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Research Article

Oscillation Criteria for Fourth-Order Nonlinear Dynamic Equations on Time Scales

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We establish some new oscillation criteria for nonlinear dynamic equation of the form $(a(t)(b(t)(c(t)x^{\Delta}(t))^{\Delta})^{\Delta})^{\Delta}+q(t)f(x(\sigma(t)))=0$ on an arbitrary time scale $\mathbf{T}$ with $\sup\mathbf{T}=\infty$, where $a(t), b(t), c(t)$ are positive rd-continuous functions. An example illustrating the importance of our result is included.

1. Introduction

A time scale $\mathbf{T}$ is an arbitrary nonempty closed set of real numbers $\mathbf{R}$ with the topology and ordering inherited from $\mathbf{R}$. The theory of time scales, which has recently received a lot of attention, was introduced by Hilger in his Ph.D thesis [1] in order to unify continuous and discrete analysis. The cases when a time scale $\mathbf{T}$ is equal to $\mathbf{R}$ or the set of all integers $\mathbf{Z}$ represent the classical theories of differential and difference equations. Many results concerning differential equations carry over quite easily to corresponding results for difference equations, while other results seem to be completely different from their continuous counterparts. The study of dynamic equations on time scales reveals such discrepancies and helps avoid proving results twice once for differential equations and once again for difference equations. The general is to prove a result for a dynamic equation where the domain of the unknown function is a time scale $\mathbf{T}$. In this way results not only related to the set of real numbers or set of integers but those pertaining to more general time scales are obtained. Therefore, not only can the theory of dynamic equations unify the theories of differential equations and difference equations, but also extends these classical cases to cases "in between," for example, to the so-called q-difference equations when $\mathbf{T}=\{1,q,q^2,\ldots,q^n,\ldots\}$, which has important applications in quantum theory (see [2]). In the last years there has been much research activity concerning the oscillation and asymptotic behavior of solutions of some dynamic equations on time scales, and we refer the reader to the paper [3–8] and the references cited therein.

Recently, Hassan in [9] studied the third-order dynamic equation

$$\left(a(t)\left\{\left[r(t)x^{\Delta}(t)\right]^{\Delta}\right\}^{\gamma}\right)^{\Delta}+f(t,x(\tau(t)))=0, \quad (1)$$

on a time scale $\mathbf{T}$, where $\gamma\geq 1$ is the quotient of odd positive integers, a and r are positive rd-continuous functions on $\mathbf{T}$, and the so-called delay function $\tau:\mathbf{T}\to\mathbf{T}$ satisfies $\tau(t)\leq t$ for $t\in\mathbf{T}$ and $\lim_{t\to\infty}\tau(t)=\infty$ and $f\in C(\mathbf{T}\times\mathbf{R},\mathbf{R})$ and obtained some oscillation criteria, which improved and extended the results that have been established in [10–12].

Li et al. in [13] also discussed the oscillation of (1), where $\gamma>0$ is the quotient of odd positive integers, $f\in C(\mathbf{T}\times\mathbf{R},\mathbf{R})$ is assumed to satisfy $uf(t,u)>0$ for $u\neq 0$, and there exists a positive rd-continuous function p on $\mathbf{T}$ such that $f(t,u)/u^{\gamma}\geq p(t)$ for $u\neq 0$. They established some new sufficient conditions for the oscillation of (1).

Wang and Xu in [14] extended the Hille and Nehari oscillation theorems to the third-order dynamic equation

$$\left(r_2(t)\left(\left(r_1(t)x^{\Delta}(t)\right)^{\Delta}\right)^{\gamma}\right)^{\Delta}+q(t)f(x(t))=0, \quad (2)$$

on a time scale $\mathbf{T}$, where $\gamma \geq 1$ is a ratio of odd positive integers and the functions $r_i(t)$ $(i = 1, 2), q(t)$ are positive real-valued rd-continuous functions defined on $\mathbf{T}$.

Erbe et al. in [15] were concerned with the oscillation of the third-order nonlinear functional dynamic equation

$$\left(a(t)\left[\left(r(t)x^{\Delta}(t)\right)^{\Delta}\right]^{\gamma}\right)^{\Delta} + f(t, x(g(t))) = 0, \quad (3)$$

on a time scale $\mathbf{T}$, where γ is the quotient of odd positive integers, a and r are positive rd-continuous functions on $\mathbf{T}$, and $g : \mathbf{T} \to \mathbf{T}$ satisfies $\lim_{t\to\infty} g(t) = \infty$ and $f \in C(\mathbf{T}\times\mathbf{R}, \mathbf{R})$. The authors obtain some new oscillation criteria and extend many known results for oscillation of third-order dynamic equations.

Qi and Yu in [16] obtained some oscillation criteria for the fourth-order nonlinear delay dynamic equation

$$x^{\Delta^4}(t) + p(t)x^{\gamma}(\tau(t)) = 0, \quad (4)$$

on a time scale $\mathbf{T}$, where γ is the ratio of odd positive integers, p is a positive real-valued rd-continuous function defined on $\mathbf{T}, \tau \in C_{\mathrm{rd}}(\mathbf{T}, \mathbf{T})$, $\tau(t) \leq t$, and $\lim_{t\to\infty}\tau(t) = \infty$.

