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Research Article

Remarks on the Blow-Up Solutions for the Critical Gross-Pitaevskii Equation

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This paper is concerned with the blow-up solutions of the critical Gross-Pitaevskii equation, which models the Bose-Einstein condensate. The existence and qualitative properties of the minimal blow-up solutions are obtained.

1. Introduction and Main Results

In this paper, we deal with the Cauchy problem of the nonlinear Schrödinger equation with a harmonic potential

$$i\phi_t + \Delta\phi - |x|^2\phi + |\phi|^{4/N}\phi = 0, \quad x \in \mathbb{R}^N, \ t \geq 0, \tag{1}$$

$$\phi(0,x) = \phi_0(x), \tag{2}$$

where $\phi = \phi(t,x)$: $[0,T) \times \mathbb{R}^N \to \mathbb{C}$ is the wave function, N is the space dimension, and Δ denotes the Laplace operator on $\mathbb{R}^N$. Equation (1) is also called Gross-Pitaevskii equation (see [1, 2]), which models the Bose-Einstein condensate (see [3, 4]). The harmonic potential $|x|^2$ describes a magnetic field. With the nonlinear term $|\phi|^{4/N}\phi$ being replaced by $|\phi|^{p-1}\phi$, it is well known that the exponent $p = 1 + 4/N$ is the minimal value for the existence of blow-up solutions (see e.g., [5, 6]). Hence (1) is called critical Gross-Pitaevskii equation.

Let us recall the classical nonlinear Schrödinger equation

$$i\psi_t + \Delta\psi + |\psi|^{4/N}\psi = 0, \quad x \in \mathbb{R}^N, \ t \geq 0, \tag{3}$$

$$\psi(0,x) = \psi_0(x). \tag{4}$$

For Cauchy problem (3)-(4), Ginibre and Velo [7] established the local existence in $H^1(\mathbb{R}^N)$. Glassey [8], Weinstein [9], and Zhang [10] proved that, for some initial data, the solutions of the Cauchy problem (3)-(4) blow up in finite time.

For the Cauchy problem (3)-(4), it is well known that there exists a minimum of L^2 norm for the initial data of blow-up solutions (see [9]). More precisely, let $Q(x)$ be the ground state, which is the unique, positive, radially symmetric solution (see [11]) of the semilinear elliptic equation

$$-\Delta u + u - |u|^{4/N}u = 0, \quad u \in H^1(\mathbb{R}^N). \tag{5}$$

Weinstein [9] proved that the solutions of the Cauchy problem (3)-(4) are globally defined if $\|\psi_0\|_{L^2} < \|Q\|_{L^2}$. On the other hand, for any $l \geq \|Q\|_{L^2}$, there exist blow-up solutions with $\|\psi_0\|_{L^2} = l$. Since then, much progress has been made on the blow-up rate and profile of the blow-up solutions of the Cauchy problem (3)-(4) (see [12–15]). In particular, based on the pseudoconformal invariance of (3) and the variational characterization of the ground, elaborate and interesting conclusions were established on the existence and profile of the minimal blow-up solution, which is the blow-up solution $\psi(t,x)$ such that $\|\psi_0\|_{L^2} = \|Q\|_{L^2}$ (see [13, 15, 16]). By using the pseudoconformal invariance of (3), Weinstein [15] constructed the explicit blow-up solution with critical mass ($\|\psi_0\|_{L^2} = \|Q\|_{L^2}$) for (3) in the form

$$(a+bt)^{-(N/2)}Q\left(\frac{x}{a+bt}\right)e^{(ib|x|^2)/4(a+bt)}e^{i(c+dt)/(a+bt)}, \tag{6}$$

where $a, b, c, d \in R$, $ad - bc = 1$, and $ab < 0$. Moreover, Weinstein proved that, for any minimal blow-up solution $\psi(t)$, the following holds:

$$\lim_{t\to T} \lambda(t)^{N/2} \psi(t, \lambda(t)(x + y(t))) = Q(x), \tag{7}$$

where T is the blow-up time and $y(t) \in \mathbb{R}^N$ and $\lambda(t) \in \mathbb{R}$ are some suitable functions.

Merle [13, 16] proved that $\psi(t, x)$ is a minimal blow-up solution of (3) if and only if there exist $\theta \in \mathbb{R}$, $\omega > 0$, $x_0 \in \mathbb{R}^N$, and $x_1 \in \mathbb{R}^N$ such that

$$\begin{aligned}\psi(t, x) = \left(\frac{\omega}{T-t}\right)^{N/2} & e^{i\theta + (i|x-x_1|^2/4(-T+t)) - (i\omega^2/(-T+t))} \\ & \times Q\left(\frac{\omega}{T-t}\left((x - x_1) - (T-t)x_0\right)\right).\end{aligned} \tag{8}$$

For the Cauchy problem (1)-(2), local well-posedness in energy space was established in Cazenave [17]. Moreover, from the result of Carles [18] and Zhang [6, 19], it is known that $\phi(t)$ is globally defined if $\|\phi_0\|_{L^2} < \|Q\|_{L^2}$. In other words, $\|\phi_0\|_{L^2} \geq \|Q\|_{L^2}$ if $\phi(t)$ blows up in finite time.

Let $\phi(t)$ and $\psi(t)$ be the solutions of the Cauchy problems (1)-(2) and (3)-(4), respectively. Under the condition of $\phi_0(x) = \psi_0(x)$, Carles [18] established a formula, which reflects the relation between $\phi(t)$ and $\psi(t)$. According to the formula, Carles [18] established the following statements.

(1) If $\phi(t)$ blows up at a finite time T_ϕ, then $T_\phi \leq \pi/2$.

(2) If $\phi(t)$ blows up at $T_\phi < \pi/2$, $\psi(t)$ blows up at time $T_\psi < \infty$.

(3) Conversely, $\psi(t)$ blows up at time $T_\psi < \infty$; then $\phi(t)$ blows up at $T_\phi < \pi/2$.

