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Research Article

The Problem of Image Recovery by the Metric Projections in Banach Spaces

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We consider the problem of image recovery by the metric projections in a real Banach space. For a countable family of nonempty closed convex subsets, we generate an iterative sequence converging weakly to a point in the intersection of these subsets. Our convergence theorems extend the results proved by Bregman and Crombez.

1. Introduction

Let $C_1, C_2, \ldots, C_r$ be nonempty closed convex subsets of a real Hilbert space H such that $\bigcap_{i=1}^r C_i \neq \emptyset$. Then, the problem of image recovery may be stated as follows: the original unknown image z is known a priori to belong to the intersection of $\{C_i\}_{i=1}^r$; given only the metric projections P_{C_i} of H onto C_i for $i = 1, 2, \ldots, r$, recover z by an iterative scheme. Such a problem is connected with the convex feasibility problem and has been investigated by a large number of researchers.

Bregman [1] considered a sequence $\{x_n\}$ generated by cyclic projections, that is, $x_0 = x \in H$, $x_1 = P_{C_1}x$, $x_2 = P_{C_2}x_1$, $x_3 = P_{C_3}x_2, \ldots$, $x_r = P_{C_r}x_{r-1}$, $x_{r+1} = P_{C_1}x_r$, $x_{r+2} = P_{C_2}x_{r+1}, \ldots$. It was proved that $\{x_n\}$ converges weakly to an element of $\bigcap_{i=1}^r C_i$ for an arbitrary initial point $x \in H$.

Crombez [2] proposed a sequence $\{y_n\}$ generated by $y_0 = y \in H$, $y_{n+1} = \alpha_0 y_n + \sum_{i=1}^r \alpha_i (y_n + \lambda_i (P_{C_i} y_n - y_n))$ for $n = 0, 1, 2, \ldots$, where $0 < \alpha_i < 1$ for all $i = 0, 1, 2, \ldots, r$ with $\sum_{i=0}^r \alpha_i = 1$ and $0 < \lambda_i < 2$ for every $i = 1, 2, \ldots, r$. It was proved that $\{y_n\}$ converges weakly to an element of $\bigcap_{i=1}^r C_i$ for an arbitrary initial point $y \in H$.

Later, Kitahara and Takahashi [3] and Takahashi and Tamura [4] dealt with the problem of image recovery by convex combinations of nonexpansive retractions in a uniformly convex Banach space E. Alber [5] took up the problem of image recovery by the products of generalized projections in a uniformly convex and uniformly smooth Banach space E whose duality mapping is weakly sequentially continuous (see also [6, 7]).

On the other hand, using the hybrid projection method proposed by Haugazeau [8] (see also [9–11] and references therein) and the shrinking projection method proposed by Takahashi et al. [12] (see also [13]), Nakajo et al. [14] and Kimura et al. [15] considered this problem by the metric projections and proved convergence of the iterative sequence to a common point of countable nonempty closed convex subsets in a uniformly convex and smooth Banach space E and in a strictly convex, smooth, and reflexive Banach space E having the Kadec-Klee property, respectively. Kohsaka and Takahashi [16] took up this problem by the generalized projections and obtained the strong convergence to a common point of a countable family of nonempty closed convex subsets in a uniformly convex Banach space whose norm is uniformly Gâteaux differentiable (see also [17, 18]). Although these results guarantee the strong convergence, they need to use metric or generalized projections onto different subsets for each step, which are not given in advance.

In this paper, we consider this problem by the metric projections, which are one of the most familiar projections to deal with. The advantage of our results is that we use projections onto the given family of subsets only, to generate the iterative scheme. Our convergence theorems extend the results of [1, 2] to a Banach space E, and they deduce the weak convergence to a common point of a countable family of nonempty closed convex subsets of E.