Grace et al. in [17] were concerned with the oscillation of the fourth-order nonlinear dynamic equation

$$x^{\Delta^4}(t) + q(t)x^{\lambda}(t) = 0, \quad (5)$$

on a time scale $\mathbf{T}$, where λ is the ratio of odd positive integers, q is a positive real-valued rd-continuous function defined on $\mathbf{T}$. They reduce the problem of the oscillation of all solutions of (5) to the problem of oscillation of two second-order dynamic equations and give some conditions ensuring that all bounded solutions of (5) are oscillatory.

Grace et al. in [18] establish some new criteria for the oscillation of fourth-order nonlinear dynamic equations

$$\left(ax^{\Delta^2}\right)^{\Delta^2}(t) + f(t, x^{\sigma}(t)) = 0, \quad t \geq t_0, \quad (6)$$

where a is a positive real-valued rd-continuous function satisfying that $\int_{t_0}^{\infty}(\sigma(s)/a(s))\Delta s < \infty$, $f : [t_0, \infty) \times \mathbf{R} \to \mathbf{R}$ is continuous satisfying $\operatorname{sgn} f(t, x) = \operatorname{sgn} x$ and $f(t, x) \leq f(t, y)$ for $x \leq y$ and $t \geq t_0$. They also investigate the case of strongly superlinear and the case of strongly sublinear equations subject to various conditions.

Agarwal et al. in [19] were concerned with oscillatory behavior of a fourth-order half-linear delay dynamic equation with damping

$$\left(r\left(x^{\Delta^3}\right)^{\gamma}\right)^{\Delta}(t) + p(t)\left(x^{\Delta^3}\right)^{\gamma}(t) + q(t)x^{\gamma}(\tau(t)) = 0, \quad (7)$$

on a time scale $\mathbf{T}$ with $\sup \mathbf{T} = \infty$, where λ is the ratio of odd positive integers, r, p, q are positive real-valued rd-continuous functions defined on $\mathbf{T}, r(t) - \mu(t)p(t) \neq 0$, $\tau \in C_{\mathrm{rd}}(\mathbf{T}, \mathbf{T})$, $\tau(t) \leq t$, and $\tau(t) \to \infty$ as $t \to \infty$. They establish some new oscillation criteria of (7).

Zhang et al. in [20] were concerned with the oscillation of a fourth-order nonlinear dynamic equation

$$\left(px^{\Delta^3}\right)^{\Delta}(t) + q(t)f(x(\sigma(t))) = 0, \quad (8)$$

on an arbitrary time scale $\mathbf{T}$ with $\sup \mathbf{T} = \infty$, where $p, q \in C_{\mathrm{rd}}(\mathbf{T}, (0, \infty))$ with $\int_{t_0}^{\infty}(1/p(s))\Delta s < \infty$ and there exists a positive constant L such that $f(y)/y \geq L$ for all $y \neq 0$; they give a new oscillation result of (8).

Motivated by the previous studies, in this paper, we will study the oscillation criteria of the following fourth-order nonlinear dynamic equation:

$$\left(a(t)\left(b(t)\left(c(t)x^{\Delta}(t)\right)^{\Delta}\right)^{\Delta}\right)^{\Delta} + q(t)f(x(\sigma(t))) = 0, \quad t \in [t_0, \infty)_{\mathbf{T}}, \quad (9)$$

where $\mathbf{T}$ is a time scale with $\sup \mathbf{T} = \infty$ and $t_0 \in \mathbf{T}$ is a constant and $[t_0, \infty)_{\mathbf{T}} = [t_0, \infty) \bigcap \mathbf{T}$. Throughout this paper, we assume that the following conditions are satisfied:

(H_1) $a, b, c, q \in C_{\mathrm{rd}}([t_0, \infty)_{\mathbf{T}}, (0, \infty))$, $b^{\Delta}(t) \geq 0$ and $c^{\Delta}(t) \geq 0$.

(H_2) $\int_{t_0}^{\infty}(1/a(s))\Delta s < \int_{t_0}^{\infty}(1/b(s))\Delta s = \int_{t_0}^{\infty}(1/c(s))\Delta s = \infty$.

(H_3) $f \in C(\mathbf{T}, \mathbf{R})$ and there exists a positive constant M such that for any $u \neq 0$, $f(u)/u \geq M$.

By a solution of (9), we mean a nontrivial real-valued function $x \in C^1_{\mathrm{rd}}([T_x, \infty)_{\mathbf{T}})$ with $T_x \geq t_0$, which has the property that $a(t)(b(t)(c(t)x^{\Delta}(t))^{\Delta})^{\Delta} \in C^1_{\mathrm{rd}}([T_x, \infty)_{\mathbf{T}})$ and satisfies (9) on $[T_x, \infty)_{\mathbf{T}}$, where C^1_{rd} is the space of differentiable functions whose derivative is rd-continuous. The solutions vanishing in some neighborhood of infinity will be excluded from our consideration. A solution $x(t)$ of (9) is said to be oscillatory if it is neither eventually positive nor eventually negative; otherwise it is called nonoscillatory.

2. Some Auxiliary Lemmas

We shall employ the following lemmas.

