

## Research Article

# Approximate Analytical Solution for the Forced Korteweg-de Vries Equation

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The forced Korteweg-de Vries (fKdV) equations are solved using Homotopy Analysis Method (HAM). HAM is an approximate analytical technique which provides a novel way to obtain series solutions of such nonlinear problems. It has the auxiliary parameter  $\hbar$ , where it is easy to adjust and control the convergence region of the series solution. Some examples of forcing terms are employed to analyse the behaviours of the HAM solutions for the different fKdV equations. Finally, this form of HAM solution is compared with the analytical soliton-type solution of fKdV equation as derived by Zhao and Guo. The results is found to be in good agreement with Zhao and Guo.

## 1. Introduction

An analytical model of Tsunami propagation was proposed by Pelinovsky et al. [1] as follows:

$$\frac{\partial \eta}{\partial t} + c \frac{\partial \eta}{\partial x} + \alpha \eta \frac{\partial \eta}{\partial x} + \beta \frac{\partial^3 u}{\partial x^3} = \frac{\partial f}{\partial x} \quad (1)$$

with

$$\alpha = \frac{3c}{2h_0}, \quad \beta = \frac{ch_0^2}{6}, \quad f = \frac{-cz}{2}, \quad (2)$$

where  $\eta = \eta(x, t)$  refers to the elevation of free water surface,  $z = z(x, t)$  represents the solid bottom,  $h$  is assumed to be the constant mean water depth, and  $c \approx \sqrt{gh}$  is the long wave speed with  $g$  being gravity acceleration. If the right-hand side of (1), that is,  $\partial f / \partial x = 0$ , then (1) will be the Korteweg-de Vries equation which is completely integrable [2]. But if  $\partial f / \partial x \neq 0$ , then the equation is difficult to be integrable where the term  $\partial f / \partial x$  is called forcing term. Therefore, (1) is known as “KdV equation with forcing term” or “forced KdV (fKdV)” equation [3].

The fKdV equation to be considered in this paper is as follows:

$$\frac{\partial u}{\partial t} + \alpha u \frac{\partial u}{\partial x} + \beta \frac{\partial^3 u}{\partial x^3} = \frac{\partial f}{\partial x}, \quad (3)$$

where  $\alpha$  and  $\beta$  are constant.

This form (3) is an alternative approach in describing the governing equations for the basic hydrodynamic model of Tsunami generation, for example, by atmospheric disturbances (Akylas [4] and Nosov and Skachko [5]). The appropriate evolution equation, which asymptotically approximates the Boussinesq equation, leads to this fKdV equation (3). The forcing term in the fKdV can be assumed to be derivable from atmospheric disturbances. Various forms of this equation have been extensively studied (see Grimshaw et al. [6], Pelinovsky et al. [7]) and numerical results show that the solution contains the set of solitary waves. Evidently, the behaviour of Tsunami waves on the open ocean is considerably more complicated than the solitary wave model; nonetheless this fKdV equation has been used in Yaacob et al. [8] as a simple mathematical model that could describe

the Tsunami waves modelling. In the present paper, the above-mentioned considerations motivate us to investigate the approximate analytical solutions of the fKdV equation by using HAM.

Up to now, to the best of our knowledge, the solutions of fKdV equation of (3) can only be obtained by numerical or perturbation techniques [3, 9, 10]. Recently, the analytical solution of fKdV equation with a certain form of forcing term has been solved by Zhao and Guo [3], using Hirota direct method. Thus the main reason of this work is to solve fKdV equation by using the homotopy analysis method (HAM) for various forcing terms, including in [3]. This approximate analytical HAM solution will be used to be compared with the analytical soliton-type solution of Zhao and Guo [3]. This analysis would in a way demonstrate the effectiveness and the potential of HAM to solve difficult nonlinear equations.

The homotopy analysis method (HAM) is an analytic method to solve nonlinear partial differential problems, which was first introduced by Liao in 1992 [11]. It is known that HAM has greater flexibility in the selection of a proper set of base functions for the solution and a much simpler way in the control of the convergence rate and region compared to perturbation approach [12–19]. Furthermore, this analytic technique does not have restriction of nonperturbation methods, such as Lyapunov's artificial small parameter method, the  $\delta$ -expansion method, and Adomian's decomposition method [20]. The analytic technique had been applied successfully in many nonlinear problems in engineering and science [21, 22], for example, the magnetohydrodynamic flows of non-Newtonian fluids over a stretching sheet [23], nonlinear progressive waves [24], free oscillations of positively damped systems with algebraically decaying amplitude [25], free oscillations of self-excited systems [26], and similarity boundary layer equations [27].

The paper is organized as follows. The HAM of the fKdV equation (3) is presented in Section 2. Section 3 briefly illustrates the HAM solution of KdV equation. Section 4 is divided into 2 parts. The first (Section 4.1) discusses the concept of HAM and is followed by some examples of external forces employed on the fKdV equation. Meanwhile, Section 4.2 compares the results of the analytical solution of [3] with the generated HAM solution. Conclusion is concisely laid out in Section 5.