There are a number of results dealing with the image recovery problem from the aspects of engineering using nonlinear functional analysis (see, e.g., [19]). Comparing with these researches, we may say that our approach is more abstract and theoretical; we adopt a general Banach space with several conditions for an underlying space, and therefore, the technique of the proofs can be applied to various mathematical results related to nonlinear problems defined on Banach spaces.

2. Preliminaries

Throughout this paper, let $\mathbb{N}$ be the set of all positive integers, $\mathbb{R}$ the set of all real numbers, E a real Banach space with norm $\|\cdot\|$, and E^* the dual of E. For $x \in E$ and $x^* \in E^*$, we denote by $\langle x, x^* \rangle$ the value of x^* at x. We write $x_n \to x$ to indicate that a sequence $\{x_n\}$ converges strongly to x. Similarly, $x_n \rightharpoonup x$ and $x_n \overset{*}{\rightharpoonup} x$ will symbolize weak and weak* convergence, respectively. We define the modulus δ_E of convexity of E as follows: δ_E is a function of $[0,2]$ into $[0,1]$ such that

$$\delta_E(\epsilon) = \inf \left\{ 1 - \frac{\|x+y\|}{2} : x, y \in E,\ \|x\| = 1, \right. \\ \left. \|y\| = 1,\ \|x-y\| \geq \epsilon \right\} \tag{1}$$

for every $\epsilon \in [0,2]$. E is called uniformly convex if $\delta_E(\epsilon) > 0$ for each $\epsilon > 0$. Let $p > 1$. E is said to be p-uniformly convex if there exists a constant $c > 0$ such that $\delta_E(\epsilon) \geq c\epsilon^p$ for every $\epsilon \in [0,2]$. It is obvious that a p-uniformly convex Banach space is uniformly convex. E is said to be strictly convex if $\|x+y\|/2 < 1$ for all $x, y \in E$ with $\|x\| = \|y\| = 1$ and $x \neq y$. We know that a uniformly convex Banach space is strictly convex and reflexive. For every $p > 1$, the (generalized) duality mapping $J_p : E \to 2^{E^*}$ of E is defined by

$$J_p x = \left\{ y^* \in E^* : \langle x, y^* \rangle = \|x\|^p,\ \|y^*\| = \|x\|^{p-1} \right\} \tag{2}$$

for all $x \in E$. When $p = 2$, J_2 is called the normalized duality mapping. We have that for $p, q > 1$, $\|x\|^p J_q x = \|x\|^q J_p x$ for all $x \in E$. E is said to be smooth if the limit

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{3}$$

exists for every $x, y \in E$ with $\|x\| = \|y\| = 1$. We know that the duality mapping J_p of E is single valued for each $p > 1$ if E is smooth. It is also known that if E is strictly convex, then the duality mapping J_p of E is one to one in the sense that $x \neq y$ implies that $J_p x \cap J_p y = \emptyset$ for all $p > 1$. If E is reflexive, then J_p is surjective, and J_p^{-1} is identical to the duality mapping $J_q^* : E^* \to 2^E$ defined by

$$J_q^* y^* = \left\{ x \in E : \langle x, y^* \rangle = \|y^*\|^q,\ \|x\| = \|y^*\|^{q-1} \right\} \tag{4}$$

for every $y^* \in E^*$, where $q \in \mathbb{R}$ satisfies $1/p + 1/q = 1$. For $p > 1$, the duality mapping J_p of a smooth Banach space E is said to be weakly sequentially continuous if $x_n \rightharpoonup x$ implies that $J_p x_n \overset{*}{\rightharpoonup} J_p x$ (see [20, 21] for details). The following is proved by Xu [22] (see also [23]).