Lemma 1. *Assume that $x(t)$ is an eventually positive solution of (9). Then there exists $t_1 \in [t_0, \infty)_{\mathbf{T}}$ sufficiently large, such that, for $t \in [t_1, \infty)_{\mathbf{T}}$, one of the following cases holds:*

(1) $x(t) > 0$, $x^{\Delta}(t) < 0$, $(c(t)x^{\Delta}(t))^{\Delta} > 0$, $(b(t)(c(t)x^{\Delta}(t))^{\Delta})^{\Delta} < 0$,

(2) $x(t) > 0$, $x^{\Delta}(t) > 0$, $(c(t)x^{\Delta}(t))^{\Delta} > 0$, $(b(t)(c(t)x^{\Delta}(t))^{\Delta})^{\Delta} < 0$,

(3) $x(t) > 0$, $x^{\Delta}(t) > 0$, $(c(t)x^{\Delta}(t))^{\Delta} > 0$, $(b(t)(c(t)x^{\Delta}(t))^{\Delta})^{\Delta} > 0$,

(4) $x(t) > 0$, $x^{\Delta}(t) > 0$, $(c(t)x^{\Delta}(t))^{\Delta} < 0$, $(b(t)(c(t)x^{\Delta}(t))^{\Delta})^{\Delta} > 0$.

Proof. Let $x(t)$ be an eventually positive solution of (9). Then there is a $t_1 \geq t_0$, sufficiently large, such that, $x(t) > 0$ for $t \geq t_1$. By (9) we have

$$\left(a(t)\left(b(t)\left(c(t)x^{\Delta}(t)\right)^{\Delta}\right)^{\Delta}\right)^{\Delta} = -q(t) f(x(\sigma(t))) \leq -Mq(t)x(\sigma(t)) < 0, \tag{10}$$

which implies that $a(t)(b(t)(c(t)x^{\Delta}(t))^{\Delta})^{\Delta}$ is decreasing and one of the following two cases holds.

(a) $(b(t)(c(t)x^{\Delta}(t))^{\Delta})^{\Delta} > 0$ for $t \geq t_1$.

(b) There is a $t_2 \geq t_1$ such that $(b(t)(c(t)x^{\Delta}(t))^{\Delta})^{\Delta} < 0$ for $t \in [t_2, \infty)_{\mathbf{T}}$.

If case (a) holds, then $b(t)(c(t)x^{\Delta}(t))^{\Delta}$ is strictly increasing on $[t_1, \infty)_{\mathbf{T}}$ and there exist the following two subcases.

(a_1) $(c(t)x^{\Delta}(t))^{\Delta} < 0$ for $t \geq t_1$.

(a_2) There exists a $t_3 \geq t_2$ such that $(c(t)x^{\Delta}(t))^{\Delta} > 0$ for $t \in [t_3, \infty)_{\mathbf{T}}$.

If subcase (a_1) holds, then we claim $x^{\Delta}(t) > 0$. If not, there exists a $t_4 \geq t_3$ such that $c(t)x^{\Delta}(t) \leq c(t_4)x^{\Delta}(t_4) < 0$ for $t \geq t_4$. Thus, we get

$$x(t) \leq x(t_4) + c(t_4)x^{\Delta}(t_4)\int_{t_4}^{t}\frac{1}{c(s)}\Delta s \longrightarrow -\infty, \tag{11}$$

which contradicts $x(t) > 0$ eventually. Therefore, we obtain case (4).

If subcase (a_2) holds, then let $b(t)(c(t)x^{\Delta}(t))^{\Delta} \geq b(t_3)(c(t_3)x^{\Delta}(t_3))^{\Delta} > 0$ we get

$$c(t)x^{\Delta}(t) \geq c(t_3)x^{\Delta}(t_3) + b(t_3)\left(c(t_3)x^{\Delta}(t_3)\right)^{\Delta} \times \int_{t_3}^{t}\frac{1}{b(s)}\Delta s \longrightarrow \infty. \tag{12}$$

Therefore, we obtain case (3).

If case (b) holds, then we claim $(c(t)x^{\Delta}(t))^{\Delta} > 0$ for $t \geq t_2$. If not, there exists a $t_5 \geq t_2$ such that $b(t)(c(t)x^{\Delta}(t))^{\Delta} \leq b(t_5)(c(t_5)x^{\Delta}(t_5))^{\Delta} < 0$ for $t \geq t_5$. Integrating this inequality from t_5 to t, we get

$$c(t)x^{\Delta}(t) \leq c(t_5)x^{\Delta}(t_5) + b(t_5)\left(c(t_5)x^{\Delta}(t_5)\right)^{\Delta} \times \int_{t_5}^{t}\frac{1}{b(s)}\Delta s \longrightarrow -\infty. \tag{13}$$

Then, there exists a $t_6 \geq t_5$ such that $c(t)x^{\Delta}(t) \leq -M < 0$ for $t \geq t_6$. Integrating this inequality from t_6 to t, we get

$$x(t) \leq x(t_6) - M\int_{t_6}^{t}\frac{1}{c(s)}\Delta s \longrightarrow -\infty, \tag{14}$$

which contradicts $x(t) > 0$ eventually. The proof is completed. □

Lemma 2 (see [12]). *Assume that there exists $T \in \mathbf{T}$ such that U satisfies*

$$U(t) > 0, \quad U^{\Delta}(t) > 0, \quad U^{\Delta\Delta}(t) > 0, \quad U^{\Delta\Delta\Delta}(t) \leq 0, \quad \text{for } t \in [T, \infty)_{\mathbf{T}}. \tag{15}$$

Then

$$\liminf_{t\to\infty}\frac{tU(t)}{h_2(t,t_0)U^{\Delta}(t)} \geq 1, \tag{16}$$

where $h_2(t,t_0) = \int_{t_0}^{t}(\tau - t_0)\Delta\tau$.