## 2. Basic Ideas of HAM

We consider differential equation in the form of  $u(r, t)$ ,

$$N[u(r, t)] = 0, \quad (4)$$

where  $N$  is the nonlinear operator and  $u(r, t)$  is the unknown function.

Using concept of HAM, we construct the so-called zeroth-order deformation equation

$$(1 - q) \ell [\varphi(r, t; q) - u_0(r, t)] = q \hbar H(r, t) N[\varphi(r, t; q)], \quad (5)$$

where  $u_0(r, t)$ , the initial guess of exact solution,  $u(r, t)$ ;  $\hbar$ , an auxiliary parameter with condition,  $\hbar \neq 0$ ;  $H$ , an auxiliary

function;  $\ell$ , an auxiliary linear operator;  $q$ , an embedding parameter with condition,  $q \in [0, 1]$ .

Note that, when  $q = 0$ , (5) holds for  $\varphi(r, t; 0) = u_0(r, t)$  and when  $q = 1$ , it holds for  $\varphi(r, t; 1) = u(r, t)$ . As  $q$  increases from 0 to 1, the solution of  $u(r, t, q)$  varies from the initial guess,  $u_0(r, t)$ , to the exact solution,  $u(r, t)$ .

By Taylor expansion of  $\varphi(r, t; q)$ , in a power series with respect to  $q$ , we obtain

$$\varphi(r, t; q) = \varphi(r, t; 0) + \sum_{m=1}^{\infty} u_m(r, t) q^m, \quad (6)$$

where

$$u_m(r, t) = \frac{1}{m!} \frac{\partial^m \varphi(r, t; q)}{\partial q^m} \bigg|_{q=0}. \quad (7)$$

With appropriate property of  $u_0(r, t)$ ,  $\hbar$ ,  $H$ , and  $\ell$ , the series of (6) is convergence at  $q = 1$ , and one has

$$u(r, t) = u_0(r, t) + \sum_{m=1}^{\infty} u_m(r, t). \quad (8)$$

Let us define the following vectors:

$$\vec{u}_n(r, t) = \{u_0(r, t), u_1(r, t), \dots, u_{n-1}(r, t), u_n(r, t)\}. \quad (9)$$

Differentiating the zeroth-order deformation equation (5)  $m$ -times with respect to the embedding parameter  $q$  and dividing it by  $m!$  and finally letting  $q = 0$ , we obtain the following  $m$ th-order deformation equation:

$$\ell[u_m(r, t) - \chi_m u_{m-1}(r, t)] = \hbar H(r, t) R_m[\vec{u}_{m-1}(r, t)], \quad (10)$$

where

$$\chi_m = \begin{cases} 1, & \text{when } m > 1, \\ 0, & \text{otherwise,} \end{cases}$$

$$R_m[\vec{u}_{m-1}(r, t)] = \frac{1}{(m-1)!} \left\{ \frac{\partial^{m-1}}{\partial q^{m-1}} N \left[ \sum_{m=0}^{\infty} u_m(r, t) q^m \right] \right\} \bigg|_{q=0}. \quad (11)$$

It is noted that HAM contains control-convergence parameter  $\hbar$  where it provides an easy way to adjust and control the convergence region. This can be done by plotting the  $\hbar$ -curves.

## 3. HAM Solution on KdV

When the forcing term of (3) becomes zero, then the equation becomes the Korteweg-de Vries (KdV) equation. Wazwaz [2] gave a form of the exact solution of KdV equation. Nazari et al. [28] used HAM to obtain a comparable approximate analytical solution of the KdV equation, which is in good

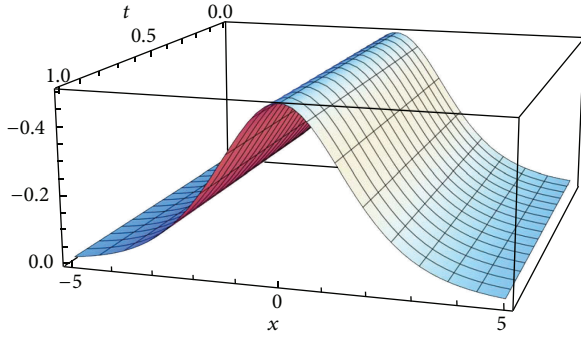


FIGURE 1: The HAM solution of KdV whereby forcing term is zero.

agreement with Wazwaz's exact solution of KdV equation. Figure 1 is given the graphical representation of the HAM solution of KdV equation, where the forcing term of (3) is zero;  $\partial f / \partial x = 0$ .

Further to this research output, we will now apply HAM to solve different fKdV equations based on various forcing terms.

#### 4. Approximate Analytical Solution for fKdV Using HAM

For HAM, the zeroth-order deformation equation is given by

$$(1 - q) \ell [u(x, t; q) - u_0(x, t)] = q \hbar H(x, t) N[u(x, t; q)] \quad (12)$$

and we use

$$u_0(x, t) = \frac{-2e^x}{(1 + e^x)^2} \quad (13)$$

as the initial guess and

$$\ell [\varphi(x, t; q)] = \frac{\partial u(x, t; q)}{\partial t} \quad (14a)$$

as the auxiliary linear operator, which satisfies

$$\ell[c] = 0, \quad (14b)$$

where  $c$  is constant.