(4) If $\phi(t)$ blows up at $T_\phi = \pi/2$, $\psi(t)$ exists globally ($T_\psi = \infty$).

Moreover, Carles studied the qualitative properties of minimal blow-up solutions $\phi(t)$ with $T_\phi < \pi/2$ (see [18, 20]). As for the minimal blow-up solutions with $T_\phi = \pi/2$, though the existence was established by the formula in [5], there is no further information on the qualitative properties obtained by the formula. Up to our knowledge, there is no result about the qualitative properties of the minimal blow-up solutions $\phi(t)$ of (1) with $T_\phi = \pi/2$.

The purpose of the present paper is to investigate the qualitative properties of the minimal blow-up solutions without any limit to the blow-up time. The formula presented in [18] is not used to carry out the objective. We follow the ideas of Merle [13, 16], as well as Weinstein [15], in which the profile and uniqueness of the minimal blow-up solutions for (3) were investigated. However, in contrast to (3), (1) loses the invariance of pseudoconformal invariance, which is very important in the arguments of [13, 15, 16]. Therefore, some appropriate modifications will be made in the argument of this work to reach our goal. In particular, we note that some techniques developed by Pang et al. [21] are adopted in this paper.

We state our main results.

Theorem 1. *There exist initial data ϕ_0 with $\|\phi_0\|_{L^2} = \|Q\|_{L^2}$ for which the solution of the Cauchy problem* (1)-(2) *blows up in a finite time.*

Theorem 2. *Let $\phi(t)$ be a blow-up solution of* (1) *with $\|\phi_0\|_{L^2} = \|Q\|_{L^2}$. Then there is $y_0 \in \mathbb{R}^N$ such that*

$$\phi(t, x) \longrightarrow \|Q\|_{L^2}^2 \delta_{y_0} \tag{9}$$

in the sense of distribution as $t \to T$.

Theorem 3. *There exists $C > 0$ such that*

$$\|\nabla\phi(t)\|_{L^2} \geq \frac{C}{T-t}, \quad \forall t \in [0, T). \tag{10}$$

Remark 4. For any blow-up solutions of (1), we know that $T \leq \pi/2$ (T is a blow-up time). When $T < \pi/2$, the formula presented in [18] is valid. For the minimal blow-up solutions with $T < \pi/2$, the conclusion of the above theorems can be found in [18]. However, there exist minimal blow-up solutions with $T = \pi/2$. For example, if the initial $\phi_0(x) = \psi_0(x) = Q(x)$, with $Q(x)$ being the solution of problem (5), the solution $\phi(t)$ of (1) will blow up at $T = \pi/2$, while the corresponding solution of (3) is a solitary wave $e^{it}Q(x)$. The minimal blow-up solutions with $T = \pi/2$ were sensible as pointed in [18].

In this paper, $L^q(\mathbb{R}^N)$, $\|\cdot\|_{L^q(\mathbb{R}^N)}$, and $\int_{\mathbb{R}^N} \cdot\, dx$ are denoted by L^q, $\|\cdot\|_{L^q}$, and $\int \cdot dx$, respectively. The various positive constants are also denoted by C.

This paper proceeds as follows. In Section 2, we establish some preliminaries. In Section 3, we give the proof of the existence and profile of the minimal blow-up solutions of (1) (Theorems 1 and 2). In Section 4, we derive the argument of the lower bound of the blow-up rate of the minimal blow-up solutions of (1) (Theorem 3).

2. Preliminaries

2.1. Local Wellposedness. The energy space of (1) was defined as

$$\Sigma := \left\{u \in H^1,\ |x|\, u \in L^2\right\}. \tag{11}$$

The inner product of the space Σ is defined as

$$\langle u, v\rangle := \int \nabla u \nabla \bar{v} + u\bar{v} + |x|^2 u\bar{v}\, dx. \tag{12}$$

The norm of Σ is denoted by $\|\cdot\|_\Sigma$. Moreover, we define an energy functional $\mathscr{E}$ on Σ by

$$\mathscr{E}(u) := \int |\nabla u|^2 + |x|^2|u|^2 - \frac{1}{1+2/N}|u|^{2+4/N} dx. \tag{13}$$

From Cazenave [17], we have the local well-posedness for the Cauchy problem of (1) follows.

Proposition 5. *For any $\phi_0 \in \Sigma$, there exist $T > 0$ and a unique solution $\phi(t, x)$ of the Cauchy problem* (1)-(2) *in* $C([0, T); \Sigma)$

such that either $T = \infty$ (global existence) or $T < \infty$ and $\lim_{t\to T}\|\phi(t)\|_{\Sigma} = \infty$ (blowup). Moreover, for any $t \in [0,T)$, it holds the conservation laws of mass

$$\|\phi(t)\|_{L^2} = \|\phi_0\|_{L^2} \tag{14}$$

and the energy

$$\mathscr{E}(\phi(t)) = \mathscr{E}(\phi_0). \tag{15}$$

2.2. Variational Characterization of the Ground State. Consider the equation

$$-\Delta u + \omega u - |u|^{4/N}u = 0, \quad u \in H^1(\mathbb{R}^N). \tag{16}$$

For (16), we set some notations such as $\mathscr{X}_\omega$ (the solution set), $\mathscr{G}_\omega$ (the ground solution set), and $\mathscr{G}$ as follows:

$$\begin{gathered} \mathscr{X}_\omega = \left\{u \in H^1;\ u \neq 0,\ -\Delta u + \omega u - |u|^{4/N}u = 0\right\}, \\ \mathscr{G}_\omega = \left\{u \in \mathscr{X}_\omega;\ S(u) \le S(v),\ \forall v \in \mathscr{X}\omega\right\}, \\ \mathscr{G} = \bigcup_{\omega\in R^+} \mathscr{G}_\omega, \end{gathered} \tag{17}$$

where $S(u) = \int (1/2)|\nabla u|^2 + (\omega/2)|u|^2 - (1/4/N + 2)|u|^{2+(4/N)}dx$.

