a separable metric space; A_n is nonempty closed subset of X. If $\{A_n\}_K$ is subsequence of $\{A_n\}$ and $K := \{n(j) : j \in \mathbb{N}\}$, then we abbreviate $\{A_{n_j}\}$ by $\{A_n\}_K$. If $K \in I$, then $\{A_n\}_K$ subsequence is called thin subsequence of $\{A_n\}$. If $K \notin I$, then $\{A_n\}_K$ subsequence is called nonthin subsequence of $\{A_n\}$.

Definition 26. Let $I \subseteq 2^{\mathbb{N}}$ be a proper ideal in $\mathbb{N}$ and let (X, d) be a separable metric space, for any nonempty closed subsets $A_k \subset X$. One has the following.

(i) $A \in X$ is said to be a Wijsman I-limit point of $\{A_n\}$ provided that there is a set $M = \{m = (m_i) : m_i < m_{i+1},\ i \in \mathbb{N}\} \subset \mathbb{N}$ such that $M \notin I$ and for each $x \in X$ $\lim_{k\to\infty} d(x, A_{m_k}) = d(x, A)$.

(ii) $A \in X$ is said to be a Wijsman I-cluster point of $\{A_n\}$ if and only if for each $\varepsilon > 0$, for each $x \in X$, we have

$$\left\{n \in \mathbb{N} : \left|d\left(x, A_n\right) - d(x, A)\right| < \varepsilon\right\} \notin I. \tag{46}$$

Denote by $I_W(\Lambda_{\{A_n\}})$, $I_W(\Gamma_{\{A_n\}})$, and $L_{\{A_n\}}$ the set of all Wijsman I-limit, Wijsman I-cluster, and Wijsman limit points of $\{A_n\}$, respectively.

For the sequences $\{A_n\}$, $I_W(\Gamma_{\{A_n\}}) \subseteq I_W(L_{\{A_n\}})$. Let $A \in I_W(\Gamma_{\{A_n\}})$. Then for each sequence $\{A_n\} \subset X$, we have $\lim_{k\to\infty} d(x, A_{m_k}) = d(x, A)$ which means that $A \in L_{\{A_n\}}$.

Theorem 27. *Let $I \subseteq 2^{\mathbb{N}}$ be a proper ideal in $\mathbb{N}$ and let (X, d) be a separable metric space. Then for each sequence $\{A_n\} \subset X$ one has $I_W(\Lambda_{\{A_n\}}) \subset I_W(\Gamma_{\{A_n\}})$.*

Proof. Let $A \in I_W(\Lambda_{\{A_n\}})$. Then, there exists $M = \{m_1 < m_2 < \cdots\} \subset \mathbb{N}$ such that $M = \{m = (m_i) : m_i < m_{i+1},\ i \in \mathbb{N}\} \notin I$ and

$$\lim_{k\to\infty} d\left(x, A_{m_k}\right) = d(x, A). \tag{47}$$

According to (47), there exists $k_0 \in \mathbb{N}$ such that for each $\varepsilon > 0$, for each $x \in X$ and $k > k_0$, $|d(x, A_{m_k}) - d(x, A)| < \varepsilon$. Hence,

$$\begin{aligned}\left\{k \in \mathbb{N} : \left|d\left(x, A_{m_k}\right) - d\left(x, A\right)\right| < \varepsilon\right\} \\ \supseteq M \setminus \left\{m_1, m_2, \ldots, m_{k_o}\right\}.\end{aligned} \tag{48}$$

Then, the set on the right hand side of (48) does not belong to I; therefore

$$\left\{k \in \mathbb{N} : \left|d\left(x, A_{m_k}\right) - d\left(x, A\right)\right| < \varepsilon\right\} \notin I \tag{49}$$

which means that $A \in I_W(\Gamma_{\{A_n\}})$. □

Theorem 28. *Let $I \subseteq 2^{\mathbb{N}}$ be a proper ideal in $\mathbb{N}$ and let (X, d) be a separable metric space. Then for each sequence $\{A_n\} \subset X$ one has $I_W(\Gamma_{\{A_n\}}) \subseteq L_{\{A_n\}}$.*

Proof. Let $A \in I_W(\Gamma_{\{A_n\}})$. Then for each $\varepsilon > 0$ and for each $x \in X$, we have

$$\left\{n \in \mathbb{N} : \left|d\left(x, A_n\right) - d\left(x, A\right)\right| < \varepsilon\right\} \notin I. \tag{50}$$

Let

$$K_n := \left\{n \in \mathbb{N} : \left|d\left(x, A_n\right) - d\left(x, A\right)\right| < \frac{1}{n}\right\} \tag{51}$$

for $n \in \mathbb{N}$. $\{K_n\}_{n=1}^{\infty}$ is decreasing sequence of infinite subsets of $\mathbb{N}$. Hence $K = \{n = (n_i) : n_i < n_{i+1},\ i \in \mathbb{N}\} \notin I$ such that $\lim_{n\to\infty} d(x, A_{n_i}) = d(x, A)$ which means that $A \in L_{\{A_n\}}$. □

Theorem 29. *Let $I \subseteq 2^{\mathbb{N}}$ a proper ideal in $\mathbb{N}$, (X, d) a separable metric space, and A_k, B_k nonempty subsets of X. If $\{A_k\} = \{B_k\}$ I-a.a.k for $k \in \mathbb{N}$, then $I_W(\Gamma_{\{A_k\}}) = I_W(\Gamma_{\{B_k\}})$ and $I_W(\Lambda_{\{A_k\}}) = I_W(\Lambda_{\{B_k\}})$.*