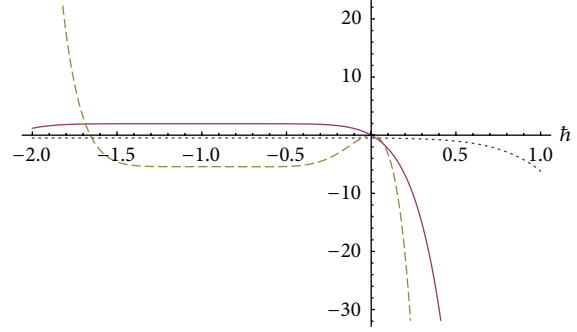
Considering

$$H(x, t) = 1,$$

$$N[u(x, t; q)] = \frac{\partial u(x, t; q)}{\partial t} + \alpha u(x, t; q) \frac{\partial u(x, t; q)}{\partial x} + \beta \frac{\partial^3 u(x, t; q)}{\partial x^3} - \frac{\partial f}{\partial x}, \quad (15)$$

the  $m$ th-order deformation problem now becomes

$$\begin{aligned} \ell [u_m(x, t) - \chi_m u_{m-1}(x, t)] \\ = q \hbar \left[ \frac{\partial u_{m-1}}{\partial t} + \alpha \left( \sum_{i=0}^{m-1} u_i \frac{\partial u_{m-1-i}}{\partial x} \right) + \beta \frac{\partial^3 u_{m-1}}{\partial x^3} - \frac{\partial f_{m-1}}{\partial x} \right] \end{aligned} \quad (16)$$

FIGURE 2: The  $\hbar$ -curves according to 10th-order approximation. Dashed point:  $u(0.01, 0.01)$ , solid line:  $\dot{u}(0.01, 0.01)$ , and dashed line:  $\ddot{u}(0.01, 0.01)$ .

with

$$u_m(x, 0) = 0 \quad \text{for } m > 1. \quad (17)$$

4.1. Using Different Forcing Terms for fKdV. For simplicity, we let  $\alpha = -6$  and  $\beta = 1$  for all the cases studied below.

4.1.1. Forcing Term,  $\partial f / \partial x = x^2$ . Consider the nonlinear KdV equation with the quadratic term  $x^2$  as the forcing term.

Then (3) is simplified as follows:

$$\frac{\partial u}{\partial t} - 6u \frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = x^2. \quad (18)$$

In this paper, MATHEMATICA<sup>®</sup> is used to solve the nonlinear differential equation (16) for the case  $\partial f / \partial x = x^2$ .

It is found that the solution for 10th-order approximation is

$$\begin{aligned} u(x, t) = & -\frac{2e^x}{(1 + e^x)^2} \\ & - \frac{\hbar t (2e^x - 2e^{2x} + x^2 + 3e^x x^2 + 3e^{2x} x^2 + e^{3x} x^2)}{(1 + e^x)^3} \\ & + \dots \end{aligned} \quad (19)$$

Liao [11] pointed out that the valid region of  $\hbar$  is a horizontal line segment. Looking at  $\hbar$ -curves in Figure 2, the valid region of convergence interval is  $-1.5 < \hbar < -0.4$ . We choose three different values of  $\hbar$  in the valid interval. The outcomes for the first case are shown in Figure 3.

The HAM solution at 10th-order approximation is presented with various values of  $\hbar$ , where convergence interval is shown in Figure 3. Figures 3(a) and 3(b) obtained for  $\hbar = -0.6$ , where the interval for Figure 3(a) is  $0 \leq t \leq 0.2$  and for Figure 3(b) is  $0 \leq t \leq 0.3$ , respectively. It is observed that there is a slight formation on the waves when time had been increased from  $t$  is 0.2 s to 0.3 s. The phenomenon is similar to the result for  $\hbar = -1$  of Figures 3(c) and 3(d).