For any $u \in \mathscr{X}_\omega$, the following two identities hold true:

$$\begin{aligned} &\int |\nabla u|^2 + \omega|u|^2 dx = \int |u|^{2+(4/N)}dx, \\ &\int (N-2)|\nabla u|^2 + N\omega|u|^2 dx \\ &\quad = \int \frac{N}{1+2/N}|u|^{2+(4/N)}dx \qquad \text{(Pohozaev's identity)}. \end{aligned} \tag{18}$$

The above two equalities imply

$$\mathscr{H}(u) = 0, \quad \forall u \in \mathscr{X}, \tag{19}$$

where

$$\mathscr{H}(u) := \int |\nabla u|^2 - \frac{1}{2/N+1}|u|^{2+(4/N)}dx. \tag{20}$$

Naturally, we get

$$u \in \mathscr{G}_\omega \Longleftrightarrow \begin{cases} u \in \mathscr{X}_\omega, \\ \|u\|_{L^2} \le \|v\|_{L^2}, \quad \forall v \in \mathscr{X}_\omega. \end{cases} \tag{21}$$

According to Cazenave [17], the set $\mathscr{G}_\omega$ can be described as

$$\mathscr{G}_\omega = \bigcup\left\{e^{i\theta}\varphi_\omega(\cdot - y);\ \theta \in R,\ y \in \mathbb{R}^N\right\}, \tag{22}$$

where φ_ω is a positive, spherically symmetric, decreasing, and real valued function.

It is of importance that Kwong [11] proved the uniqueness for the solution $Q(x)$ of the problem

$$\begin{gathered} -\Delta u + u - |u|^{4/N}u = 0, \quad u \in H^1, \\ u(x) = u(|x|), \\ u(x) > 0. \end{gathered} \tag{23}$$

Noticing the fact that $Q(x) = \varphi_\omega|_{\omega=1}$, it is easy to check that

$$\varphi_\omega = \omega^{N/4}Q\left(\omega^{1/2}x\right) \in \mathscr{G}_\omega, \quad \|\varphi_\omega\|_{L^2} = \|Q\|_{L^2}. \tag{24}$$

It follows from (21), (22), and (24) that

$$u \in \mathscr{G}_\omega \Longleftrightarrow \begin{cases} u \in \mathscr{X}_\omega, \\ \|u\|_{L^2} = \|Q\|_{L^2}, \end{cases} \tag{25}$$

$$\begin{aligned} \mathscr{G}_\omega &= \bigcup\left\{e^{i\theta}\varphi_\omega(\cdot - y);\ \theta \in R,\ y \in \mathbb{R}^N\right\}, \\ &= \bigcup\left\{e^{i\theta}\omega^{N/4}Q\left(\omega^{1/2}(\cdot - y)\right);\ \theta \in R,\ y \in \mathbb{R}^N\right\}. \end{aligned} \tag{26}$$

With functional $\mathscr{H}$ defined by (20), we now introduce the following constrained minimization problem

$$\mathscr{I}(\|Q\|_{L^2}) \equiv \inf\left\{\mathscr{H}(f) \mid f \in H^1,\ \|f\|_{L^2} = \|Q\|_{L^2}\right\}. \tag{27}$$

Now, we claim that

$$u \in \mathscr{G} \Longleftrightarrow u \text{ is a solution to the minimization problem (27)}. \tag{28}$$

In fact, $(N/(N+2))\|Q\|_{L^2}^{4/N}$ is the minimum of the functional (see Kwong [11] or Weinstein [9])

$$I(\psi) = \frac{\|\nabla\psi\|_{L^2}^2\|\psi\|_{L^2}^{4/N}}{\|\psi\|_{L^{2+4/N}}^{2+4/N}}, \quad \psi \in H^1, \tag{29}$$

which derives the Gagliardo-Nirenberg inequality

$$\|\psi\|_{L^{2+4/N}}^{2+4/N} \le \frac{N+2}{N}\left(\frac{\|\psi\|_{L^2}}{\|Q\|_{L^2}}\right)^{4/N}\|\nabla\psi\|_{L^2}^2. \tag{30}$$

The inequality (30) implies the following lemma on the functional $\mathscr{H}$.

Lemma 6 (see Weinstein [9]). *For any $f \in H^1$, one has*

$$\left[1 - \left(\frac{\|f\|_{L^2}}{\|Q\|_{L^2}}\right)^{4/N}\right]\|\nabla f\|_{L^2}^2 \le \mathscr{H}(f). \tag{31}$$

Lemma 6 implies that

$$\mathscr{H}(f) \ge 0, \quad \text{if } \|f\|_{L^2} \le \|Q\|_{L^2}. \tag{32}$$

It follows from (19), (27), and (32) that

$$\mathscr{I}(\|Q\|_{L^2}) = 0. \tag{33}$$

Hence, from (19) and (25), it holds that

$$u \in \mathscr{G} \Longrightarrow u \quad \text{is a solution to the minimization problem (27).} \tag{34}$$

On the other hand, if u is a minimizer of the variational problem of (27), it solves the Euler-Lagrange equation (16). So $u \in \mathscr{X}_\omega$ for some $\omega > 0$, and by (27) and (25), we know $u \in \mathscr{G}_\omega \subset \mathscr{G}$. This implies that

$$u \in \mathscr{G} \Longleftarrow u \quad \text{is a solution to the minimization problem (27).} \tag{35}$$

Hence (28) holds true.

Putting together (22), (25), and (28), we summarize the variational characterization.

Proposition 7. *Each of the following three statements is equivalent:*

(i) $u \in \bigcup_{\omega \in \mathbb{R}^+} \mathscr{G}_\omega$,

(ii) u *is a solution to the minimizing problem* $\min\{\mathscr{H}(u), \|u\|_{L^2} = \|Q\|_{L^2}\}$,

(iii) $u = e^{i\theta}\omega^{N/4}Q(\omega^{1/2}(x - x_0))$, *for some* $\theta \in \mathbb{R}$, $\omega \in \mathbb{R}^+$, *and* $x_0 \in \mathbb{R}^N$.

