Research Article

On the Generalized Krätzel Transform and Its Extension to Bohemian Spaces

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We investigate the Krätzel transform on certain class of generalized functions. We propose operations that lead to the construction of desired spaces of generalized functions. The Krätzel transform is extended and some of its properties are obtained.

1. Introduction

In recent years, integral transforms of Bohemian have comprised an active area of research. Several integral transforms are extended to various spaces of Bohemian, especially, that permit a factorization property of Fourier convolution type.

On the other hand, several integral transforms that have not permitted a factorization property of Fourier convolution type are also extended to various spaces of Bohemian. In the sequence of these integral transforms, the Krätzel transform [1]

$$\left(\mathcal{K}_\nu^\rho f \right)(x) = \int_{\mathbb{R}_+} \mathcal{Z}_\rho^\nu (xy) f(y) \, dy, \tag{1}$$

where

$$\mathcal{Z}_\rho^\nu (xy) = \int_{\mathbb{R}_+} t^{\nu-1} e^{-t^\rho - xy/t} \, dt, \tag{2}$$

$\rho > 0 \ (\in \mathbb{N}), \nu \in \mathbb{C}, x > 0$ is extended to certain space of ultra-Bohemian, denoted by $\mathscr{S}_+(l^r, \alpha, (\alpha_i), a)$ and $\mathscr{S}_+(l^r, \alpha, \{\alpha_i\}, a)$, $1 \le r \le \infty$, respectively.

Another form of the Krätzel transform was initially introduced in [2], and defined as a generalization of the Laplace transform in [3] as

$$\left(l_\nu^{(\rho)} f \right)(x) = \int_{\mathbb{R}_+} \lambda_\nu^{(\rho)} (xy) f(y) \, dy, \tag{3}$$

where

$$\lambda_\nu^{(\rho)}(x) = \frac{(2\pi)^{((\rho-1)/2)\rho(1/2)}}{T(\nu+1-(1/\rho))} \\ \times \left(\frac{x}{\rho} \right)^{\rho\nu} \int_1^\infty \left(t^\rho - 1 \right)^{\nu-(1/\rho)} e^{-xt} \, dt, \tag{4}$$

for $\rho \in \mathbb{N}$ and $\operatorname{Re} \nu > -1 + (1/\rho), x > 0$. The $l_\nu^{(\rho)}$ transform was extended to generalized functions in [3] and to distributions in [4]. By $l^\rho(X)$, we denote the space of equivalence classes of measurable functions $f : X \to \mathbb{R}$ such that

$$\int_X |f|^\rho \, d\mu < \infty, \tag{5}$$

where two measurable functions are equivalent when they are equal to μ a.e.

For $l^\rho(X)$, the l^ρ-norm is defined as

$$\|f\|_{l^\rho} = \left(\int_X |f|^\rho \, d\mu \right)^{1/\rho}. \tag{6}$$

The Banach space $l_{\rho,\mu}$ of Lebesgue measurable functions is defined by [3, page 446]

$$\|f\|_{\rho,\mu} = \left\| x^{-\mu} f \right\|_\rho < \infty. \tag{7}$$

Due to [3, Proposition 2.1], we state the following theorem.

Theorem 1. *Let $1 \le \rho \le \infty, \mu, \ v \in \mathbb{C}, \ \rho > 0, (1/\rho) + (1/\rho') = 1$ and $\operatorname{Re}\mu > -(1/\rho') - \min\{0, \rho\operatorname{Re} v\}$ and then $l_v^{(\rho)}$ is a continuous linear mapping from $l_{\rho,\mu}$ into $l_{\rho,2/(\rho-\mu-1)}$.*

Note that we always assume that the hypothesis of Theorem 1 is satisfied. By $\curlyvee$ we denote the Mellin-type convolution product of first order defined by [5]

$$(f \curlyvee \varphi)(y) = \int_{\mathbb{R}_+} f\left(yt^{-1}\right) t^{-1}\varphi(t)\, dt. \tag{8}$$

By $\mathscr{D}(\mathbb{R}_+)$, or simply $\mathscr{D}$, denote the Schwartz space of test functions of compact support defined on $\mathbb{R}_+$. Then we have the following definition.

Definition 2. Let $f, g \in l_{\rho,\mu}$. For f and g, we define the operation $\otimes$ given by

$$(f \otimes g)(x) = \int_{\mathbb{R}_+} f(xt)\, g(t)\, dt. \tag{9}$$

Then we easily see that the operations in (8) and (9) are very basic for the next construction of the desired Bohemian spaces.