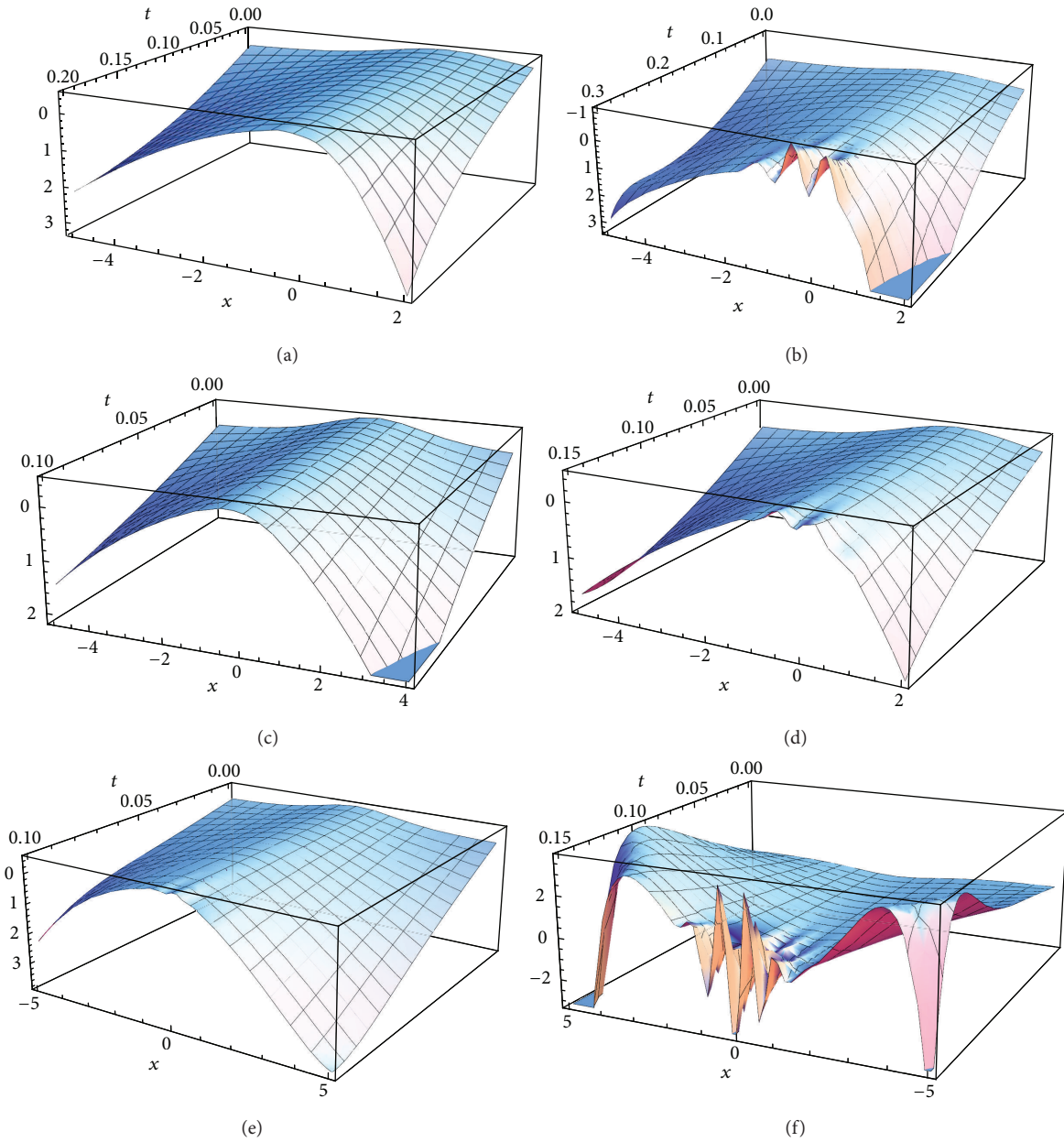


FIGURE 3: The HAM solution at 10th-order approximation with different convergence values of  $\hbar$ . (a and b) for  $\hbar = -0.6$ , (c and d) for  $\hbar = -1.0$ , and (e and f) for  $\hbar = -1.4$ .

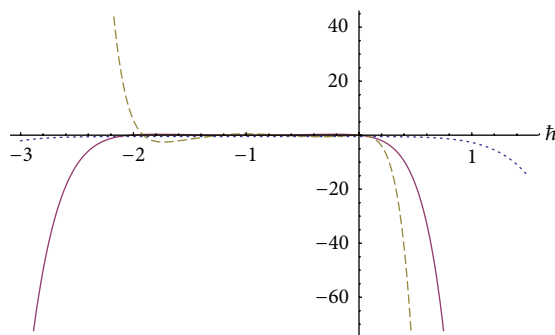


FIGURE 4: The  $\hbar$ -curves according to 9th-order approximation. Dashed point:  $u(0.01, 0.01)$ , solid line:  $\dot{u}(0.01, 0.01)$ , and dashed line:  $\ddot{u}(0.01, 0.01)$ .

It is noted that the wave is normal with no turbulence in Figure 3(e) where then  $\hbar$  value is chosen to be 1.4. Comparing Figures 3(e) and 3(f) with the same value of  $\hbar$ , it is observed that the wave exhibits anomalous behaviour when time is increased. For all cases above, we found that the wave displays certain chaotic manner when we increase the time. Figure 3 shows clearly that the wave changes in abnormal mode corresponding to the quadratic force in the right-hand side of (18).

**4.1.2. Forcing Term,  $\partial f / \partial x = \sin(x)$ .** Consider the nonlinear KdV equation with the quadratic term  $\sin(x)$  as the forcing term.