Theorem 1 (Xu [22]). *Let E be a smooth Banach space and $p > 1$. Then, E is p-uniformly convex if and only if there exists a constant $c > 0$ such that $\|x+y\|^p \geq \|x\|^p + p\langle y, J_p x\rangle + c\|y\|^p$ holds for every $x, y \in E$.*

Remark 2. For a p-uniformly convex and smooth Banach space E, we have that the constant c in the theorem above satisfies $c \leq 1$. Let

$$c_0 = \sup \left\{ c > 0 : \|x+y\|^p \geq \|x\|^p + p \left\langle y, J_p x \right\rangle \right. \\ \left. + c\|y\|^p \ \forall x, y \in E \right\}. \tag{5}$$

Then, there exists a positive real sequence $\{c_n\}$ such that $\lim_{n\to\infty} c_n = c_0$ and $\|x+y\|^p \geq \|x\|^p + p\langle y, J_p x\rangle + c_n\|y\|^p$ for all $x, y \in E$ and $n \in \mathbb{N}$. So, we get $\|x+y\|^p \geq \|x\|^p + p\langle y, J_p x\rangle + c_0\|y\|^p$ for every $x, y \in E$. Therefore, c_0 is the maximum of constants. In the case of Hilbert spaces, the normalized duality mapping J_2 is the identity mapping and $c_0 = 1$.

Let E be a smooth Banach space and $p > 1$. The function $\phi_p : E \times E \to \mathbb{R}$ is defined by

$$\phi_p(y, x) = \|y\|^p - p\left\langle y, J_p x \right\rangle + (p-1)\|x\|^p \tag{6}$$

for every $x, y \in E$. We have $\phi_p(x, y) \geq 0$ for all $x, y \in E$ and $\phi_p(z, x) + \phi_p(x, y) = \phi_p(z, y) + p\langle x - z, J_p x - J_p y\rangle$ for every $x, y, z \in E$. It is known that if E is strictly convex and smooth, then, for $x, y \in E$, $\phi_p(y, x) = \phi_p(x, y) = 0$ if and only if $x = y$. Indeed, suppose that $\phi_p(y, x) = \phi_p(x, y) = 0$. Then, since

$$\begin{aligned} 0 &= \phi_p(y, x) + \phi_p(x, y) \\ &= p\left(\|x\|^p + \|y\|^p - \left\langle x, J_p y\right\rangle - \left\langle y, J_p x\right\rangle\right) \\ &\geq p\left(\|x\|^p + \|y\|^p - \|x\|\|y\|^{p-1} - \|y\|\|x\|^{p-1}\right) \\ &= p\left(\|x\| - \|y\|\right)\left(\|x\|^{p-1} - \|y\|^{p-1}\right) \geq 0, \end{aligned} \tag{7}$$

we have $\|x\| = \|y\|$. It follows that $\langle y, J_p x\rangle = p^{-1}(\|y\|^p + (p-1)\|x\|^p - \phi_p(y, x)) = \|y\|^p$ and $\|J_p x\| = \|x\|^{p-1} = \|y\|^{p-1}$, which implies that $J_p y = J_p x$. Since J_p is one to one, we have $x = y$ (see also [17]). We have the following result from Theorem 1.

Lemma 3. *Let $p > 1$ and E be a p-uniformly convex and smooth Banach space. Then, for each $x, y \in E$,*

$$\phi_p(x, y) \geq c_0 \|x - y\|^p \quad (8)$$

holds, where c_0 is maximum in Remark 2.

Proof. Let $x, y \in E$. By Theorem 1, we have

$$\|x\|^p \geq \|y\|^p + p \langle x - y, J_p y \rangle + c_0 \|x - y\|^p, \quad (9)$$

where c_0 is maximum in Remark 2. Hence, we get

$$\begin{aligned} \phi_p(x, y) &= \|x\|^p - \|y\|^p - p \langle x - y, J_p y \rangle \\ &\geq c_0 \|x - y\|^p, \end{aligned} \quad (10)$$

which is the desired result. □

Let C be a nonempty closed convex subset of a strictly convex and reflexive Banach space E, and let $x \in E$. Then, there exists a unique element $x_0 \in C$ such that $\|x_0 - x\| = \inf_{y \in C} \|y - x\|$. Putting $x_0 = P_C x$, we call P_C the metric projection onto C (see [24]). We have the following result [25, p. 196] for the metric projection.