2.3. Lemmas

Lemma 8 (see Zhang [6]). *Let* $\phi_0 \neq 0$, *the initial datum of Cauchy problem* (1)-(2), *satisfy*

$$\mathscr{E}(\phi_0) \leq \int |x|^2 |\phi_0|^2 dx; \tag{36}$$

then $\phi(t)$ *blows up in a finite time.*

Consider the constrained minimization problem

$$I(\alpha) \equiv \min\left\{\mathscr{H}(f) \mid f \in H^1, \|f\|_{L^2} = \alpha\right\}. \tag{37}$$

For $I(\alpha)$, we cite a lemma in [15].

Lemma 9 (see Weinstein [15]). *(a) Consider* $I(\alpha) = 0$ *or* $I(\alpha) = -\infty$.

(b) Let $\alpha < \|Q\|_{L^2}$ *and* u_n *be a minimizing sequence; then it holds that* $I(\alpha) = 0$ *and* $u_n \rightharpoonup 0$ *weakly in* H^1.

Now, we recall some lemmas on the compactness.

Lemma 10 (see Brezis and Lieb [22]). *Let* $f \in L^1_{\text{loc}}$, $\|\nabla f\|_{L^2} \leq C$, *and* $\mu(|f| > \varepsilon) \geq \delta > 0$. *Then there exists a shift* $T_y f(x) = f(x + y)$ *such that, for some constant* $\alpha = \alpha(C, \delta, \varepsilon)$,

$$\mu\left(B(0,1) \cap \left[T_y g > \frac{\varepsilon}{2}\right]\right) > \delta. \tag{38}$$

Lemma 11 (see Lieb [23]). *Let* f_j *be a uniformly bounded sequence of functions in* $W^{1,p}$ *with* $1 < p < \infty$. *Assume further that there are positive constant* C *and* η *satisfying* $\mu(|f_j| > \eta) \geq C$. *Then there exists a sequence* $y_j \in \mathbb{R}^N$ *such that*

$$f_j(\cdot + y_j) \rightharpoonup f \neq 0 \quad \text{weakly in } W^{1,p}. \tag{39}$$

Lemma 12. *Let* θ *be a real-valued function on* $\mathbb{R}^N$ *and* $v \in H^1(\mathbb{R}^N)$ *with* $\|v\|_{L^2} \leq \|Q\|_{L^2}$. *Then*

$$\left|\Im \int \bar{v}(x) \nabla\theta(x)\, dx\right| \leq \left(2\mathscr{H}(v) \int |v(x)|^2 |\nabla\theta(x)|^2 dx\right)^{1/2}. \tag{40}$$

Proof. It follows from (30) and $\|v\|_{L^2} \leq \|Q\|_{L^2}$ that

$$\mathscr{H}\left(e^{i\alpha\theta} v\right) \geq 0 \tag{41}$$

for all real numbers α. On the other hand, it has

$$\mathscr{H}\left(e^{i\alpha\theta} v\right) = \alpha^2 \int |v|^2 |\nabla\theta|^2 dx - \alpha \int \Im(v\nabla\bar{v})\nabla\theta dx + \mathscr{H}(v). \tag{42}$$

Thus the discriminant of the equation in α must be negative or null and the desired inequality follows. □

Lemma 13. *There is a constant* c_0 *such that*

$$\int |x|^2 |\phi(t,x)|^2 dx \leq c_0. \tag{43}$$

Proof. Setting $J(t) = \int |x|^2 |\phi(t,x)|^2 dx$, we have

$$J'(t) = 2\Im \int \bar{\phi}\nabla\phi dx, \qquad J''(t) = 4\mathscr{E}(\phi) - 4J(t). \tag{44}$$

It follows that

$$J(t) = (J(0) - \mathscr{E}(0))\cos t + J'(0)\sin t + \mathscr{E}(0), \tag{45}$$

which implies the conclusion. □

Lemma 14 (see [16, page 433]). *Let* $u_n \in H^1$, $c_0 > 0$, *and* $R_0 > 0$, *for arbitrary* n, *satisfy*

$$\begin{gathered} \mathscr{H}(u_n) \leq c_0, \\ \|u_n\|_{L^2} \leq \|Q\|_{L^2}, \\ \|\nabla u_n\|_{L^2} \longrightarrow \infty, \\ \int_{|x|>r_0} |u_n|^2 dx \leq \varepsilon(n), \end{gathered} \tag{H}$$

where $\varepsilon(n) > 0$ *depends only on* n. *Then, it holds that*

$$\int_{|x|>4r_0} |\nabla u_n|^2 dx \leq A, \tag{46}$$

with $A = A(r_0, c_0 > 0)$.

3. Profile of the Minimal Blow-Up Solution

Now we prove the existence of the minimal blow-up solutions.

Proof of Theorem 1. Setting $\phi_0 = \phi(c,\lambda) = c\lambda^{N/2}Q(\lambda x)$ with λ being arbitrary positive real number and c being complex number satisfying $|c| = 1$, then

$$\|\phi_0\|_{L^2} = \|Q\|_{L^2}. \tag{47}$$

From (15) and (19), the corresponding energy is

$$\begin{aligned}\mathscr{E}(\phi_0) &= \left(1 - |c|^{4/N}\right)|c|^2\lambda^2 \int |\nabla Q|^2 dx \\ &\quad + \int |x|^2|\phi_0|^2 dx = \int |x|^2|\phi_0|^2 dx.\end{aligned} \tag{48}$$

Thus Lemma 8 infers that $\phi(t,x)$ blows up in a finite time. □

Employing the concentration compactness lemma, we can prove the following proposition which is crucial to the study of the blow-up profile (Theorem 2).