Theorem 3. *Let $f \in l_{\rho,\mu}$ and $\varphi \in \mathscr{D}$, then $l_v^{(\rho)}(f \curlyvee \varphi)(x) = ((l_v^{(\rho)} f) \otimes \varphi)(x)$.*

Proof. Let $x > 0$. For $f \in l_{\rho,\mu}$ and $\varphi \in \mathscr{D}$ we, by (3) and (8), get that

$$
\begin{aligned}
& l_v^{(\rho)}\left(f \curlyvee \varphi\right)(x) \\
& = \int_{\mathbb{R}_+} \left(\int_{\mathbb{R}_+} f\left(yt^{-1}\right) t^{-1}\varphi(t)\, dt \right) \lambda_v^{(\rho)}(xy)\, dy \\
& = \int_{\mathbb{R}_+} \left(\int_{\mathbb{R}_+} f\left(yt^{-1}\right) t^{-1}\varphi(t)\, dt \right).
\end{aligned}
\tag{10}
$$

A change of variables $y = tz$ in (10) implies

$$
\begin{aligned}
l_v^{(\rho)}\left(f \curlyvee \varphi\right)(x) & = \int_{\mathbb{R}_+} \left(\int_{\mathbb{R}_+} f(z)\, \lambda_v^{(\rho)}(xtz)\, dz \right) \varphi(t)\, dt \\
& = \int_{\mathbb{R}_+} \left(l_v^{(\rho)} f\right)(xt)\, \varphi(t)\, dt.
\end{aligned}
\tag{11}
$$

Hence, by (9), we have

$$l_v^{(\rho)}\left(f \curlyvee \varphi\right)(x) = \left(\left(l_v^{(\rho)} f\right) \otimes \varphi\right)(x). \tag{12}$$

This completes the proof of theorem. $\qquad\square$

2. General Bohemian

The structure necessary for the construction of Bohemian consists of the following:

(1) a group X and a commutative semigroup $(Y, *)$;

(2) a operation $\odot : X \times Y \to X$ such that $x \odot (v_1 * v_2) = (x \odot v_1) \odot v_2$, for all $x \in X$ and $v_1, v_2, \in Y$;

(3) a collection $\Delta \subset Y^{\mathbb{N}}$ such that

    (i) for all $(v_n), (\sigma_n) \in \Delta$, we have $(v_n * \sigma_n) \in \Delta$,

    (ii) If $x \odot v_n = y \odot v_n$, then $x = y$ where $x, y \in X, (v_n) \in \Delta$, and $n \in \mathbb{N}$.

Then the set Δ that satisfies (i) and (ii) is called the set of all delta sequences.

Let $A = \{(x_n, v_n) : x_n \in X, (v_n) \in \Delta, x_n \odot v_m = x_m \odot v_n, \text{for all } m, n \in \mathbb{N}\}$. Then we say $(x_n, v_n) \sim (y_n, \sigma_n)$ if there are $(x_n, v_n), (y_n, \sigma_n) \in A$, such that $x_n \odot \sigma_m = y_m \odot v_n$, for all $m, n \in \mathbb{N}$. The relation $\sim$ is an equivalence relation in A. The space of equivalence classes in A is called the space of Bohemian and denoted by $\mathbb{BH}$. Each element of $\mathbb{BH}$ is called Bohemian. Then the convergence in $\mathbb{BH}$ is defined as follows.

(1) $(h_n) \in \mathbb{BH}$ is said to be δ-convergent to $h \in \mathbb{BH}$, denoted by $h_n \xrightarrow{\delta} h$ as $n \to \infty$, if $\exists$ a delta sequence (v_n) such that $(h_n \odot v_n), (h \odot v_n) \in X$, and $(h_n \odot v_k) \to (h \odot v_k) \in X$ as $n \to \infty$, for all $k, n \in \mathbb{N}$.

(2) $(h_n) \in \mathbb{BH}$ is said to be Δ-convergent to $h \in \mathbb{BH}$, denoted by $h_n \xrightarrow{\Delta} h$ as $n \to \infty$, if $\exists$ a $(v_n) \in \Delta$ such that $(h_n - h) \odot v_n \in X$, for all $n \in \mathbb{N}$, and $(h_n - h) \odot v_n \to 0 \in X$ as $n \to \infty$.

The following theorem is equivalent to the statement of δ-convergence.

Theorem 4. *$h_n \xrightarrow{\delta} h \in \mathbb{BH}$ as $n \to \infty$ if and only if there are $f_{n,k}, f_k \in X$ and $v_k \in \Delta$ such that $h_n = [f_{n,k}/v_k], h = [f_k/v_k]$ and, for all $k \in \mathbb{N}$, $f_{n,k} \to f_k \in X$ as $n \to \infty$. See [6–17] for more details.*

3. The Spaces $\mathbb{BH}_1(l_{\rho,\mu})$ and $\mathbb{BH}_2(l_{\rho,2/(\rho-\mu-1)})$

In this section we construct the space $\mathbb{BH}(l_{\rho,\mu}, (\mathscr{D}, \curlyvee), \curlyvee, \Delta)$ (or $\mathbb{BH}_1(l_{\rho,\mu})$) and the space $\mathbb{BH}(l_{\rho,2/(\rho-\mu-1)}, (\mathscr{D}, \curlyvee), \otimes, \Delta)$ (or $\mathbb{BH}_2(l_{\rho,2/(\rho-\mu-1)})$) of Bohemian and give their properties.

At the first step, we prove the following connecting theorem.