Lemma 4. *Let C be a nonempty closed convex subset of a strictly convex, reflexive, and smooth Banach space E, and let $x \in E$. Then, $y = P_C x$ if and only if $\langle y - z, J_2(x - y) \rangle \geq 0$ for all $z \in C$, where P_C is the metric projection onto C.*

Remark 5. For $p > 1$, it holds that $\|x\| J_p x = \|x\|^{p-1} J_2 x$ for every $x \in E$. Therefore, under the same assumption as Lemma 4, we have that $y = P_C x$ if and only if $\langle y - z, J_p(x - y) \rangle \geq 0$ for all $z \in C$.

3. Main Results

Firstly, we consider the iteration of Crombez's type and get the following result.

Theorem 6. *Let $p, q > 1$ be such that $1/p + 1/q = 1$. Let $\{C_n\}_{n \in \mathbb{N}}$ be a family of nonempty closed convex subsets of a p-uniformly convex and smooth Banach space E whose duality mapping J_p is weakly sequentially continuous. Suppose that $\bigcap_{n \in \mathbb{N}} C_n \neq \emptyset$. Let $\lambda_{n,k} \in]0, (1 + 1/(p-1))^{p-1} c_0[$ and $\alpha_{n,k} \in [0, 1]$ for all $n \in \mathbb{N}$ and $k = 1, 2, \ldots, n$ with $\sum_{k=1}^{n} \alpha_{n,k} = 1$ for every $n \in \mathbb{N}$, where c_0 is maximum in Remark 2. Let $\{x_n\}$ be a sequence generated by $x_1 = x \in E$ and*

$$x_{n+1} = J_q^* \left(\sum_{k=1}^{n} \alpha_{n,k} \left(J_p x_n - \lambda_{n,k} J_p \left(x_n - P_{C_k} x_n \right) \right) \right) \quad (11)$$

for every $n \in \mathbb{N}$. If $0 < \liminf_{n \to \infty} \lambda_{n,k} \leq \limsup_{n \to \infty} \lambda_{n,k} < (1 + 1/(p-1))^{p-1} c_0$ and $\liminf_{n \to \infty} \alpha_{n,k} > 0$ for each $k \in \mathbb{N}$, then $\{x_n\}$ converges weakly to a point in $\bigcap_{n=1}^{\infty} C_n$.

Proof. Let $y_{n,k} = J_q^*(J_p x_n - \lambda_{n,k} J_p(x_n - P_{C_k} x_n))$ for $n \in \mathbb{N}$ and $k = 1, 2, \ldots, n$. Then, for $z \in \bigcap_{n \in \mathbb{N}} C_n$, we obtain

$$\begin{aligned} &\phi_p(z, y_{n,k}) - \phi_p(z, x_n) \\ &\quad = -\phi_p(y_{n,k}, x_n) + p \langle y_{n,k} - z, J_p y_{n,k} - J_p x_n \rangle \\ &\quad = -\phi_p(y_{n,k}, x_n) - p \lambda_{n,k} \langle y_{n,k} - z, J_p(x_n - P_{C_k} x_n) \rangle \\ &\quad = -\phi_p(y_{n,k}, x_n) - p \lambda_{n,k} \langle y_{n,k} - x_n, J_p(x_n - P_{C_k} x_n) \rangle \\ &\qquad - p \lambda_{n,k} \langle x_n - z, J_p(x_n - P_{C_k} x_n) \rangle \end{aligned} \quad (12)$$

for all $n \in \mathbb{N}$ and $k = 1, 2, \ldots, n$. Using Remark 5 with that $z \in C_k$, we get