Proposition 15. *Let $\phi(t) \in C([0,T),\Sigma)$ be a blow-up solution of the Cauchy problem (1)-(2) and T is the blow-up time. Set $\lambda(t) = \|\nabla Q\|_{L^2}/\|\nabla\phi(t)\|_{L^2}$ and $(S_\lambda\phi)(x,t) = \lambda^{N/2}\phi(\lambda x,t)$. If*

$$\|\phi_0\|_{L^2} = \|Q\|_{L^2}, \tag{49}$$

it holds that

$$S_{\lambda(t)}\phi(\cdot + y(t), t)\, e^{i\gamma(t)} \longrightarrow Q(\cdot) \quad \text{in } H^1, \text{ as } t \longrightarrow T \tag{50}$$

with $y(t) \in \mathbb{R}^N$ and $\gamma(t) \in \mathbb{R}$.

Proof. Let $t_k \to T$. We choose $\lambda_k = \lambda(t_k)$ to satisfy

$$\left\|\nabla S_{\lambda_k}\phi(\cdot + y_k, t_k)\right\|_{L^2} = \lambda_k\|\nabla\phi(\cdot + y_k, t_k)\|_{L^2} = \|\nabla Q\|_{L^2}. \tag{51}$$

Setting $\phi_k \equiv S_{\lambda_k}\phi(\cdot + y_k, t_k)$, noticing that $\|\phi(t_k)\|_{L^2}$ tends to ∞ as $t_k \to T$, $\lambda_k \to 0$, and

$$\|\phi_k\|_{L^2} = \|\phi(t_k)\|_{L^2} = \|\phi_0\|_{L^2}, \tag{52}$$

we know that ϕ_k is uniformly bounded in H^1 and there is a weakly convergent subsequence ϕ_{k_j} such that

$$\begin{aligned}\phi_{k_j} &\rightharpoonup \phi \quad \text{in } L^2, \\ \phi_{k_j} &\rightharpoonup \phi \quad \text{in } H^1.\end{aligned} \tag{53}$$

We note that

$$\begin{aligned}\mathscr{H}\left(\phi_{k_j}\right) = \lambda_{k_j}^2\mathscr{H}\left(\phi\left(t_{k_j}\right)\right) \le \lambda_{k_j}^2\mathscr{E}(\phi_0) \longrightarrow 0,& \\ j \longrightarrow \infty.&\end{aligned} \tag{54}$$

Since we have assumed $\|\phi_0\|_{L^2} = \|Q\|_{L^2}$, by (52), (54), and (31), we know that ϕ_k is a minimizing sequence for the variational problem (27).

Next, we will prove that the minimizing sequence ϕ_k has a subsequence ϕ_{k_j} and a family y_j such that $\phi_{k_j}(\cdot - y_k)$ has a strong limit in H^1. To see this, we need to make use of the concentration-compactness lemma (Lions [24]) which means that ϕ_{k_j} has one of three properties: vanishing, dichotomy, and compactness.

Vanishing. For every $M < \infty$, one has

$$\lim_{j\to\infty}\sup_{y\in\mathbb{R}^N}\int_{y_j+B_r(M)}\left|\phi_{k_j}(x)\right|^2 dx = 0. \tag{55}$$

Dichotomy. There exist a constant $\alpha \in (0,\|Q\|_{L^2})$ and sequences ψ_j^1 and ψ_j^2, bounded in H^1, such that, for all $\varepsilon > 0$, there exists $j_0 > 0$ such that for $j > j_0$

$$\begin{gathered}\left|\left\|\psi_j^1\right\|_{L^2} - \alpha\right| \le \varepsilon, \qquad \left|\left\|\psi_j^2\right\|_{L^2} - \left(\|Q\|_{L^2} - \alpha\right)\right| \le \varepsilon, \\ \left\|\phi_{k_j} - \psi_j^1 - \psi_j^2\right\|_{H^1} \le \varepsilon, \\ \left\|\phi_{k_j} - \psi_j^1 - \psi_j^2\right\|_{L^p} \le \varepsilon \quad \text{for } 2 \le p < \frac{2N}{N-2}, \\ \text{distance}\left(\operatorname{supp}\psi_j^1, \operatorname{supp}\psi_j^2\right) \longrightarrow \infty.\end{gathered} \tag{56}$$

Compactness. There exists y_j in $\mathbb{R}^N$. For any $\varepsilon > 0$, we can find $M < \infty$ such that

$$\int_{y_j+B_r(M)}\left|\phi_{k_j}\right|^2 dx \ge \|Q\|_{L^2}^2 - \varepsilon. \tag{57}$$

Now, we exclude the cases of vanishing and dichotomy.

Exclusion of Vanishing. By (52), (51), and (54) there are $C_1 > 0$ and $C_2 > 0$ such that

$$\|\phi_k\|_{L^2}^2 \le C_1, \qquad \|\phi_k\|_{L^{2+4/N}}^{2+4/N} \ge C_2 > 0. \tag{58}$$

By the boundness of $\|\phi_k\|_{H^1}$ and the Sobolev inequality, there exist $\gamma > 2 + 4/N$ and $C_3 > 0$ such that