Proof. If $\{A_k\} = \{B_k\}$ a.a.k for $k \in \mathbb{N}$, then

$$K := \left\{k \in \mathbb{N} : A_k \neq B_k\right\} \in I \tag{52}$$

Let $A \in I_W(\Gamma_{\{A_k\}})$. For each $\varepsilon > 0$ and for each $x \in X$ we have

$$\left\{k \in \mathbb{N} : \left|d\left(x, A_k\right) - d\left(x, A\right)\right| < \varepsilon\right\} \notin I, \tag{53}$$

$\forall \varepsilon > 0$. If $\{A_k\} = \{B_k\}$ I-a.a.k, then $\{k \in \mathbb{N} : |d(x, B_k) - d(x, A)| < \varepsilon\} \notin I$ which means that $A \in I_W(\Gamma_{\{B_k\}})$; hence $I_W(\Gamma_{\{A_k\}} \subset I_W(\Gamma_{\{B_k\}})$. Similarly we can also prove that $I_W(\Gamma_{\{B_k\}}) \subset I_W(\Gamma_{\{A_k\}}$. So we have $I_W(\Gamma_{\{A_k\}} = I_W(\Gamma_{\{B_k\}})$.

Now, we show that $I_W(\Lambda_{\{A_k\}}) = I_W(\Lambda_{\{B_k\}})$. Let $A \in I_W(\Lambda_{\{A_k\}})$. Then there exists a set $M = \{m = (m_i) : m_i < m_{i+1},\ i \in \mathbb{N}\} \subset \mathbb{N}$ such that $M \notin I$ and

$$\begin{gathered}\lim_{k\to\infty} d\left(x, A_{m_k}\right) = d\left(x, A\right), \\ M = \left\{k : k \in M \text{ and } A_k \neq B_k\right\} \\ \cup \left\{k : k \in M \text{ and } A_k = B_k\right\},\end{gathered} \tag{54}$$

$M \notin I$, and hence $\{k : k \in M \text{ and } A_k = B_k\} \notin I$. Then there exists

$$P = \left\{p = \left(p_i\right) : p_i < p_{i+1},\ i \in \mathbb{N}\right\} \notin I \tag{55}$$

such that

$$\lim_{k\to\infty} d\left(x, B_{p_k}\right) = d\left(x, A\right) \tag{56}$$

which means that $A \in I_W(\Lambda_{\{B_k\}})$. Similarly we can also prove that $I_W(\Lambda_{\{B_k\}}) \subset I_W(\Lambda_{\{A_k\}})$. Therefore we have $I_W(\Lambda_{\{A_k\}}) = I_W(\Lambda_{\{B_k\}})$. □

References

[1] J.-P. Aubin and H. Frankowska, *Set-Valued Analysis*, vol. 2, Birkhäuser, Boston, Mass, USA, 1990.

[2] M. Baronti and P. L. Papini, "Convergence of sequences of sets," in *Methods of Functional Analysis in Approximation Theory*, vol. 76, pp. 135–155, Birkhäuser, Basel, Switzerland, 1986.

[3] G. Beer, "Convergence of continuous linear functionals and their level sets," *Archiv der Mathematik*, vol. 52, no. 5, pp. 482–491, 1989.

[4] G. Beer, "On convergence of closed sets in a metric space and distance functions," *Bulletin of the Australian Mathematical Society*, vol. 31, no. 3, pp. 421–432, 1985.

[5] J. Borwein and J. Vanderwerff, "Dual Kadec-Klee norms and the relationships between Wijsman, slice, and Mosco convergence," *The Michigan Mathematical Journal*, vol. 41, no. 2, pp. 371–387, 1994.

[6] O. Kisi and F. Nuray, "On S_{λ}^{L} (I)-asymptotically statistical equivalence of sequences of sets," *Mathematical Analysis*, vol. 2013, Article ID 602963, 6 pages, 2013.

[7] Y. Sonntag and C. Zălinescu, "Set convergences: an attempt of classification," *Transactions of the American Mathematical Society*, vol. 340, no. 1, pp. 199–226, 1993.

[8] R. A. Wijsman, "Convergence of sequences of convex sets, cones and functions," *Bulletin of the American Mathematical Society*, vol. 70, pp. 186–188, 1964.

[9] R. A. Wijsman, "Convergence of sequences of convex sets, cones and functions. II," *Transactions of the American Mathematical Society*, vol. 123, pp. 32–45, 1966.

[10] R. C. Buck, "Generalized asymptotic density," *The American Journal of Mathematics*, vol. 75, pp. 335–346, 1953.

[11] H. Fast, "Sur la convergence statistique," *Colloquium Mathematicae*, vol. 2, pp. 241–244, 1951.

[12] I. J. Schoenberg, "The integrability of certain functions and related summability methods," *The American Mathematical Monthly*, vol. 66, pp. 361–375, 1959.

[13] P. Kostyrko, M. Mačaj, T. Šalát, and M. Sleziak, "I-convergence and extremal I-limit points," *Mathematica Slovaca*, vol. 55, no. 4, pp. 443–464, 2005.

[14] P. Kostyrko, T. Šalát, and W. Wilczyński, "I-convergence," *Real Analysis Exchange*, vol. 26, no. 2, pp. 669–685, 2000.

[15] A. R. Freedman and J. J. Sember, "Densities and summability," *Pacific Journal of Mathematics*, vol. 95, no. 2, pp. 293–305, 1981.

[16] F. Nuray and B. E. Rhoades, "Statistical convergence of sequences of sets," *Fasciculi Mathematici*, no. 49, pp. 87–99, 2012.