$$\begin{aligned} &\langle x_n - z, J_p(x_n - P_{C_k} x_n) \rangle \\ &\quad = \langle x_n - P_{C_k} x_n, J_p(x_n - P_{C_k} x_n) \rangle \\ &\qquad + \langle P_{C_k} x_n - z, J_p(x_n - P_{C_k} x_n) \rangle \\ &\quad \geq \|x_n - P_{C_k} x_n\|^p \end{aligned} \quad (13)$$

for every $n \in \mathbb{N}$ and $k = 1, 2 \ldots, n$. Thus, by Lemma 3 we have

$$\begin{aligned} &\phi_p(z, y_{n,k}) - \phi_p(z, x_n) \\ &\quad \leq -c_0 \|y_{n,k} - x_n\|^p \\ &\qquad - p \lambda_{n,k} \langle y_{n,k} - x_n, J_p(x_n - P_{C_k} x_n) \rangle \\ &\qquad - p \lambda_{n,k} \|x_n - P_{C_k} x_n\|^p \\ &\quad \leq -c_0 \|y_{n,k} - x_n\|^p + p \lambda_{n,k} \|y_{n,k} - x_n\| \|x_n - P_{C_k} x_n\|^{p-1} \\ &\qquad - p \lambda_{n,k} \|x_n - P_{C_k} x_n\|^p \end{aligned} \quad (14)$$

for each $n \in \mathbb{N}$ and $k = 1, 2, \ldots, n$. Since it holds that

$$st \leq \frac{1}{\beta} \frac{s^p}{p} + \beta^{q-1} \frac{t^q}{q} \quad (15)$$

for $s, t \geq 0$, $p, q > 1$ with $1/p + 1/q = 1$, and $\beta > 0$, we have

$$\begin{aligned} &\|y_{n,k} - x_n\| \|x_n - P_{C_k} x_n\|^{p-1} \\ &\quad \leq \frac{1}{\beta_k p} \|y_{n,k} - x_n\|^p \\ &\qquad + \beta_k^{1/(p-1)} \frac{p-1}{p} \|x_n - P_{C_k} x_n\|^p \end{aligned} \quad (16)$$

for every $k \in \mathbb{N}$, $\beta_k > 0$ and $n \geq k$. Therefore, it follows that

$$\begin{aligned} &\phi_p(z, y_{n,k}) - \phi_p(z, x_n) \\ &\quad \leq \left(\frac{\lambda_{n,k}}{\beta_k} - c_0 \right) \|y_{n,k} - x_n\|^p \\ &\qquad + \lambda_{n,k} \left((p-1) \beta_k^{1/(p-1)} - p \right) \|x_n - P_{C_k} x_n\|^p \end{aligned} \quad (17)$$

for every $n \in \mathbb{N}$, $k = 1, 2, \ldots, n$, and $\beta_k > 0$. Since

$$\begin{aligned}\phi_p(z, x_{n+1}) &= \|z\|^p - p\left\langle z, \sum_{k=1}^{n}\alpha_{n,k}J_p y_{n,k}\right\rangle \\ &\quad + (p-1)\left\|\sum_{k=1}^{n}\alpha_{n,k}J_p y_{n,k}\right\|^{p/(p-1)} \\ &\le \|z\|^p - p\sum_{k=1}^{n}\alpha_{n,k}\langle z, J_p y_{n,k}\rangle \\ &\quad + (p-1)\sum_{k=1}^{n}\alpha_{n,k}\|y_{n,k}\|^p \\ &= \sum_{k=1}^{n}\alpha_{n,k}\phi_p(z, y_{n,k})\end{aligned} \tag{18}$$

for every $n \in \mathbb{N}$, we have

$$\begin{aligned}&\phi_p(z, x_{n+1}) - \phi_p(z, x_n) \\ &\quad \le \sum_{k=1}^{n}\alpha_{n,k}\left(\frac{\lambda_{n,k}}{\beta_k} - c_0\right)\|y_{n,k} - x_n\|^p \\ &\quad\quad + \sum_{k=1}^{n}\alpha_{n,k}\lambda_{n,k}\left((p-1)\beta_k^{1/(p-1)} - p\right)\left\|x_n - P_{C_k}x_n\right\|^p\end{aligned} \tag{19}$$