Theorem 5. *Let $f \in l_{\rho,\mu}$ and $\varphi, \psi \in \mathscr{D}$; then, $f \otimes (\varphi \curlyvee \psi) = (f \otimes \varphi) \otimes \psi$.*

Proof. Let $f \in l_{\rho,\mu}$ and $\varphi, \psi \in \mathscr{D}$; then, applying the Fubinitz theorem yields

$$
\begin{aligned}
& \left(f \otimes (\varphi \curlyvee \psi)\right)(x) \\
& = \int_{\mathbb{R}_+} f(xt)\, (\varphi \curlyvee \psi)(t)\, dt \\
& = \int_{\mathbb{R}_+} f(xt) \left(\int_{\mathbb{R}_+} \varphi\left(\frac{t}{y}\right) \psi(y)\, y^{-1}\, dy \right) dt \\
& = \int_{\mathbb{R}_+} \left(\int_{\mathbb{R}_+} f(xt)\, \varphi\left(\frac{t}{y}\right) y^{-1}\, dy \right) \psi(y)\, dy.
\end{aligned}
\tag{13}
$$

The change of variables $t = yz$ implies $dt = y\,dz$ and further

$$(f \otimes (\varphi \vee \psi))(x) = \int_{\mathbb{R}_+} \left(\int_{\mathbb{R}_+} f(xyz) \varphi(z)\,dz \right) \psi(y)\,dy. \tag{14}$$

Therefore,

$$(f \otimes (\varphi \vee \psi))(x) = \int_{\mathbb{R}_+} (f \otimes \varphi)(xy) \psi(y)\,dy. \tag{15}$$

Hence, (15) implies

$$(f \otimes (\varphi \vee \psi))(x) = ((f \otimes \varphi) \otimes \psi)(x). \tag{16}$$

This completes the proof of the theorem. □

Next, forthcoming theorems prove the existence of the space $\mathbb{BH}_1(l_{\rho,\mu})$.

Theorem 6. *Let $f \in l_{\rho,\mu}$ and $\varphi \in \mathscr{D}$ and then $f \vee \varphi \in l_{\rho,\mu}$.*

Proof. For each $f \in l_{\rho,\mu}$ and $\varphi \in \mathscr{D}$, we have

$$\left\| x^\mu (f \vee \varphi)(x) \right\|_{\rho,\mu}^\rho = \int_{\mathbb{R}_+} \left| x^\mu \int_{\mathbb{R}_+} f(xt) \varphi(t)\,dt \right|^\rho dx. \tag{17}$$

By Jensen's theorem, we write

$$\left\| x^\mu (f \vee \varphi)(x) \right\|_{\rho,\mu}^\rho$$
$$\leq \int_{\mathbb{R}_+} \int_{\mathbb{R}_+} \left| x^\mu \int_{\mathbb{R}_+} f(xt) \varphi(t)\,dt \right|^\rho dx$$
$$\leq \int_{\mathbb{R}_+} \int_{\mathbb{R}_+} \left| x^\mu f(xt) \right|^\rho |\varphi(t)|\,dt\,dx \tag{18}$$
$$\leq \|f\|_{\rho,\mu}^\rho \int_K |\varphi(t)|\,dt,$$

where $K = [a, b], b > a > 0$ is a compact subset containing the support of φ. Hence, from (18), we get

$$\left\| x^\mu (f \vee \varphi)(x) \right\|_{\rho,\mu}^\rho \leq \|f\|_{\rho,\mu}^\rho M(b-a), \tag{19}$$

where M is certain positive real number. This completes the proof of the theorem. □

Theorem 7. *Let $f \in l_{\rho,\mu}$ and $\varphi, \psi \in \mathscr{D}$ and then*

(i) $f \vee (\varphi + \psi) = f \vee \varphi + f \vee \psi$;

(ii) $(\alpha f) \vee \varphi = \alpha(f \vee \varphi)$;

(iii) *let $(f_n) \in l_{\rho,\mu}$ and $f_n \to f$ as $n \to \infty$ then for $\varphi \in \mathscr{D}$, and we have $f_n \vee \varphi \to f \vee \varphi$ as $n \to \infty$;*

(iv) $f \vee (\varphi \vee \psi) = (f \vee \varphi) \vee \psi$.

Proof of Part (i), (ii), and (iii) follows from properties of integral operators $\int$. Similarly, the proof of Part (iv) is straightforward from the properties of $\vee$ proved in [5].

Following theorem is straightforward, and detailed proof is omitted.

Theorem 8. *Let $f \in l_{\rho,\mu}$ and $(\mu_n) \in \Delta$ then $f \vee \mu_n \to f$ as $n \to \infty$.*

Proof. Since $\mathscr{D}$ is a dense subspace of l_ρ then it is a dense subspace of $l_{\rho,\mu}$. Hence, there can be found $\psi \in \mathscr{D}$ such that

$$\|f - \psi\|_{\rho,\mu} < \varepsilon \tag{20}$$

for $\varepsilon > 0$. Also, from (18), we have

$$\left\| (f - \psi) \vee \mu_n \right\|_{\rho,\mu}^\rho \leq \|f\|_{\rho,\mu}^\rho \|\mu_n\|_{l^1}. \tag{21}$$

Let $g(t) = y^\mu \psi(yt^{-1})t^{-1}$ then; $g(t) \in \mathscr{D}$ and hence uniformly continuous on $\mathbb{R}_+$.