3. The Main Result

Now we state and prove our main result.

Theorem 3. *Assume that one of the following conditions holds:*

$$\int_{t_0}^{\infty}\frac{Q(s)}{b(s)}\Delta s = \infty, \tag{17}$$

$$\int_{t_0}^{\infty}\frac{1}{c(u)}\int_{u}^{\infty}\frac{Q(s)}{b(s)}\Delta s\Delta u = \infty, \tag{18}$$

$$\limsup_{t\to\infty}\int_{t_0}^{t}\left[Mq(v)\int_{\sigma(v)}^{\infty}\frac{1}{c(u)}\int_{u}^{\infty}\frac{Q(s)}{b(s)}\Delta s\Delta u - \frac{\int_{v}^{\infty}(Q(s)/b(s))\Delta s}{4c(v)\int_{\sigma(v)}^{\infty}(1/c(u))\int_{u}^{\infty}(Q(s)/b(s))\Delta s\Delta u}\right] \times \Delta v = \infty. \tag{19}$$

If there exist two positive functions $\alpha, \beta \in C_{\mathrm{rd}}^{1}([T_0, \infty)_{\mathbf{T}}, (0, +\infty))$ such that for all sufficiently large $t_1 \in [t_0, \infty)_{\mathbf{T}}$, and $t_4 > t_3 > t_2 > t_1$, and some constant $d \in (0,1)$,

$$\limsup_{t\to\infty}\int_{t_0}^{t}\left[\left(dMq(s)Q(\sigma(s))h_2(\sigma(s),t_0) \times \int_{t_1}^{\sigma(s)}\frac{1}{b(u)}\Delta u \times (\sigma(s)c(s))^{-1}\right) - \frac{1}{4Q(\sigma(s))a(s)}\right]\Delta s = \infty, \tag{20}$$

$$\limsup_{t\to\infty}\int_{t_4}^{t}\left[Mq(s)\alpha^{\sigma}(s)\times\int_{t_3}^{\sigma(s)}\left(\int_{t_2}^{z}\left(\frac{\int_{t_1}^{v}(1/a(u))\Delta u}{b(v)}\Delta v\right)\times(c(z))^{-1}\right)\Delta z\times\left(\int_{t_1}^{\sigma(s)}\frac{1}{a(z)}\Delta z\right)^{-1}-\frac{\left[\left(\alpha^{\Delta}(s)\right)_{+}\right]^{2}a(s)\int_{t_1}^{\sigma(s)}(1/a(z))\Delta z}{4\alpha^{\sigma}(s)\int_{t_1}^{s}(1/a(z))\Delta z}\right]\times\Delta s=\infty, \tag{21}$$

$$\limsup_{t\to\infty}\int_{t_0}^{t}\left[\frac{M\beta^{\sigma}(s)}{b(s)}\int_{s}^{\infty}\frac{\int_{u}^{\infty}q(v)\Delta v}{a(u)}\Delta u-\frac{\left[\left(\beta^{\Delta}(s)\right)_{+}\right]^{2}c(s)\int_{t_1}^{\sigma(s)}(1/c(u))\Delta u}{4\beta^{\sigma}(s)\int_{t_1}^{s}(1/c(u))\Delta u}\right]\times\Delta s=\infty, \tag{22}$$

where

$$Q(t):=\int_{t}^{\infty}\frac{1}{a(s)}\Delta s,\qquad f_{+}(t):=\max\{0,f(t)\}. \tag{23}$$

Then, every solution $x(t)$ of (9) *is oscillatory.*

Proof. Assume that (9) has a nonoscillatory solution $x(t)$ on $[t_0,\infty)_{\mathbf{T}}$. Then, without loss of generality, there is a $t_1\geq t_0$, sufficiently large, such that $x(t)>0$ for $t\geq t_1$. By Lemma 1, there exist the following four possible cases:

(1) $x(t)>0, x^{\Delta}(t)<0, (c(t)x^{\Delta}(t))^{\Delta}>0, (b(t)(c(t)x^{\Delta}(t))^{\Delta})^{\Delta}<0$,
(2) $x(t)>0, x^{\Delta}(t)>0, (c(t)x^{\Delta}(t))^{\Delta}>0, (b(t)(c(t)x^{\Delta}(t))^{\Delta})^{\Delta}<0$,
(3) $x(t)>0, x^{\Delta}(t)>0, (c(t)x^{\Delta}(t))^{\Delta}>0, (b(t)(c(t)x^{\Delta}(t))^{\Delta})^{\Delta}>0$,
(4) $x(t)>0, x^{\Delta}(t)>0, (c(t)x^{\Delta}(t))^{\Delta}<0, (b(t)(c(t)x^{\Delta}(t))^{\Delta})^{\Delta}>0$.