$$\|\phi_k\|_{L^\gamma}^\gamma \le C_3. \tag{59}$$

Now, we show the existence of positive constants ε and δ such that

$$\mu(|\phi_k| > \varepsilon) \ge \delta > 0. \tag{60}$$

Indeed, from (58) and (59), for sufficiently small $\varepsilon > 0$, we get

$$\begin{aligned} c_2 &\le \int |\phi_k|^{2+4/N} dx \\ &= \int_{\{|\phi_k|<\varepsilon\}} |\phi_k|^{2+4/N} dx \\ &\quad + \int_{\{\varepsilon<|\phi_k|<(1/\varepsilon)\}} |\phi_k|^{2+4/N} dx + \int_{\{|\phi_k|>(1/\varepsilon)\}} |\phi_k|^{2+4/N} dx \\ &\le \frac{C_2}{4C_1} \int_{\{|\phi_k|<\varepsilon\}} |\phi_k|^2 dx \\ &\quad + \int_{\{\varepsilon<|\phi_k|<(1/\varepsilon)\}} |\phi_k|^{2+4/N} dx + \frac{C_2}{4C_3} \int_{\{|\phi_k|>(1/\varepsilon)\}} |\phi_k|^{\gamma} dx \\ &\le \frac{C_2}{4C_1} \|\phi_k\|_{L^2}^2 + \int_{\{\varepsilon<|\phi_k|<(1/\varepsilon)\}} |\phi_k|^{2+4/N} dx \\ &\quad + \frac{C_2}{4C_3} \|\phi_k\|_{L^\gamma}^{\gamma} dx \\ &\le \frac{C_2}{2} + \mu\left(|\phi_k| > \varepsilon\right) \left(\frac{1}{\varepsilon}\right)^{2+4/N}. \end{aligned} \tag{61}$$

Thus we know that (60) with $\delta = (C_2/2)\varepsilon^{2+4/N}$ is valid. From (60) and Lemma 10, there exist α and y_k satisfying

$$\mu\left(\{|x| \le 1\} \cap \{|\phi_k(\cdot + y_k)|\} > \frac{\varepsilon}{2}\right) > \delta. \tag{62}$$

Thus,

$$\int_{|x|\le 1} \left|\phi_{k_j}(\cdot + y_j)\right|^2 dx \ge \left(\frac{\varepsilon}{2}\right)^2 \delta, \tag{63}$$

which excludes the occurrence of vanishing.

Exclusion of Dichotomy. Suppose by contradiction that dichotomy occurs. Then, by the same argument as that in the case of *vanishing* we can get

$$0 < \upsilon < \mu\left\{\theta < \left|\psi_j^1\right|\right\}, \tag{64}$$

where θ and υ are two constants and ψ_j^1 is bounded in H^1. Hence, by Lemma 11, there are a subsequence $\psi_{j_r}^1$ and a sequence y_r such that

$$\psi_{j_r}^1(\cdot + y_r) \rightharpoonup \psi \ne 0 \quad \text{in } H^1. \tag{65}$$

Using (56) gives rise to

$$\begin{aligned} 0 = I\left(\|Q\|_{L^2}\right) &\ge \liminf_{r\to\infty} \mathscr{H}\left(\psi_{j_r}^1\right) + \liminf_{r\to\infty} \mathscr{H}\left(\psi_{j_r}^2\right) \\ &= \liminf_{r\to\infty} \mathscr{H}\left(\psi_{j_r}^1\right). \end{aligned} \tag{66}$$

On the other hand, the fact $\|\psi_{j_r}^1\|_{L^2} < \|Q\|_{L^2}$ implies with Lemma 6 that

$$\liminf_{r\to\infty} \mathscr{H}\left(\psi_{j_r}^1\right) \ge 0. \tag{67}$$

Thus, for any fixed n^*, it has

$$\begin{aligned} 0 = I\left(\|Q\|_{L^2}\right) &\ge \liminf_{r\to\infty} \mathscr{H}\left(\psi_{j_r}^1\right) = \sup_n \inf_{r\ge n} \mathscr{H}\left(\psi_{j_r}^1\right) \\ &\ge \inf_{r\ge n^*} \mathscr{H}\left(\psi_{j_r}^1\right). \end{aligned} \tag{68}$$

We can then extract a minimizing subsequence, which we rename it by $\psi_{j_r}^1$; that is, $\lim_{r\to\infty} \mathscr{H}(\psi_{j_r}^1) = 0$. Using Lemma 9 yields

$$\psi_{j_r}^1 \longrightarrow 0, \tag{69}$$

which is impossible from (65).

Occurrence of Compactness. It follows from the previous arguments that compactness occurs. By (57), we get

$$\|Q\|_{L^2}^2 - \varepsilon \le \int_{y_j+B(M)} \left|\phi_{k_j}\right|^2 dx \le \int \left|\phi_{k_j}\right|^2 dx \le \|Q\|_{L^2}^2. \tag{70}$$

For $\phi_{k_j}(\cdot + y_j)$ being bounded in $H^1(\mathbb{R}^N)$, there exist $\phi \in H^1(\mathbb{R}^N)$ and a subsequence, which we again label it by ϕ_{k_j}, such that

$$\phi_{k_j}\left(\cdot + y_j\right) \rightharpoonup \phi \quad \text{in } H^1. \tag{71}$$

Given $M > 0$, the embedding $H^1(\mathbb{R}^N) \hookrightarrow L^2(\{|x| \le r\})$ is compact and

$$\int_{|x|\le r} |\phi|^2 dx = \lim_{j\to\infty} \int_{x_m+B(r)} \left|\phi_{k_j}\right|^2 dx. \tag{72}$$

Making use of (70) derives

$$\int_{\mathbb{R}^N} |\phi|^2 dx \ge \|Q\|_{L^2}^2 - \varepsilon \tag{73}$$

for any $\varepsilon > 0$. Hence, it holds that

$$\int_{\mathbb{R}^N} |\phi|^2 dx = \|Q\|_{L^2}^2. \tag{74}$$

It follows that

$$\phi_{k_j}\left(\cdot + y_j\right) \longrightarrow \phi \quad \text{in } L^2, \tag{75}$$

which implies with the Gagliardo-Nirenberg inequality (30) that

$$\phi_{k_j}\left(\cdot + y_j\right) \longrightarrow \phi \quad \text{in } L^{2+4/N}. \tag{76}$$

To show $\phi_{k_j} \to \phi$ in H^1, we only need to show that $\|\nabla\phi\|_{L^2} = \|\nabla Q\|_{L^2}$.