Therefore, there is $\delta > 0$ such that

$$|g(y) - g(x)| < \varepsilon \quad \text{whenever} \quad |y - x| < \delta. \tag{22}$$

Thus, using (22), we get

$$\left\| (\psi \times \mu_n - \psi)(y) \right\|_{\rho,\mu}^\rho$$
$$= \int_{\mathbb{R}_+} \left| y^\mu (\psi \times \mu_n - \psi)(y) \right|^\rho dy$$
$$\leq \int_{\mathbb{R}_+} \int_{\mathbb{R}_+} \left| y^\mu \left(\psi(yt^{-1})t^{-1}\mu_n(t) \right. \right.$$
$$\left. \left. - \psi(y) \right) \mu_n(t) \right|^\rho dt\,dy \tag{23}$$
$$\leq \int_{\mathbb{R}_+} \int_{\mathbb{R}_+} \left| y^\mu \left(\psi(yt^{-1})t^{-1} - \psi(y) \right) \right|^\rho$$
$$\times |\mu_n(t)|\,dt\,dy$$
$$= \int_{\mathbb{R}_+} \int_{\mathbb{R}_+} |g(y) - g(1)|^\rho |\mu_n(t)|\,dt\,dy$$
$$= \int_{\mathbb{R}_+} \int_{\mathbb{R}_+} \varepsilon^\rho |\mu_n(t)|\,dt\,dy.$$

By (22), $\operatorname{supp} \mu_n(t) \to 0$ as $n \to \infty$ implies that there can be found $N \in \mathbb{N}$, such that $\operatorname{supp} \mu_n \subseteq [0, \delta]$, for all $n \geq N$. Further, the fact that the function ψ is of compact support thus this implies that $\operatorname{supp} \psi(y) \subseteq K = [a, b]$, where K is a compact subset of $\mathbb{R}_+$. Thus, from (23), we write

$$\|\psi \vee \mu_n - \psi\|_{\rho,\mu}^\rho \leq \varepsilon^\rho \int_a^b \int_0^\delta M\,dt \tag{24}$$
$$= \varepsilon^\rho (b-a)(2\delta) M.$$

Now, we have

$$\|f \times \mu_n - f\|_{\rho,\mu} \leq \left\| (f - \psi) \vee \mu_n \right\|_{\rho,\mu}$$
$$+ \|\psi \times \mu_n - \psi\|_{\rho,\mu} + \left\| (f - \psi) \right\|_{\rho,\mu}. \tag{25}$$

On using (20), (21) and (24) prove that

$$\|f \times \mu_n - f\|_{\rho,\mu} \longrightarrow 0 \quad \text{as } \epsilon \longrightarrow 0. \tag{26}$$

Thus the theorem is proved. Then the Bohemian space $\mathbb{BH}_1(l_{\rho,\mu})$ is therefore constructed. The operations such as sum and multiplication by a scalar of two Bohemian in $\mathbb{BH}_1(l_{\rho,\mu})$ are defined in a natural way

$$\left[\frac{(f_n)}{(\mu_n)}\right] + \left[\frac{(g_n)}{(\tau_n)}\right] = \left[\frac{(f_n \curlyvee \tau_n + g_n \curlyvee \mu_n)}{(\mu_n \curlyvee \tau_n)}\right],$$

$$\alpha\left[\frac{(f_n)}{(\mu_n)}\right] = \left[\frac{(af_n)}{(\mu_n)}\right], \tag{27}$$

where α is a complex number.

Similarly, the operation $\curlyvee$ and differentiation are defined by

$$\left[\frac{(f_n)}{(\mu_n)}\right] \curlyvee \left[\frac{(g_n)}{(\tau_n)}\right] = \left[\frac{(f_n \curlyvee g_n)}{(\mu_n \curlyvee \tau_n)}\right],$$

$$\mathscr{D}^\alpha\left[\frac{(f_n)}{(\mu_n)}\right] = \left[\frac{(\mathscr{D}^\alpha f_n)}{(\mu_n)}\right]. \tag{28}$$

Now constructing the space $\mathbb{BH}_2(l_{\rho,2/(\rho-\mu-1)})$ follows from theorems which were used for constructing the space $\mathbb{BH}_1(l_{\rho,\mu})$. Therefore, the corresponding proofs of Theorems 10 and 11 are omitted. □

Theorem 9. *Let $f \in l_{\rho,2/(\rho-\mu-1)}$ and $\varphi \in \mathscr{D}$ and then $f \otimes \varphi \in l_{\rho,2/(\rho-\mu-1)}$.*

Theorem 10. *Let $f \in l_{\rho,2/(\rho-\mu-1)}$, $\varphi, \psi \in \mathscr{D}$ and then the following hold:*

    (i) $f \otimes (\varphi + \psi) = f \otimes \varphi + f \otimes \psi$;

    (ii) $(\alpha f) \otimes \varphi = \alpha(f \otimes \varphi)$;

    (iii) if $f_n \to f$ in $l_{\rho,2/(\rho-\mu-1)}$, as $n \to \infty$ then $f_n \otimes \varphi \to f \otimes \varphi$ as $n \to \infty$;

    (iv) if $(\mu_n) \in \Delta$, then $f \otimes \mu_n \to f$ as $n \to \infty$.