If case (1) holds, then

$$\left(a(t)\left(b(t)\left(c(t)x^{\Delta}(t)\right)^{\Delta}\right)^{\Delta}\right)^{\Delta}=-q(t)f(x(\sigma(t)))\leq-Mq(t)x(\sigma(t))<0, \tag{24}$$

which implies that $a(t)(b(t)(c(t)x^{\Delta}(t))^{\Delta})^{\Delta}$ is decreasing on $[t_1,\infty)_{\mathbf{T}}$, and so

$$a(s)\left(b(s)\left(c(s)x^{\Delta}(s)\right)^{\Delta}\right)^{\Delta}\leq a(t)\left(b(t)\left(c(t)x^{\Delta}(t)\right)^{\Delta}\right)^{\Delta}\quad\text{for } s\geq t\geq t_0. \tag{25}$$

Dividing the previous inequality by $a(s)$ and integrating the resulting inequality from t to l, we get

$$b(l)\left(c(l)x^{\Delta}(l)\right)^{\Delta}\leq b(t)\left(c(t)x^{\Delta}(t)\right)^{\Delta}+a(t)\left(b(t)\left(c(t)x^{\Delta}(t)\right)^{\Delta}\right)^{\Delta}\times\int_{t}^{l}\frac{1}{a(s)}\Delta s. \tag{26}$$

Let $l\to\infty$, we obtain

$$b(t)\left(c(t)x^{\Delta}(t)\right)^{\Delta}\geq-a(t)\left(b(t)\left(c(t)x^{\Delta}(t)\right)^{\Delta}\right)^{\Delta}Q(t). \tag{27}$$

Hence, there exists a constant $m>0$ such that

$$b(t)\left(c(t)x^{\Delta}(t)\right)^{\Delta}\geq mQ(t). \tag{28}$$

Integrating (28) from t_0 to t, we get

$$c(t)x^{\Delta}(t)-c(t_0)x^{\Delta}(t_0)\geq m\int_{t_0}^{t}\frac{Q(s)}{b(s)}\Delta s, \tag{29}$$

which implies that

$$\int_{t_0}^{t}\frac{Q(s)}{b(s)}\Delta s\leq-\frac{c(t_0)x^{\Delta}(t_0)}{m}, \tag{30}$$

which contradicts assumption (17).

Integrating (28) from t to ∞, we get

$$-c(t)x^{\Delta}(t)\geq m\int_{t}^{\infty}\frac{Q(s)}{b(s)}\Delta s. \tag{31}$$

Integrating the previous inequality from t_0 to t gives

$$x(t_0)-x(t)\geq m\int_{t_0}^{t}\frac{1}{c(u)}\int_{u}^{\infty}\frac{Q(s)}{b(s)}\Delta s\Delta u, \tag{32}$$

which implies

$$\int_{t_0}^{t}\frac{1}{c(u)}\int_{u}^{\infty}\frac{Q(s)}{b(s)}\Delta s\Delta u\leq\frac{x(t_0)}{m}, \tag{33}$$

which contradicts assumption (18).

Let $A(t)=a(t)(b(t)(c(t)x^{\Delta}(t))^{\Delta})^{\Delta}$. Integrating (27) from t to ∞ gives

$$-c(t)x^{\Delta}(t)\geq\int_{t}^{\infty}-\frac{A(s)Q(s)}{b(s)}\Delta s\geq-A(t)\int_{t}^{\infty}\frac{Q(s)}{b(s)}\Delta s. \tag{34}$$

Integrating (34) from t to ∞, we get

$$x(t) \geq -\int_t^\infty \frac{A(u)}{c(u)} \int_u^\infty \frac{Q(s)}{b(s)} \Delta s \Delta u \geq -A(t) \int_t^\infty \frac{1}{c(u)} \int_u^\infty \frac{Q(s)}{b(s)} \Delta s \Delta u. \tag{35}$$

Set

$$R(t) = \frac{A(t)}{x(t)} \quad \text{for } t \in [t_1, \infty)_{\mathbf{T}}. \tag{36}$$

Then, $R(t) < 0$ for $t \in [t_1, \infty)_{\mathbf{T}}$ and

$$R^\Delta(t) = \frac{A^\Delta(t)}{x^\sigma(t)} - \frac{A(t)x^\Delta(t)}{x(t)x^\sigma(t)} \leq -Mq(t) - \frac{A(t)x^\Delta(t)}{x(t)x^\sigma(t)}. \tag{37}$$

By (34), we get

$$R^\Delta(t) \leq -Mq(t) - \frac{A^2(t)}{c(t)x(t)x^\sigma(t)} \int_t^\infty \frac{Q(s)}{b(s)} \Delta s \leq -Mq(t) - \frac{A^2(t)}{c(t)x^2(t)} \int_t^\infty \frac{Q(s)}{b(s)} \Delta s. \tag{38}$$

Combining (36) with (38) gives

$$R^\Delta(t) \leq -Mq(t) - \frac{R^2(t)}{c(t)} \int_t^\infty \frac{Q(s)}{b(s)} \Delta s. \tag{39}$$

In view of (35), we get

$$R(t) \int_t^\infty \frac{1}{c(u)} \int_u^\infty \frac{Q(s)}{b(s)} \Delta s \Delta u \geq -1. \tag{40}$$

From (39), we obtain

$$\begin{aligned} R^\Delta(t) &\int_{\sigma(t)}^\infty \frac{1}{c(u)} \int_u^\infty \frac{Q(s)}{b(s)} \Delta s \Delta u \\ &\leq -Mq(t) \int_{\sigma(t)}^\infty \frac{1}{c(u)} \int_u^\infty \frac{Q(s)}{b(s)} \Delta s \Delta u \\ &\quad - \frac{R^2(t)}{c(t)} \int_t^\infty \frac{Q(s)}{b(s)} \Delta s \int_{\sigma(t)}^\infty \frac{1}{c(u)} \int_u^\infty \frac{Q(s)}{b(s)} \Delta s \Delta u. \end{aligned} \tag{41}$$