for all $n \in \mathbb{N}$ and $\beta_1, \beta_2, \ldots, \beta_n > 0$. Since $\lambda_{n,k} \in]0, (1+1/(p-1))^{p-1}c_0[$, $\alpha_{n,k} \in [0, 1]$ for all $n \in \mathbb{N}$ and $k = 1, 2, \ldots, n$,

$$\begin{gathered}0 < \liminf_{n\to\infty}\lambda_{n,k} \le \limsup_{n\to\infty}\lambda_{n,k} < \left(1 + \frac{1}{(p-1)}\right)^{p-1}c_0, \\ \liminf_{n\to\infty}\alpha_{n,k} > 0\end{gathered} \tag{20}$$

for each $k \in \mathbb{N}$, we can choose $\beta_k > 0$ for every $k \in \mathbb{N}$ such that $\alpha_{n,k}(\lambda_{n,k}/\beta_k - c_0) \le 0$, $\alpha_{n,k}\lambda_{n,k}((p-1)\beta_k^{1/(p-1)} - p) \le 0$ for all $n \ge k$ and

$$\begin{gathered}\limsup_{n\to\infty}\alpha_{n,k}\left(\frac{\lambda_{n,k}}{\beta_k} - c_0\right) < 0, \\ \limsup_{n\to\infty}\alpha_{n,k}\lambda_{n,k}\left((p-1)\beta_k^{1/(p-1)} - p\right) < 0\end{gathered} \tag{21}$$

for each $k \in \mathbb{N}$. Hence, there exists $\lim_{n\to\infty}\phi_p(z, x_n)$ for every $z \in \bigcap_{n\in\mathbb{N}} C_n$ and

$$\lim_{n\to\infty}\|y_{n,k} - x_n\| = \lim_{n\to\infty}\left\|x_n - P_{C_k}x_n\right\| = 0 \tag{22}$$

for all $k \in \mathbb{N}$. It follows from Lemma 3 that $\{x_n\}$ is bounded. Let $\{x_{n_i}\}$ and $\{x_{m_j}\}$ be subsequences of $\{x_n\}$ such that $x_{n_i} \rightharpoonup u_1$ and $x_{m_j} \rightharpoonup u_2$. Then, we get $\|x_{n_i} - P_{C_k}x_{n_i}\| \to 0$ which implies that $u_1 \in C_k$ for every $k \in \mathbb{N}$. In the same way, we also have $u_2 \in C_k$ for every $k \in \mathbb{N}$. Let $\lim_{n\to\infty}\phi_p(u_1, x_n) = \mu_1$ and $\lim_{n\to\infty}\phi_p(u_2, x_n) = \mu_2$. Since

$$\begin{aligned}\mu_1 - \mu_2 &= \lim_{i\to\infty}\left(\phi_p(u_1, x_{n_i}) - \phi_p(u_2, x_{n_i})\right) \\ &= \|u_1\|^p - \|u_2\|^p + p\lim_{i\to\infty}\langle u_2 - u_1, J_p x_{n_i}\rangle\end{aligned} \tag{23}$$

and J_p is weakly sequentially continuous, we have

$$\begin{aligned}\mu_1 - \mu_2 &= \|u_1\|^p - \|u_2\|^p + p\langle u_2 - u_1, J_p u_1\rangle \\ &= -\phi_p(u_2, u_1).\end{aligned} \tag{24}$$

Similarly, we obtain $\mu_2 - \mu_1 = -\phi_p(u_1, u_2)$. So, we get $\phi_p(u_1, u_2) + \phi_p(u_2, u_1) = 0$, that is, $u_1 = u_2$. Therefore, $\{x_n\}$ converges weakly to a point in $\bigcap_{n\in\mathbb{N}} C_n$. □

Using the idea of [9, p. 256], we also have the following result by the iteration of Bregman's type.