Theorem 11. *Let $f \in l_{\rho,2/(\rho-\mu-1)}$ and $\varphi, \psi \in \mathscr{D}$ and then $f \otimes (\varphi \curlyvee \psi) = (f \otimes \varphi) \otimes \psi$.*

Proof of this theorem is similar to that of Theorem 6. Thus the space $\mathbb{BH}_2(l_{\rho,2/(\rho-\mu-1)})$ can be regarded as Bohemian space.

4. Krätzel Transform of Bohemian

By aid of Theorem 4, we have the right to define the Krätzel transform of $[(f_n)/(\mu_n)] \in \mathbb{BH}_1(l_{\rho,\mu})$ as the Bohemian

$$\widehat{l_v^{(\rho)}}\left[\frac{(f_n)}{(\mu_n)}\right] = \left[\frac{(l_v^{(\rho)} f_n)}{(\mu_n)}\right] \tag{29}$$

is embedded in the space $\mathbb{BH}_2(l_{\rho,2/(\rho-\mu-1)})$.

Theorem 12. *The mapping $\widehat{l_v^{(\rho)}}$: $\mathbb{BH}_1(l_{\rho,\mu})$ $\to$ $\mathbb{BH}_2(l_{\rho,2/(\rho-\mu-1)})$ is*

    (i) well defined,

    (ii) linear.

Proof. Let $[(f_n)/(\mu_n)], [(g_n)/(\psi_n)] \in \mathbb{BH}_1(l_{\rho,\mu})$ be such that $[(f_n)/(\mu_n)] = [(g_n)/(\psi_n)]$, and then $f_n \curlyvee \psi_m = g_m \curlyvee \mu_n = g_n \curlyvee \mu_m$. Using Theorem 4 implies $l_v^{(\rho)} f_n \otimes \psi_m = l_v^{(\rho)} g_n \otimes \mu_m$, for all n, m. The idea of quotient of sequences in $\mathbb{BH}_2(l_{\rho,2/(\rho-\mu-1)})$ implies that

$$\frac{l_v^{(\rho)} f_n}{\mu_n} \text{ is equivalent to } \frac{l_v^{(\rho)} g_n}{\psi_n}. \tag{30}$$

That is,

$$\left[\frac{(l_v^{(\rho)} f_n)}{(\mu_n)}\right] = \left[\frac{(l_v^{(\rho)} g_n)}{(\psi_n)}\right]. \tag{31}$$

To prove part (ii) of the theorem, if $[(f_n)/(\mu_n)], [(g_n)/(\psi_n)] \in \mathbb{BH}_1(l_{\rho,\mu})$, then

$$\widehat{l_v^{(\rho)}}\left(\left[\frac{(f_n)}{(\mu_n)}\right] + \left[\frac{(g_n)}{(\psi_n)}\right]\right) = \left[\frac{(l_v^{(\rho)} f_n \otimes \psi_n + l_v^{(\rho)} g_n \otimes \mu_n)}{(\mu_n \otimes \psi_n)}\right]. \tag{32}$$

Hence

$$\widehat{l_v^{(\rho)}}\left(\left[\frac{(f_n)}{(\mu_n)}\right] + \left[\frac{(g_n)}{(\psi_n)}\right]\right) = \widehat{l_v^{(\rho)}}\left[\frac{(f_n)}{(\mu_n)}\right] + \widehat{l_v^{(\rho)}}\left[\frac{(g_n)}{(\psi_n)}\right]. \tag{33}$$

Also, if $\alpha \in \mathbb{C}$, then

$$\alpha\widehat{l_v^{(\rho)}}\left[\frac{(f_n)}{(\mu_n)}\right] = \alpha\left[\frac{(l_v^{(\rho)} f_n)}{(\mu_n)}\right] = \left[\frac{(l_v^{(\rho)}(\alpha f_n))}{(\mu_n)}\right]. \tag{34}$$

Hence

$$\alpha\widehat{l_v^{(\rho)}}\left[\frac{(f_n)}{(\mu_n)}\right] = \widehat{l_v^{(\rho)}}\left(\alpha\left[\frac{(f_n)}{(\mu_n)}\right]\right). \tag{35}$$

This completes the proof. □

Definition 13. Let $[(l_v^{(\rho)} f_n)/(\mu_n)] \in \mathbb{BH}_2(l_{\rho,2/(\rho-\mu-1)})$. We define the inverse of $\widehat{l_v^{(\rho)}}$ transform of the Bohemian $[(l_v^{(\rho)} f_n)/(\mu_n)]$ as

$$\widehat{l_v^{(\rho)}}^{-1}\left[\frac{(l_v^{(\rho)} f_n)}{(\mu_n)}\right] = \left[\frac{(l_v^{(\rho)})^{-1}(l_v^{(\rho)} f_n)}{(\mu_n)}\right] = \left[\frac{(f_n)}{(\mu_n)}\right], \tag{36}$$

for each $(\mu_n) \in \Delta$.