Integrating (41) from t_1 to t gives

$$\begin{aligned} R(t) &\int_t^\infty \frac{1}{c(u)} \int_u^\infty \frac{Q(s)}{b(s)} \Delta s \Delta u \\ &- R(t_1) \int_{t_1}^\infty \frac{1}{c(u)} \int_u^\infty \frac{Q(s)}{b(s)} \Delta s \Delta u \\ &+ \int_{t_1}^t Mq(v) \int_{\sigma(v)}^\infty \frac{1}{c(u)} \int_u^\infty \frac{Q(s)}{b(s)} \Delta s \Delta u \Delta v \\ \leq &-\int_{t_1}^t \frac{R(v)}{c(v)} \int_v^\infty \frac{Q(s)}{b(s)} \Delta s \Delta v \\ &- \int_{t_1}^t \frac{R^2(v)}{c(v)} \int_v^\infty \frac{Q(s)}{b(s)} \Delta s \int_{\sigma(v)}^\infty \frac{1}{c(u)} \\ &\times \int_u^\infty \frac{Q(s)}{b(s)} \Delta s \Delta u \Delta v \\ \leq &\int_{t_1}^t \frac{\int_v^\infty (Q(s)/b(s)) \Delta s}{4c(v) \int_{\sigma(v)}^\infty (1/c(u)) \int_u^\infty (Q(s)/b(s)) \Delta s \Delta u} \Delta v, \end{aligned} \tag{42}$$

which implies

$$\begin{aligned} &\int_{t_1}^t \left[Mq(v) \int_{\sigma(v)}^\infty \frac{1}{c(u)} \int_u^\infty \frac{Q(s)}{b(s)} \Delta s \Delta u \right. \\ &\qquad \left. - \frac{\int_v^\infty (Q(s)/b(s)) \Delta s}{4c(v) \int_{\sigma(v)}^\infty (1/c(u)) \int_u^\infty (Q(s)/b(s)) \Delta s \Delta u} \right] \Delta v \\ &\quad \leq R(t_1) \int_{t_1}^\infty \frac{1}{c(u)} \int_u^\infty \frac{Q(s)}{b(s)} \Delta s \Delta u \\ &\qquad - R(t) \int_t^\infty \frac{1}{c(u)} \int_u^\infty \frac{Q(s)}{b(s)} \Delta s \Delta u \\ &\quad \leq R(t_1) \int_{t_1}^\infty \frac{1}{c(u)} \int_u^\infty \frac{Q(s)}{b(s)} \Delta s \Delta u + 1. \end{aligned} \tag{43}$$

Which contradicts assumption (19).

If case (2) holds, then set

$$R(t) = \frac{A(t)}{b(t)\left(c(t)x^\Delta(t)\right)^\Delta} \quad \text{for } t \in [t_1, \infty)_{\mathbf{T}}, \tag{44}$$

and $R(t) < 0$ for $t \in [t_1, \infty)_{\mathbf{T}}$ and

$$\begin{aligned} R^\Delta(t) \leq &\frac{A^\Delta(t)}{\left(b(cx^\Delta)^\Delta\right)(\sigma(t))} \\ &- \frac{A(t)\left(b(cx^\Delta)^\Delta\right)^\Delta(t)}{\left(b(cx^\Delta)^\Delta\right)(t)\left(b(cx^\Delta)^\Delta\right)(\sigma(t))} \\ \leq &-M\frac{q(t)}{b(\sigma(t))} \frac{x(\sigma(t))}{(cx^\Delta)^\Delta(\sigma(t))} - \frac{R^2(t)}{a(t)}. \end{aligned} \tag{45}$$

On the other hand, let $U(t) = \int_{t_1}^{t} u(s)\Delta s$ for $t \in [t_1,\infty)_{\mathbf{T}}$, where $u(t) = c(t)x^{\Delta}(t)$; it is easy to check that $U(t) > 0$, $U^{\Delta}(t) > 0$, $U^{\Delta\Delta}(t) > 0$. In view of

$$\left(b(t)u^{\Delta}(t)\right)^{\Delta} = b^{\Delta}(t)u^{\Delta}(t) + b^{\sigma}(t)u^{\Delta\Delta}(t) < 0, \tag{46}$$

by (H_1), we get

$$U^{\Delta\Delta\Delta}(t) = u^{\Delta\Delta}(t) < 0. \tag{47}$$

Therefore, by Lemma 2, for any $d \in (0,1)$, there exists $t_d \in [t_1,\infty)_{\mathbf{T}}$ such that

$$\frac{tU(t)}{h_2(t,t_0)U^{\Delta}(t)} \geq d \quad \text{for } t \in [t_d,\infty)_{\mathbf{T}}. \tag{48}$$

Then, we see that

$$\frac{\int_{t_1}^{t} c(s)x^{\Delta}(s)\Delta s}{c(t)x^{\Delta}(t)} \geq \frac{dh_2(t,t_0)}{t}. \tag{49}$$

Since

$$\begin{aligned}\int_{t_1}^{t} u(s)\Delta s &= \int_{t_1}^{t} c(s)x^{\Delta}(s)\Delta s \\ &= c(t)x(t) - c(t_1)x(t_1) - \int_{t_1}^{t} c^{\Delta}(s)x^{\sigma}(s)\Delta s,\end{aligned} \tag{50}$$

we get

$$c(t)x(t) \geq \int_{t_1}^{t} c(s)x^{\Delta}(s)\Delta s. \tag{51}$$

In view of (49), we obtain that for all $t \in [t_d,\infty)_{\mathbf{T}}$,

$$\frac{x(t)}{x^{\Delta}(t)} = \frac{c(t)x(t)}{c(t)x^{\Delta}(t)} \geq \frac{\int_{t_1}^{t} c(s)x^{\Delta}(s)\Delta s}{c(t)x^{\Delta}(t)} \geq \frac{dh_2(t,t_0)}{t}. \tag{52}$$