Theorem 7. *Let $p, q > 1$ be such that $1/p + 1/q = 1$. Let I be a countable set and $\{C_j\}_{j\in I}$ a family of nonempty closed convex subsets of a p-uniformly convex and smooth Banach space E whose duality mapping J_p is weakly sequentially continuous. Suppose that $\bigcap_{j\in I} C_j \ne \emptyset$. Let $\lambda_n \in]0, (1+1/(p-1))^{p-1}c_0[$ for all $n \in \mathbb{N}$, where c_0 is maximum in Remark 2, and let $\{x_n\}$ be a sequence generated by $x_1 = x \in E$ and*

$$x_{n+1} = J_q^*\left(J_p x_n - \lambda_n J_p\left(x_n - P_{C_{i(n)}}x_n\right)\right) \tag{25}$$

for every $n \in \mathbb{N}$, where the index mapping $i : \mathbb{N} \to I$ satisfies that, for every $j \in I$, there exists $M_j \in \mathbb{N}$ such that $j \in \{i(n), \ldots, i(n+M_j-1)\}$ for each $n \in \mathbb{N}$. If $0 < \liminf_{n\to\infty}\lambda_n \le \limsup_{n\to\infty}\lambda_n < (1+1/(p-1))^{p-1}c_0$, then, $\{x_n\}$ converges weakly to a point in $\bigcap_{j\in I} C_j$.

Proof. Let $z \in \bigcap_{j\in I} C_j$. As in the proof of Theorem 6, we have

$$\begin{aligned}&\phi_p(z, x_{n+1}) - \phi_p(z, x_n) \\ &\quad \le \left(\frac{\lambda_n}{\beta} - c_0\right)\|x_{n+1} - x_n\|^p \\ &\quad\quad + \lambda_n\left((p-1)\beta^{1/(p-1)} - p\right)\left\|x_n - P_{C_{i(n)}}x_n\right\|^p\end{aligned} \tag{26}$$

for all $n \in \mathbb{N}$ and $\beta > 0$. Since $\lambda_n \in]0, (1+1/(p-1))^{p-1}c_0[$ for all $n \in \mathbb{N}$ and $0 < \liminf_{n\to\infty}\lambda_n \le \limsup_{n\to\infty}\lambda_n < (1+1/(p-1))^{p-1}c_0$, we can find that $\beta > 0$ such that

$$\begin{gathered}\limsup_{n\to\infty}\left(\frac{\lambda_n}{\beta} - c_0\right) < 0, \\ \limsup_{n\to\infty}\lambda_n\left((p-1)\beta^{1/(p-1)} - p\right) < 0.\end{gathered} \tag{27}$$

Then, there exists $\lim_{n\to\infty}\phi_p(z, x_n)$ for every $z \in \bigcap_{i\in I} C_i$ and

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = \lim_{n\to\infty}\left\|x_n - P_{C_{i(n)}}x_n\right\| = 0. \tag{28}$$

So, we have that $\{x_n\}$ is bounded from Lemma 3. Let $\{x_{n_k}\}$ be a subsequence of $\{x_n\}$ such that $x_{n_k} \rightharpoonup u$. For fixed $j \in I$, there exists a strictly increasing sequence $\{m_k\} \subset \mathbb{N}$ such that $n_k \leq m_k \leq n_k + M_j - 1$ and $i(m_k) = j$ for every $k \in \mathbb{N}$. It follows that

$$\left\| x_{m_k} - x_{n_k} \right\| \leq \sum_{l=n_k}^{n_k+M_j-1} \left\| x_{l+1} - x_l \right\| \tag{29}$$

for all $k \in \mathbb{N}$ which implies that $x_{m_k} \rightharpoonup u$. Since $\lim_{k\to\infty} \|x_{m_k} - P_{C_j} x_{m_k}\| = 0$, $u \in C_j$ for every $j \in I$. So, we get $u \in \bigcap_{j\in I} C_j$. As in the proof of Theorem 6, using that J_p is weakly sequentially continuous, we get that $\{x_n\}$ converges weakly to a point in $\bigcap_{j\in I} C_j$. □