Theorem 14. $\widehat{l_v^{(\rho)}}$: $\mathbb{BH}_1(l_{\rho,\mu})$ $\to$ $\mathbb{BH}_2(l_{\rho,2/(\rho-\mu-1)})$ *is an isomorphism.*

Proof. Let $\widehat{l_v^{(\rho)}}[(f_n)/(\mu_n)] = \widehat{l_v^{(\rho)}}[(g_n)/(\psi_n)] \in \mathbb{BH}_2(l_{\rho,2/(\rho-\mu-1)})$ and then by (29) we get $l_v^{(\rho)} f_n \otimes \psi_m = l_v^{(\rho)} g_m \otimes \mu_n$. Therefore, Theorem 4 implies

$$l_v^{(\rho)}(f_n \curlyvee \psi_m) = l_v^{(\rho)}(g_m \curlyvee \mu_n). \tag{37}$$

Thus $f_n \curlyvee \psi_m = g_m \curlyvee \mu_n$, for all $m, n \in \mathbb{N}$. The concept of quotients of equivalent classes of $\mathbb{BH}_1(l_{\rho,\mu})$ then gives

$$\left[\frac{(f_n)}{(\mu_n)} \right] = \left[\frac{(g_n)}{(\psi_n)} \right]. \tag{38}$$

This proves that $\widehat{l_v^{(\rho)}}$ is injective.

To show that $l_v^{(\rho)}$ is surjective, let $[(l_v^{(\rho)} f_n)/(\mu_n)] \in \mathbb{BH}_2(l_{\rho,2/(\rho-\mu-1)})$. Then $(l_v^{(\rho)} f_n)/(\mu_n)$ is a quotient of sequences in $\mathbb{BH}_2(l_{\rho,2/(\rho-\mu-1)})$. Hence, $l_v^{(\rho)} f_n \otimes \mu_m = l_v^{(\rho)} f_m \otimes \mu_n$, for all $m, n \in \mathbb{N}$. Once again, Theorem 4 implies that $l_v^{(\rho)}(f_n \curlyvee \mu_m) = l_v^{(\rho)}(f_m \curlyvee \mu_n)$. Hence $[(f_n)/(\mu_n)] \in \mathbb{BH}_1(l_{\rho,\mu})$ satisfies

$$l_v^{(\rho)} \left[\frac{(f_n)}{(\mu_n)} \right] = \left[\frac{(l_v^{(\rho)} f_n)}{(\mu_n)} \right]. \tag{39}$$

This completes the proof of the theorem. $\qquad\square$

Theorem 15. *Let* $\psi \in \mathscr{D}$ *and* $[(l_v^{(\rho)} f_n)/(\mu_n)] \in \mathbb{BH}_2(l_{\rho,2/(\rho-\mu-1)})$ *then one has*

(i) $\widehat{l_v^{(\rho)}}^{-1}([(l_v^{(\rho)} f_n)/(\mu_n)] \otimes \psi) = [(f_n)/(\mu_n)] \curlyvee \psi$;

(ii) $\widehat{l_v^{(\rho)}}([(f_n)/(\mu_n)] \curlyvee \psi) = [(l_v^{(\rho)} f_n)/(\mu_n)] \otimes \psi$.

Proof. By using (29), we have

$$\widehat{l_v^{(\rho)}}^{-1}\left(\left[\frac{(l_v^{(\rho)} f_n)}{(\mu_n)} \right] \otimes \psi \right) = \left[\frac{(l_v^{(\rho)})^{-1}\left((l_v^{(\rho)} f_n) \otimes \psi \right)}{(\mu_n)} \right]. \tag{40}$$

Theorem 4 then gives

$$\widehat{l_v^{(\rho)}}^{-1}\left(\left[\frac{(l_v^{(\rho)} f_n)}{(\mu_n)} \right] \otimes \psi \right) = \left[\frac{(f_n) \curlyvee \psi}{(\mu_n)} \right] = \left[\frac{(f_n)}{(\mu_n)} \right] \curlyvee \psi. \tag{41}$$

Hence the part (i) of the theorem is proved. The proof of part (ii) is similar thus we omit the details. This completes the proof of the theorem. $\qquad\square$

Theorem 16. *The mappings*

(i) $\widehat{l_v^{(\rho)}} : \mathbb{BH}_1(l_{\rho,\mu}) \to \mathbb{BH}_2(l_{\rho,2/(\rho-\mu-1)})$ *are continuous with respect to* δ *and* Δ-*convergence.*

(ii) $\widehat{l_v^{(\rho)}}^{-1} : \mathbb{BH}_2(l_{\rho,2/(\rho-\mu-1)}) \to \mathbb{BH}_1(l_{\rho,\mu})$ *are continuous with respect to* δ *and* Δ-*convergence.*

Proof. At first, let us show that $\widehat{l_v^{(\rho)}}$ and $\widehat{l_v^{(\rho)}}^{-1}$ are continuous with respect to δ-convergence.