On the other hand, there exists $t_2 \geq t_d$ such that for any $t \in [t_2,\infty)_{\mathbf{T}}$,

$$\begin{aligned}c(t)x^{\Delta}(t) &= c(t_2)x^{\Delta}(t_2) \\ &\quad + \int_{t_2}^{t} \frac{b(s)\left(c(s)x^{\Delta}(s)\right)^{\Delta}}{b(s)}\Delta s \\ &\geq b(t)\left(c(t)x^{\Delta}(t)\right)^{\Delta}\int_{t_2}^{t}\frac{1}{b(s)}\Delta s.\end{aligned} \tag{53}$$

It follows from (52) and (53) that

$$\begin{aligned}x(t) &\geq \frac{dh_2(t,t_0)}{t}x^{\Delta}(t) \\ &\geq \frac{dh_2(t,t_0)\int_{t_2}^{t}(1/b(s))\Delta s}{tc(t)}b(t)\left(c(t)x^{\Delta}(t)\right)^{\Delta}.\end{aligned} \tag{54}$$

Combining (45) with (54) gives

$$R^{\Delta}(t) \leq -\frac{Mdq(t)h_2(\sigma(t),t_0)\int_{t_2}^{\sigma(t)}(1/b(s))\Delta s}{\sigma(t)c(\sigma(t))} - \frac{R^2(t)}{a(t)}. \tag{55}$$

By (27) we get

$$R(t)Q(t) \geq -1. \tag{56}$$

Multiplying both sides of (55) with t replaced by s, by $Q^{\sigma}(s)$, and integrating with respect to s from t_2 to t $(t \geq t_2)$, one gets

$$\begin{aligned}&\int_{t_2}^{t} R^{\Delta}(s)Q^{\sigma}(s)\Delta s \\ &\quad\leq -\int_{t_2}^{t}\frac{Mdq(s)h_2(\sigma(s),t_0)\int_{t_2}^{\sigma(s)}(1/b(u))\Delta u}{\sigma(s)c(\sigma(s))} \\ &\qquad \times Q^{\sigma}(s)\Delta s \\ &\qquad -\int_{t_2}^{t}\frac{R^2(s)}{a(s)}Q^{\sigma}(s)\Delta s.\end{aligned} \tag{57}$$

Thus,

$$\begin{aligned}R(t)Q(t) &\leq R(t_2)Q(t_2) \\ &\quad -\int_{t_2}^{t}\frac{Mdq(s)h_2(\sigma(s),t_0)\int_{t_2}^{\sigma(s)}(1/b(u))\Delta u}{\sigma(s)c(\sigma(s))} \\ &\qquad \times Q^{\sigma}(s)\Delta s \\ &\quad -\int_{t_2}^{t}\left[\frac{R(s)}{a(s)} + \frac{R^2(s)}{a(s)}Q^{\sigma}(s)\right]\Delta s \\ &\leq R(t_2)Q(t_2) \\ &\quad -\int_{t_2}^{t}\left[\frac{Mdq(s)h_2(\sigma(s),t_0)\int_{t_2}^{\sigma(s)}(1/b(u))\Delta u}{\sigma(s)c(\sigma(s))}\right. \\ &\qquad \left.\times Q^{\sigma}(s) - \frac{1}{4Q(\sigma(s))a(s)}\right]\Delta s,\end{aligned} \tag{58}$$

which implies that

$$\begin{aligned}&\int_{t_2}^{t}\left[\frac{Mdq(s)h_2(\sigma(s),t_0)\int_{t_2}^{\sigma(s)}(1/b(u))\Delta u}{\sigma(s)c(\sigma(s))}\right. \\ &\qquad \left.\times Q^{\sigma}(s) - \frac{1}{4Q(\sigma(s))a(s)}\right]\Delta s \\ &\quad \leq R(t_2)Q(t_2) - R(t)Q(t) \leq R(t_2)Q(t_2) + 1,\end{aligned} \tag{59}$$

which contradicts assumption (20).

If case (3) holds, then since

$$b(t)\left(c(t)x^{\Delta}(t)\right)^{\Delta} \geq \int_{t_1}^{t}\left(b(s)\left(c(s)x^{\Delta}(s)\right)^{\Delta}\right)^{\Delta}\Delta s \geq A(t)\int_{t_1}^{t}\frac{1}{a(s)}\Delta s, \tag{60}$$

we have

$$\left(\frac{b(t)\left(c(t)x^{\Delta}(t)\right)^{\Delta}}{\int_{t_1}^{t}(1/a(s))\Delta s}\right)^{\Delta} \leq 0. \tag{61}$$