Suppose that the index set I is a finite set $\{0, 1, 2, \ldots, N-1\}$. For the cyclic iteration, the index mapping i is defined by $i(j) = j \bmod N$ for each $j \in I$. Clearly it satisfies the assumption in Theorem 7. In the case where the index set I is countably infinite, that is, $I = \mathbb{N}$, one of the simplest examples of $i : \mathbb{N} \to \mathbb{N}$ can be defined as follows:

$$i(n) = \begin{cases} 1 & (n = 2m - 1 \text{ for some } m \in \mathbb{N}), \\ 2 & (n = 2(2m - 1) \text{ for some } m \in \mathbb{N}), \\ 3 & (n = 4(2m - 1) \text{ for some } m \in \mathbb{N}), \\ \ldots, & \\ k & \left(n = 2^{k-1}(2m - 1) \text{ for some } m \in \mathbb{N}\right), \\ \ldots. & \end{cases} \tag{30}$$

Then, the assumption in Theorem 7 is satisfied by letting $M_j = 2^j$ for each $j \in I = \mathbb{N}$.

4. Deduced Results

Since a real Hilbert space H is 2-uniformly convex and the maximum c_0 in Remark 2 is equal to 1, we get the following results. At first, we have the following theorem which generalizes the results of [2] by Theorem 6.

Theorem 8. *Let $\{C_n\}_{n\in\mathbb{N}}$ be a family of nonempty closed convex subsets of H such that $\bigcap_{n\in\mathbb{N}} C_n \neq \emptyset$. Let $\lambda_{n,k} \in]0, 2[$ and $\alpha_{n,k} \in [0, 1]$ for all $n \in \mathbb{N}$ and $k = 1, 2, \ldots, n$ with $\sum_{k=1}^{n} \alpha_{n,k} = 1$ for every $n \in \mathbb{N}$. Let $\{x_n\}$ be a sequence generated by $x_1 = x \in H$ and*

$$x_{n+1} = \sum_{k=1}^{n} \alpha_{n,k} \left(x_n - \lambda_{n,k} \left(x_n - P_{C_k} x_n \right) \right) \tag{31}$$

for every $n \in \mathbb{N}$. If it holds that $0 < \liminf_{n\to\infty} \lambda_{n,k} \leq \limsup_{n\to\infty} \lambda_{n,k} < 2$ and $\liminf_{n\to\infty} \alpha_{n,k} > 0$ for each $k \in \mathbb{N}$, then, $\{x_n\}$ converges weakly to a point in $\bigcap_{n=1}^{\infty} C_n$.

Next, we have the following theorem which extends the result of [1] by Theorem 7.

Theorem 9. *Let I be a countable set and $\{C_j\}_{j\in I}$ a family of nonempty closed convex subsets of H such that $\bigcap_{j\in I} C_j \neq \emptyset$. Let $\lambda_n \in]0, 2[$ for all $n \in \mathbb{N}$, and let $\{x_n\}$ be a sequence generated by $x_1 = x \in H$ and*

$$x_{n+1} = x_n - \lambda_n \left(x_n - P_{C_{i(n)}} x_n \right) \tag{32}$$

for every $n \in \mathbb{N}$, where the index mapping $i : \mathbb{N} \to I$ satisfies that, for every $j \in I$, there exists $M_j \in \mathbb{N}$ such that $j \in \{i(n), \ldots, i(n+M_j-1)\}$ for each $n \in \mathbb{N}$. If $0 < \liminf_{n\to\infty} \lambda_n \leq \limsup_{n\to\infty} \lambda_n < 2$, then, $\{x_n\}$ converges weakly to a point in $\bigcap_{j\in I} C_j$.

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