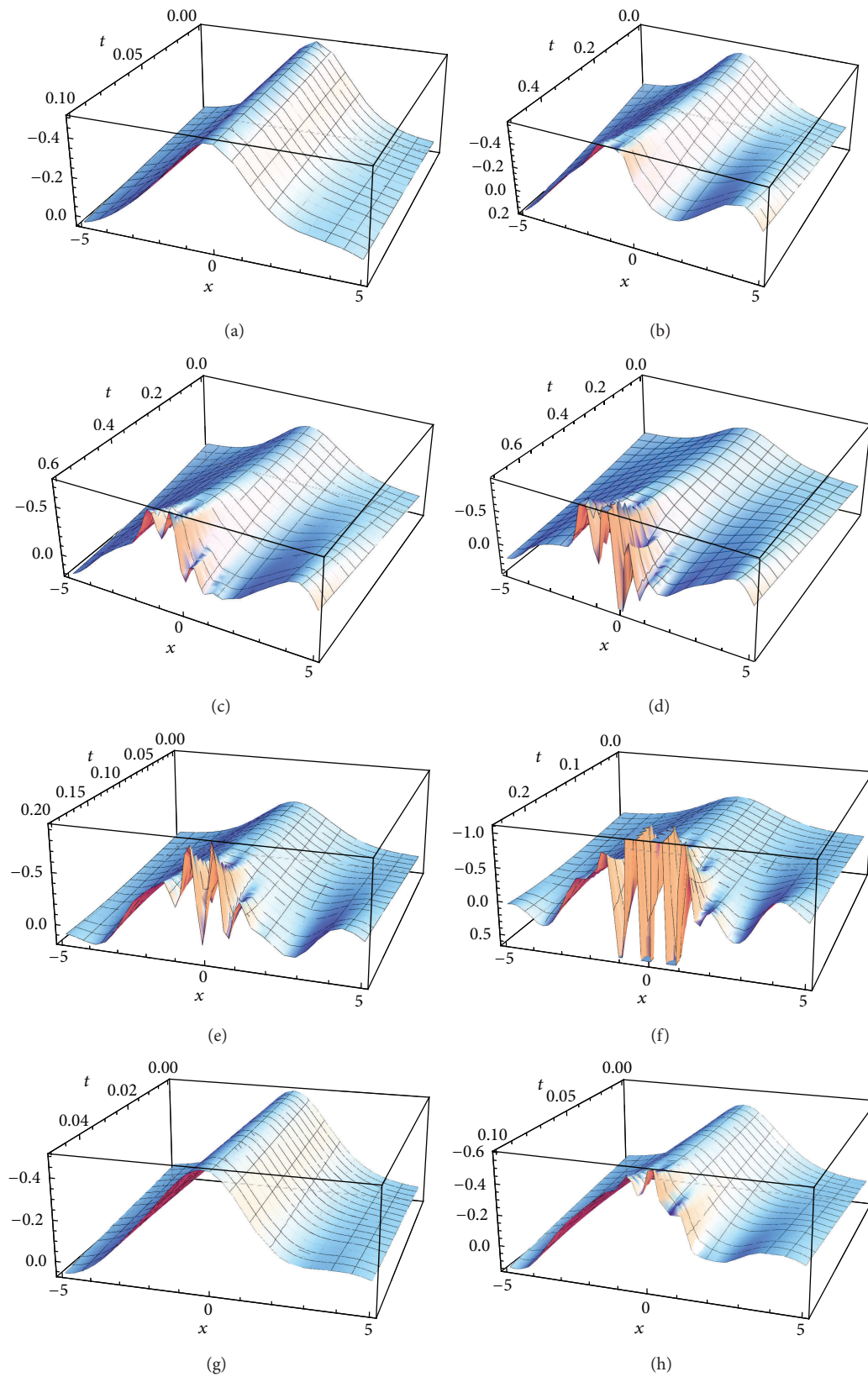


FIGURE 5: The HAM solution at 9th-order approximation with different convergence values of  $\hbar$ . (a, b, c, and d) for  $\hbar = -0.3$ , (e and f) for  $\hbar = -1.0$ , and (g and h) for  $\hbar = -1.7$ .

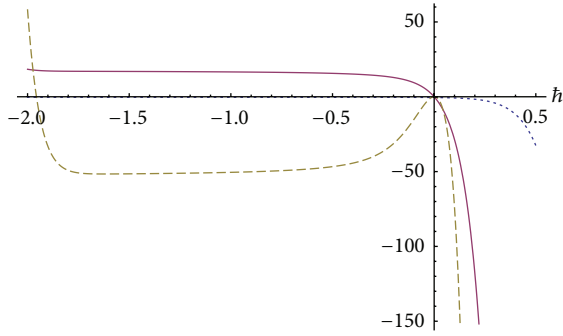


FIGURE 6: The  $\hbar$ -curves according to 17th-order approximation. Dashed point:  $u(0.01, 0.01)$ , solid line:  $\dot{u}(0.01, 0.01)$ , and dashed line:  $\ddot{u}(0.01, 0.01)$ .