Let $\beta_n \xrightarrow{\delta} \beta$ in $\mathbb{BH}_1(l_{\rho,\mu})$ as $n \to \infty$ and then we show that $\widehat{l_v^{(\rho)}}\beta_n \to \widehat{l_v^{(\rho)}}\beta$ as $n \to \infty$. By virtue of Theorem 5, we can find $f_{n,k}$ and f_k in $l_{\rho,\mu}$ such that

$$\beta_n = \left[\frac{f_{n,k}}{\mu_k} \right], \qquad \beta = \left[\frac{f_k}{\mu_k} \right] \tag{42}$$

such that $f_{n,k} \to f_k$ as $n \to \infty$ for every $k \in \mathbb{N}$. Hence, $l_v^{(\rho)} f_{n,k} \to l_v^{(\rho)} f_k \in l_{\rho,2/(\rho-\mu-1)}$ as $n \to \infty$. Thus,

$$\left[\frac{l_v^{(\rho)} f_{n,k}}{\mu_k} \right] \longrightarrow \left[\frac{l_v^{(\rho)} f_k}{\mu_k} \right] \in \mathbb{BH}_2\left(l_{\rho,2/(\rho-\mu-1)} \right) \tag{43}$$

as $n \to \infty$.

To prove the Part (ii), Let $g_n, g \in \mathbb{BH}_2(l_{\rho,2/(\rho-\mu-1)})$ be such that $g_n \xrightarrow{\delta} g$ as $n \to \infty$. Then, once again, by Theorem 5, $g_n = [l_v^{(\rho)} f_{n,k}/\mu_k]$ and $g = [l_v^{(\rho)} f_k/\mu_k]$ for some $f_{n,k}, f_k \in l_{\rho,\mu}$ and $l_v^{(\rho)} f_{n,k} \to l_v^{(\rho)} f_k$ as $n \to \infty$. Hence $[f_{n,k}/\mu_k] \to [f_k/\mu_k]$ as $n \to \infty$.

Using (36), we get

$$\widehat{l_v^{(\rho)}}^{-1} \left[\frac{l_v^{(\rho)} f_{n,k}}{\mu_k} \right] \longrightarrow \widehat{l_v^{(\rho)}}^{-1} \left[\frac{l_v^{(\rho)} f_k}{\mu_k} \right] \quad \text{as } n \to \infty. \tag{44}$$

Now, we establish that $\widehat{l_v^{(\rho)}}$ and $\widehat{l_v^{(\rho)}}^{-1}$ are continuous with respect to Δ-convergence.

Let $\beta_n \xrightarrow{\Delta} \beta$ in $\mathbb{BH}_1(l_{\rho,\mu})$ as $n \to \infty$. Then, there exist $f_n \in l_{\rho,\mu}$ and $(\mu_n) \in \Delta$ such that $(\beta_n - \beta) \curlyvee \mu_n = [((f_n) \curlyvee \mu_k)/\mu_k]$ and $f_n \to 0$ as $n \to \infty$. By applying (29) we get

$$\widehat{l_v^{(\rho)}} \left((\beta_n - \beta) \curlyvee \mu_n \right) = \left[\frac{l_v^{(\rho)} \left((f_n) \curlyvee \mu_k \right)}{\mu_k} \right]. \tag{45}$$

Hence, we have $\widehat{l_v^{(\rho)}}((\beta_n - \beta) \curlyvee \mu_n) = [((l_v^{(\rho)} f_n) \otimes \mu_k)/\mu_k] = l_v^{(\rho)} f_n \to 0$ as $n \to \infty$ in $l_{\rho,2/(\rho-\mu-1)}$.

Therefore

$$\widehat{l_v^{(\rho)}} \left((\beta_n - \beta) \curlyvee \mu_n \right) = \left(\widehat{l_v^{(\rho)}}\beta_n - \widehat{l_v^{(\rho)}}\beta \right) \otimes \mu_n \longrightarrow 0 \tag{46}$$

$$\text{as } n \longrightarrow \infty.$$

Hence, $\widehat{l_v^{(\rho)}}\beta_n \xrightarrow{\Delta} \widehat{l_v^{(\rho)}}\beta$ as $n \to \infty$.

Finally, let $g_n \xrightarrow{\Delta} g \in \mathbb{BH}_2(l_{\rho,2/(\rho-\mu-1)})$ as $n \to \infty$ and then we find $l_v^{(\rho)} f_k \in l_{\rho,2/(\rho-\mu-1)}$ such that $(g_n - g) \otimes \mu_n = [(l_v^{(\rho)} f_k \otimes \mu_k)/\mu_k]$ and $l_v^{(\rho)} f_k \to 0$ as $n \to \infty$ for some $(\mu_n) \in \Delta$.

Now, using (36), we obtain

$$\widehat{l_v^{(\rho)}}^{-1} \left((g_n - g) \otimes \mu_n \right) = \left[\frac{(l_v^{(\rho)})^{-1} \left(l_v^{(\rho)} f_k \otimes \mu_k \right)}{\mu_k} \right]. \tag{47}$$

Theorem 4 implies

$$\widehat{l_v^{(\rho)}}^{-1} \left((g_n - g) \otimes \mu_n \right) = \left[\frac{(f_n) \curlyvee \mu_k}{\mu_k} \right] = f_n \longrightarrow 0 \tag{48}$$

$$\text{as } n \longrightarrow \infty \text{ in } l_{\rho,\mu}.$$

Thus

$$\widehat{l_v^{(\rho)}}^{-1} \left((g_n - g) \otimes \mu_n \right)$$

$$= \left(\widehat{l_v^{(\rho)}}^{-1} g_n - \widehat{l_v^{(\rho)}}^{-1} g \right) \curlyvee \mu_n \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{49}$$

From this, we find that $\widehat{l_v^{(\rho)}}^{-1} g_n \xrightarrow{\Delta} \widehat{l_v^{(\rho)}}^{-1} g$ as $n \to \infty$ in $\mathbb{BH}_1(l_{\rho,\mu})$.