From (51) and (54), we know that

$$\begin{aligned} 0 &= \lim_{t\to T} \mathscr{H}\left(\phi_{\phi_{k_j}}\right) \\ &= \|\nabla Q\|_{L^2} - \frac{1}{2/N+1} \lim_{t\to T} \int \left|\phi_{k_j}\right|^{4/N+2} dx \\ &= \|\nabla Q\|_{L^2} - \frac{1}{2/N+1} \lim_{t\to T} \int |\phi|^{4/N+2} dx. \end{aligned} \tag{77}$$

Hence, $\|\nabla\phi\|_{L^2} < \|\nabla Q\|_{L^2}$ derives $\mathscr{E}(\phi) < 0$. This contradicts Lemma 6 and the fact $\phi \neq 0$.

Since ϕ solves the minimizing problem (27), it satisfies the Euler-Lagrange equation (16). Noticing the fact $\|\nabla|\phi|\|_{L^2} \leq \|\nabla\phi\|_{L^2}$, we infer that $|\phi|$ is also a solution to problem (27). Thus it is a nonnegative solution of (16). It follows from $\|\phi\|_{L^2} = \|Q\|_{L^2}$, $\|\nabla\phi\|_{L^2} = \|\nabla Q\|_{L^2}$, and Proposition 7 that

$$\phi = Q\left(\cdot + y_j\right) e^{i\gamma} \tag{78}$$

for some $y \in \mathbb{R}^N$ and $\gamma \in \mathbb{R}$. By redefining the sequence y_j, we can set $\gamma = 0$. □

Proof of Theorem 2. It follows from Proposition 15 that

$$\lambda^N(t)\left|\phi(t, \lambda(t)(x + x(t)))\right|^2 \longrightarrow |Q(x)|^2 \quad \text{in } L^1 \text{ as } t \longrightarrow T, \tag{79}$$

$$\left|\phi(t, x + x(t))\right|^2 \longrightarrow \|Q\|_{L^2}^2 \delta_{x=0} \quad \text{as } t \longrightarrow T. \tag{80}$$

Using Lemma 13 derives that

$$\limsup_{t \to T} |x(t)| \leq \frac{\sqrt{C_0}}{\|Q\|_{L^2}}. \tag{81}$$

Hence we have a positive constant r_0 such that

$$\forall t \in [0, T), \qquad |x(t)| \leq r_0. \tag{82}$$

$$\begin{aligned}
&\int_{B(0,r)} |\phi(t,x)|^2 x\,dx \\
&\quad = \int_{B(0,r)} |\phi(t,x)|^2 (x - x(t))\,dx \\
&\qquad + \int_{B(0,r)} |\phi(t,x)|^2 x(t)\,dx \\
&\quad = \int_{B(-x(t),r)} \left|\phi(t, y + x(t))\right|^2 y\,dy \\
&\qquad + \int_{B(-x(t),r)} \left|\phi(t, y + x(t))\right|^2 x(t)\,dy.
\end{aligned} \tag{83}$$

From (82), for arbitrary $r > r_0$, there is a $\delta > 0$ such that $B(0,\delta) \subset B(-x(t), r)$. The formula (80) implies that

$$\int_{B(0,r)} |\phi(t,x)|^2 x\,dx - \int |Q(x)|^2 x(t)\,dx = 0. \tag{84}$$

On the other hand, Lemma 13 implies that

$$\int_{|x|>r} |\phi(t,x)|^2 x\,dx \leq \frac{C_0}{r}. \tag{85}$$

Thus

$$\lim_{t \to T} \left\{ \int |\phi(t,x)|^2 x\,dx - \int |Q(x)|^2 dx\, x(t) \right\} = 0. \tag{86}$$

By Lemma 12, we obtain

$$\begin{aligned}
&\frac{d}{dt}\left| \int |\phi(t,x)|^2 x\,dx \right| \\
&\quad = \left| 2\Im \int \overline{\phi}(t,x)\nabla\phi(t,x)\,dx \right| \\
&\quad = 2\Im \sum_{j=1}^{N} \int \left| \overline{\phi}(t,x)\nabla\phi(t,x) \cdot \nabla\theta_j(x)\,dx \right| \\
&\quad \leq 2\sum_{j=1}^{N} \left(2\mathscr{H}(\phi(t)) \int |\phi(t,x)|^2 \left|\nabla\theta_j(x)\right|^2 dx \right)^{1/2} \leq C,
\end{aligned} \tag{87}$$

where $\theta_j(x) = x_j$. Hence there exists $x_1 \in \mathbb{R}^N$ such that

$$\lim_{t \to T} \int |\phi(t,x)|^2 x\,dx = -\left(\int |Q(x)|^2 dx \right) x_1. \tag{88}$$

Combining (86) with (88), we know that $x(t) \to -x_1$ as $t \to T$ and we have

$$|u(t,x)| \longrightarrow \|Q\|_{L^2}^2 \delta_{x=x_1}. \tag{89}$$

□

4. Blow-Up Rate

To establish the lower bound of the blow-up rate, we use the following proposition.

Proposition 16. *Letting y_0 be the blow-up point determined in Theorem 2, it has*

$$\lim_{t \to T} \int \left|x - y_0\right|^2 |\phi(t,x)|^2 dx = 0. \tag{90}$$

Proof. Let us define a positive function $h(x) \in C^1(\mathbb{R}^N)$ such that

$$h(x) = h(|x|) = \begin{cases} = 0, & |x| < 1, \\ > 0, & 1 < |x| < 2, \\ = \dfrac{|x|^2}{4}, & |x| > 2, \end{cases} \tag{91}$$

and $h_A(x) = A^2 h(x/A)$ for $A > 0$ and it is valid that

$$\left|\nabla h_A(x)\right|^2 \leq C h_A(x), \quad \forall x \in \mathbb{R}^N. \tag{92}$$