The fKdV equation is as follows:

$$\frac{\partial u}{\partial t} - 6u \frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = \sin(x). \quad (20)$$

Similarly MATHEMATICA<sup>®</sup> is used to solve the nonlinear differential equation (16) for the case  $\partial f / \partial x = \sin(x)$ .

The solution at 9th-order approximation is

$$\begin{aligned} u(x, t) = & -\frac{2e^x}{(1+e^x)^2} \\ & -\hbar t (2e^x - 2e^{2x} + \sin(x) + 3e^x \sin(x) \\ & + 3e^{2x} \sin(x) + e^{3x} \sin(x)) \\ & \times ((1+e^x)^3)^{-1} + \dots \end{aligned} \quad (21)$$

Liao [11] pointed out that the horizontal line segment will be the valid region of  $\hbar$ . Based on Figure 4, the admissible interval will be  $-1.7 < \hbar < -0.3$ , so we choose  $\hbar = -0.3$ ,  $\hbar = -1$ , and  $\hbar = -1.7$ .

The outcomes for the  $\sin(x)$  force term are shown in Figure 5.

Figure 5 shows the HAM solution at 9th-order approximation with three different convergent values of  $\hbar$ . Figures 5(a), 5(b), 5(c), and 5(d) are the outcomes for different intervals of time less than  $\hbar = -0.3$ . The interval for Figure 5(a) is  $0 \leq t \leq 0.1$  and the wave portrays a normal wave without any disturbance. The wave seems to be disturbed and the transition seen clearly in Figures 5(b) and 5(c) after the time is increased. In Figure 5(d), the time is increased to 0.7 s and it is observed that the wave starts to fluctuate. Upon considering  $\hbar = -1$ , the output is shown in Figures 5(e) and 5(f) where the suddenness and large energy of waves are created in short period of time. Figures 5(g) and 5(h) are the outcomes of  $\hbar = -1.7$  for  $0 \leq t \leq 0.06$  and  $0 \leq t \leq 0.1$  where the waves are compressed and propagate in short time. As a whole it can be concluded that the wave changes tremendously over time with the sine force incorporated in fKdV equation.

4.1.3. *Forcing Term,  $\partial f / \partial x = e^x$ .* Consider the nonlinear KdV equation with the quadratic term  $e^x$  as the forcing term.

The fKdV equation is as follows:

$$\frac{\partial u}{\partial t} - 6u \frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = e^x. \quad (22)$$

Similarly MATHEMATICA<sup>®</sup> is used to solve the nonlinear differential equation (16) for the case  $\partial f / \partial x = e^x$ .

The solution at 17th-order approximation is

$$\begin{aligned} u(x, t) = & -\frac{2e^x}{(1+e^x)^2} - \frac{e^x (3 + e^x + 3e^{2x} + e^{3x}) \hbar t}{(1+e^x)^3} \\ & - \frac{e^x \hbar t}{2(1+e^x)^4} (8 + 16e^x + 20e^{2x} + 16e^{3x} \\ & + 4e^{4x} + 6\hbar + 8e^x \hbar + 8e^{2x} \hbar \\ & + 8e^{3x} \hbar + 2e^{4x} \hbar + 3\hbar t + 20e^x \hbar t \\ & + 32e^{2x} \hbar t + 4e^{3x} \hbar t + e^{4x} \hbar t) + \dots \end{aligned} \quad (23)$$

Based on Figure 6, the accepted convergence interval will be  $-1.7 \leq \hbar \leq -0.5$ . For further deliberation, the convergence values of  $\hbar$  are chosen as,  $\hbar = -0.5$ ,  $\hbar = -1$ , and  $\hbar = -1.7$ .

The outcomes for the  $e^x$  force term are shown in Figure 7.

Figure 7 presents the solution of HAM at 17th-order approximation with different convergence values of  $\hbar$ . Figures 7(a) and 7(b) are observed for  $\hbar = -0.5$  under the time interval of  $0 \leq t \leq 0.001$  and  $0 \leq t \leq 0.05$ , respectively. The wave is undisturbed when  $t$  is 0.001 s but the wave changes over time. This similar scenario occurs in Figure 7(c), upon using  $\hbar = -1.0$ . When  $\hbar = -1.7$  is used the outcome is depicted in Figure 7(d) for the interval of  $0 \leq t \leq 0.01$ . It is found that the wave steadily rises over at a point in Figure 7(d).

4.2. *Comparison of fKdV HAM Solution with Zhao and Guo's Analytical Solution.* In 2009, Zhao and Guo [3] gave a number of analytical solutions corresponding to the fKdV equation by using Hirota's direct method. In this paper, we will compare the analytical solution of [3] with the corresponding HAM solution. This comparison would again reveal the effectiveness and potential of HAM to solve difficult nonlinear equations.