This completes the proof of the theorem. $\square$

Theorem 17. *The $\widehat{l_v^{(\rho)}}$ transform is consistent with the classical transform $l_v^{(\rho)}$.*

Proof. For every $f \in l_{\rho,\mu}$, let β be its representative in $\mathbb{BH}_1(l_{\rho,\mu})$ and then $\beta = [(f \curlyvee (\mu_n))/(\mu_n)]$ where $(\mu_n) \in \Delta$, for all $n \in \mathbb{N}$. Since (μ_n) is independent from the representative, for all $n \in \mathbb{N}$, therefore

$$\widehat{l_v^{(\rho)}}(\beta) = \widehat{l_v^{(\rho)}}\left(\left[\frac{f \curlyvee (\mu_n)}{(\mu_n)}\right]\right) = \left[\frac{l_v^{(\rho)} f \otimes (\mu_n)}{(\mu_n)}\right], \quad (50)$$

which is the representative of $l_v^{(\rho)} f \in l_{\rho,2/(\rho-\mu-1)}$. Thus this completes the proof. $\square$

Theorem 18. *Let $[(g_n)/(\psi_n)] \in \mathbb{BH}_2(l_{\rho,2/(\rho-\mu-1)})$ and then a necessary and sufficient condition for $[(g_n)/(\psi_n)]$ to be in the range of $\widehat{l_v^{(\rho)}}$ is that g_n belongs to range of $l_v^{(\rho)}$, for all $n \in \mathbb{N}$.*

Proof. Let $[(g_n)/(\psi_n)]$ be in the range of $\widehat{l_v^{(\rho)}}$ and then it is clear that g_n belongs to the range of $l_v^{(\rho)}$, for all $n \in \mathbb{N}$.

To establish the converse, let g_n be in the range of $l_v^{(\rho)}$, for all $n \in \mathbb{N}$. Then there is $f_n \in l_{\rho,\mu}$ such that $l_v^{(\rho)} f_n = g_n$, $n \in \mathbb{N}$. Since $[(g_n)/(\psi_n)] \in \mathbb{BH}_2(l_{\rho,2/(\rho-\mu-1)})$,

$$g_n \otimes \psi_m = g_m \otimes \psi_n, \quad (51)$$

for all $m, n \in \mathbb{N}$, therefore,

$$l_v^{(\rho)}(f_n \curlyvee \varphi_n) = l_v^{(\rho)}(f_m \curlyvee \varphi_n), \quad \forall m, n \in \mathbb{N}, \quad (52)$$

where $f_n \in l_{\rho,\mu}$ and $\varphi_n \in \Delta$, for all $n \in \mathbb{N}$. Then it follows that we get $f_n \curlyvee \varphi_m = f_m \curlyvee \varphi_n$, for all $m, n \in \mathbb{N}$. Thus f_n/φ_n is a quotient of sequences in $\mathbb{BH}_1(l_{\rho,\mu})$. Therefore, we have $[(f_n)/(\varphi_n)] \in \mathbb{BH}_1(l_{\rho,\mu})$ and

$$\widehat{l_v^{(\rho)}}\left(\left[\frac{(f_n)}{(\varphi_n)}\right]\right) = \left[\frac{(g_n)}{(\psi_n)}\right]. \quad (53)$$

Hence the theorem is proved. $\square$

Theorem 19. *If $\beta = [(f_n)/(\varphi_n)] \in \mathbb{BH}_1(l_{\rho,\mu})$ and $\gamma = [(\kappa_n)/(\mu_n)] \in \mathbb{BH}_1(l_{\rho,\mu})$, then one has*

$$\widehat{l_v^{(\rho)}}\left(\left[\frac{(f_n)}{(\varphi_n)}\right] \curlyvee \left[\frac{(\kappa_n)}{(\mu_n)}\right]\right) = \left[\frac{(l_v^{(\rho)} f_n)}{(\varphi_n)}\right] \otimes \left[\frac{(\kappa_n)}{(\mu_n)}\right]. \quad (54)$$

Proof. Let the hypothesis of the theorem be satisfied for some $[(f_n)/(\varphi_n)]$ and $[(\kappa_n)/(\mu_n)]$. Therefore,

$$\widehat{l_v^{(\rho)}}\left(\left[\frac{(f_n)}{(\varphi_n)}\right] \curlyvee \left[\frac{(\kappa_n)}{(\mu_n)}\right]\right) = \widehat{l_v^{(\rho)}}\left(\left[\frac{(f_n \curlyvee \kappa_n)}{(\varphi_n \curlyvee \mu_n)}\right]\right)$$

$$= \left[\frac{((l_v^{(\rho)} f_n) \otimes \kappa_n)}{(\varphi_n \otimes \mu_n)}\right] \quad (55)$$

$$= \left[\frac{(l_v^{(\rho)} f_n)}{(\varphi_n)}\right] \otimes \left[\frac{(\kappa_n)}{(\mu_n)}\right].$$

This completes the proof. $\square$

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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