Carrying out direct computation and using Hölder's inequality, we have

$$\begin{aligned}
&\left|\frac{d}{dt}\int|\phi(t,x)|^2 h_A(x-y_0)\,dx\right| \\
&\quad=\left|2\Im\sum_{j=1}^{N}\int\overline{\phi}(t,x)\nabla\phi(t,x)\cdot\nabla h_A(x-y_0)\,dx\right| \\
&\quad\le C\left(\int_{|x-y_0|\ge A}|\nabla\phi(t,x)|^2dx\right)^{1/2} \\
&\qquad\times\left(\int|\phi(t,x)|^2\nabla h_A(x-y_0)\,dx\right)^{1/2} \\
&\quad\le C\left(\int_{|x-y_0|\ge A}|\nabla\phi(t,x)|^2dx\right)^{1/2} \\
&\qquad\times\left(\int|\phi(t,x)|^2 h_A(x-y_0)\,dx\right)^{1/2},
\end{aligned} \tag{93}$$

which implies

$$\begin{aligned}
&\left|\frac{d}{dt}\left(\int|\phi(t,x)|^2 h_A(x-y_0)\,dx\right)^{1/2}\right| \\
&\quad\le C\left(\int_{|x-y_0|\ge A}|\nabla\phi(t,x)|^2dx\right)^{1/2}.
\end{aligned} \tag{94}$$

Integrating on both sides gives rise to

$$\begin{aligned}
&\sup_{t\in[0,T)}\left(\int|\phi(t,x)|^2 h_A(x-y_0)\,dx\right)^{1/2} \\
&\quad\le\left(\int|\phi_0(x)|^2 h_A(x-y_0)\,dx\right)^{1/2} \\
&\qquad+C\int_0^T\left(\int_{|x-y_0|\ge A}|\nabla\phi(s,x)|^2dx\right)^{1/2}ds.
\end{aligned} \tag{95}$$

From the fact $\phi_0\in\Sigma$, we have

$$\begin{aligned}
&\sup_{t\in[0,T)}\left(\int|\phi(t,x)|^2 h_A(x-y_0)\,dx\right)^{1/2} \\
&\quad\le\varepsilon(A)+C\int_0^T\left(\int_{|x-y_0|\ge A}|\nabla\phi(s,x)|^2dx\right)^{1/2}ds.
\end{aligned} \tag{96}$$

By the virtue of Lemma 14 and Proposition 16, there exist A_1 and $C_2>0$ such that

$$\int_{|x-y_0|\ge A_1}\left(|\nabla\phi(s,x)|^2dx\right)^{1/2}ds\le C_2,\quad\forall s\in[0,T). \tag{97}$$

Using the dominated convergence theorem, we infer that

$$\lim_{A\to\infty}\int_0^T\left(\int_{|x-y_0|\ge A}|\nabla\phi(s,x)|^2dx\right)^{1/2}ds=0. \tag{98}$$

Thus, it holds that

$$\lim_{A\to\infty}\sup_{t\in[0,T)}\left(\int_{|x-y_0|\ge A}|\phi(t,x)|^2|x-y_0|^2dx\right)=0, \tag{99}$$

which implies that there is $a_\varepsilon>0$ such that, for $\forall t\in[0,T)$,

$$\int_{|x-y_0|\ge a_\varepsilon}|x-y_0|^2|\phi(t,x)|^2dx\le\frac{\varepsilon}{2}. \tag{100}$$

The identity $\|\phi(t)\|_{L^2}=\|\phi_0\|_{L^2}=\|Q\|_{L^2}$ shows that

$$\begin{aligned}
&\int_{|x-y_0|\le b_\varepsilon}|x-y_0|^2|\phi(t,x)|^2dx\le b_\varepsilon^2\|Q\|_{L^2} \\
&\quad\le\frac{\varepsilon}{2},\quad\text{for } b_\varepsilon^2=\frac{\varepsilon}{2\|Q\|_{L^2}}.
\end{aligned} \tag{101}$$

In addition, we have

$$\begin{aligned}
&\int_{b_\varepsilon\le|x-y_0|\le a_\varepsilon}|x-y_0|^2|\phi(t,x)|^2dx \\
&\quad\le a_\varepsilon^2\int_{b_\varepsilon\le|x-y_0|\le a_\varepsilon}|\phi(t,x)|^2dx.
\end{aligned} \tag{102}$$

Using Theorem 2 yields

$$\lim_{t\to T}\int_{b_\varepsilon\le|x-y_0|\le a_\varepsilon}|x-y_0|^2|\phi(t,x)|^2dx=0. \tag{103}$$

In conclusion, for all $\varepsilon>0$, we have shown that

$$\lim_{t\to T}\int|x-y_0|^2|\phi(t,x)|^2dx\le\frac{\varepsilon}{2}+\frac{\varepsilon}{2}\le\varepsilon. \tag{104}$$

□

Now, we establish the lower bound of the blow-up rate.

Proof of Theorem 3. Simple calculation yields

$$\begin{aligned}
&\frac{d}{dt}\int|x-y_0|^2|\phi(t,x)|^2dx \\
&\quad=4\Im\int(x-y_0)\phi(t,x)\nabla\overline{\phi}(t,x).
\end{aligned} \tag{105}$$

Therefore, the inequality (40) in the case $\theta(x)=|x-y_0|^2$ implies that

$$\left|\frac{d}{dt}\left(\int|x-y_0|^2|\phi(t,x)|^2dx\right)^{1/2}\right|\le C. \tag{106}$$

Integrating from t to T, by Proposition 16, we obtain

$$\left|\left(\int|x-y_0|^2|\phi(t,x)|^2dx\right)^{1/2}\right|\le C(T-t). \tag{107}$$

Combining the above inequality and the following inequality

$$\begin{aligned}
&\left(\int|\phi(t,x)|^2dx\right)^2 \\
&\quad\le\left(\int|x-y_0|^2|\phi(t,x)|^2dx\right)\left(\int|\nabla\phi(t,x)|^2dx\right),
\end{aligned} \tag{108}$$

we get the result

$$\left\|\nabla\phi(t)\right\|_{L^2} \geq \frac{\|Q\|_{L^2}}{C(T-t)}. \tag{109}$$

□

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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