The fKdV equation used in [3] is as follows:

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + \lambda \frac{\partial^3 u}{\partial x^3} = \varphi \frac{\partial f}{\partial x} \quad (24)$$

with

$$f = \frac{12k\lambda}{\varphi} \left( k^3 (4\lambda - \phi) - \frac{\partial a(t)}{\partial t} \right) \quad (25)$$

$$\times \operatorname{sech}^2(k(x - \phi k^2 t) - a(t)),$$

$$a(t) = b \arctan(At) + b_1 t + b_0. \quad (26)$$

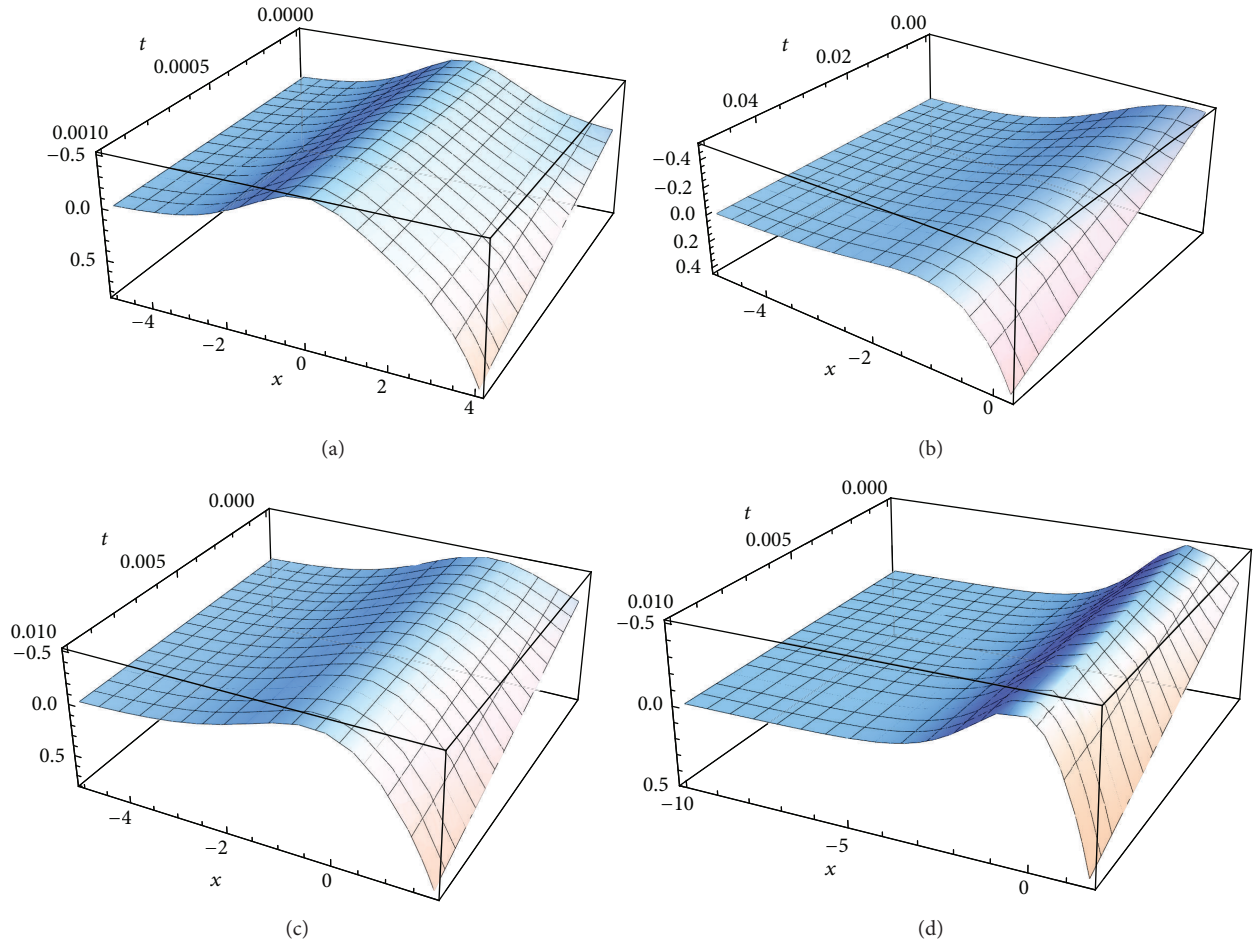


FIGURE 7: The HAM solution at 17th-order approximation with different convergence values of  $\hbar$ . (a and b) for  $\hbar = -0.5$ , (c) for  $\hbar = -1.0$ , and (d) for  $\hbar = -1.7$ .

The parameters chosen by Zhao and Guo [3] and the same input are used in our HAM procedure and are shown in Figure 8, that are

$$\begin{aligned} k = 1, \quad \phi = 4, \quad \lambda = \frac{1}{4}, \quad \varphi = 2, \\ A = 3, \quad b = -1, \quad b_1 = -3, \quad b_0 = 0. \end{aligned} \quad (27)$$

Figure 8 shows the comparison between the graphs of analytical solution of [3] with the HAM solution. Due to very high computational and some limitations on technologies being used, we have simplified (25), by making the term  $t$  as constant in the forcing term. For purpose of comparison with the result of [3], we have chosen to show four cases at  $t = 0.25$  s,  $0.5$  s,  $0.75$  s, and  $1$  s. It should be noted that, since the time has been fixed as constant in the right-hand side term, then for each case of different  $\hbar$ , the ensuing different orders of approximation are obtained. HAM solution shows perfect agreement at  $u(x, 1)$  with the analytical solution of [3]. As a whole, based on the graphs obtained, HAM solution shows good approximation with the analytical solution of [3].

## 5. Conclusion

In this paper, HAM is used to get the approximate analytical solutions of forced KdV equations for different applied forcing terms. With the existence of auxiliary parameter  $\hbar$ , HAM provides us with a straightforward way to control the convergence of approximation series whereby it makes a noticeable difference between HAM and the existing perturbation methods. Figures 2–7 show distinct fluctuations in the wave profiles of the HAM solutions, with various forcing terms being incorporated in the fKdV equations. Nonetheless, Figure 8 shows very close approximation of HAM solution with the analytical solution of Zhao and Guo [3]. These outcomes bring to light that HAM has flexibility and potential to solve complicated nonlinear problems. Since fKdV equation is associated with modelling of Tsunami propagation, as proposed by Pelinovsky et al. [1], we are hopeful that this work would have some added insight into more complicated Tsunami propagation models. In order to achieve more practical oriented results, there are suggestions that further work could be considered under data-driven (measurements) framework, for example, a recent overview can be seen from [29].

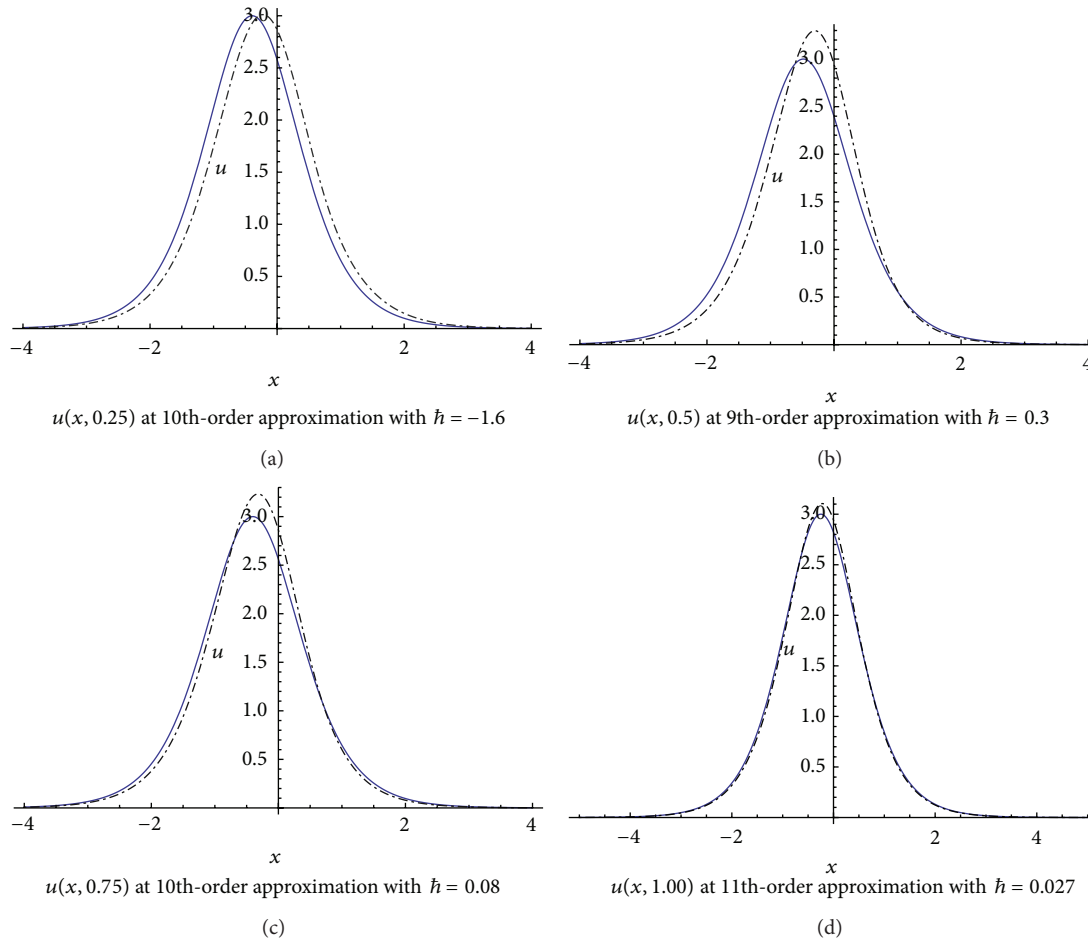


FIGURE 8: MATHEMATICA<sup>®</sup> graphs depicting  $u(x, 0.25)$ ,  $u(x, 0.50)$ ,  $u(x, 0.75)$ , and  $u(x, 1)$  against  $x$ . Solid line is for analytical solution of [3], and dash line is the HAM solution.

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