Hence, there exists $t_2 \in [t_1,\infty)_{\mathbf{T}}$ such that

$$\begin{aligned} c(t)x^{\Delta}(t) &= c(t_2)x^{\Delta}(t_2) + \int_{t_2}^{t}\frac{b(s)\left(c(s)x^{\Delta}(s)\right)^{\Delta}}{\int_{t_1}^{s}(1/a(u))\Delta u} \times \frac{\int_{t_1}^{s}(1/a(u))\Delta u}{b(s)}\Delta s \\ &\geq \frac{b(t)\left(c(t)x^{\Delta}(t)\right)^{\Delta}}{\int_{t_1}^{t}(1/a(s))\Delta s}\int_{t_2}^{t}\frac{\int_{t_1}^{s}(1/a(u))\Delta u}{b(s)}\Delta s, \end{aligned} \tag{62}$$

which implies that

$$\left(\frac{c(t)x^{\Delta}(t)}{\int_{t_2}^{t}\left(\int_{t_1}^{s}(1/a(u))\Delta u/b(s)\right)\Delta s}\right)^{\Delta} \leq 0. \tag{63}$$

Hence, there exists $t_3 \in [t_2,\infty)_{\mathbf{T}}$ such that

$$\begin{aligned} x(t) &= x(t_3) + \int_{t_3}^{t}\frac{c(s)x^{\Delta}(s)}{\int_{t_2}^{s}\left(\int_{t_1}^{v}(1/a(u))\Delta u/b(v)\right)\Delta v} \times \frac{\int_{t_2}^{s}\left(\int_{t_1}^{v}(1/a(u))\Delta u/b(v)\right)\Delta v}{c(s)}\Delta s \\ &\geq \frac{c(t)x^{\Delta}(t)}{\int_{t_2}^{t}\left(\int_{t_1}^{s}(1/a(u))\Delta u/b(s)\right)\Delta s} \times \int_{t_3}^{t}\frac{\int_{t_2}^{s}\left(\int_{t_1}^{v}(1/a(u))\Delta u/b(v)\right)\Delta v}{c(s)}\Delta s. \end{aligned} \tag{64}$$

Combining (62) with (64) gives

$$x(t) \geq \frac{\int_{t_3}^{t}\left(\int_{t_2}^{s}\left(\int_{t_1}^{v}(1/a(u))\Delta u/b(v)\right)\Delta v/c(s)\right)\Delta s}{\int_{t_1}^{t}(1/a(s))\Delta s} \times b(t)\left(c(t)x^{\Delta}(t)\right)^{\Delta}. \tag{65}$$

Write

$$R(t) = \alpha(t)\frac{A(t)}{b(t)\left(c(t)x^{\Delta}(t)\right)^{\Delta}} \quad \text{for } t \in [t_1,\infty)_{\mathbf{T}}. \tag{66}$$

Thus, $R(t) > 0$ and for any $t \in [t_1,\infty)_{\mathbf{T}}$,

$$\begin{aligned} R^{\Delta}(t) &= \alpha^{\Delta}(t)\frac{A(t)}{b(t)\left(c(t)x^{\Delta}(t)\right)^{\Delta}} + \alpha^{\sigma}(t)\left(\frac{A(t)}{b(t)\left(c(t)x^{\Delta}(t)\right)^{\Delta}}\right)^{\Delta} \\ &= \frac{\alpha^{\Delta}(t)}{\alpha(t)}R(t) + \alpha^{\sigma}(t) \times \left(A^{\Delta}(t)b(t)\left(c(t)x^{\Delta}(t)\right)^{\Delta} - A(t)\left(b(t)\left(c(t)x^{\Delta}(t)\right)^{\Delta}\right)^{\Delta}\right) \\ &\quad \times \left(b(t)\left(c(t)x^{\Delta}(t)\right)^{\Delta}\left(b\left(cx^{\Delta}\right)^{\Delta}\right)(\sigma(t))\right)^{-1} \\ &\leq \frac{\left(\alpha^{\Delta}(t)\right)_{+}}{\alpha(t)}R(t) + \alpha^{\sigma}(t)\frac{A^{\Delta}(t)}{\left(b\left(cx^{\Delta}\right)^{\Delta}\right)(\sigma(t))} - \alpha^{\sigma}(t)\frac{A(t)\left(b(t)\left(c(t)x^{\Delta}(t)\right)^{\Delta}\right)^{\Delta}}{b(t)\left(c(t)x^{\Delta}(t)\right)^{\Delta}\left(b\left(cx^{\Delta}\right)^{\Delta}\right)(\sigma(t))} \\ &= \frac{\left(\alpha^{\Delta}(t)\right)_{+}}{\alpha(t)}R(t) + \alpha^{\sigma}(t)\frac{A^{\Delta}(t)}{\left(b\left(cx^{\Delta}\right)^{\Delta}\right)(\sigma(t))} - \alpha^{\sigma}(t)\frac{R^{2}(t)}{a(t)\alpha^{2}(t)}\frac{\left(b\left(cx^{\Delta}\right)^{\Delta}(t)\right)}{\left(b\left(cx^{\Delta}\right)^{\Delta}\right)(\sigma(t))}. \end{aligned} \tag{67}$$

By (61) and (65), we get

$$\begin{aligned} R^{\Delta}(t) &\leq -Mq(t)\alpha^{\sigma}(t)\frac{x^{\sigma}(t)}{\left(b\left(cx^{\Delta}\right)^{\Delta}\right)(\sigma(t))} + \frac{\left(\alpha^{\Delta}(t)\right)_{+}}{\alpha(t)}R(t) - \alpha^{\sigma}(t)\frac{R^{2}(t)}{a(t)\alpha^{2}(t)}\frac{\left(b\left(cx^{\Delta}\right)^{\Delta}\right)(t)}{\left(b\left(cx^{\Delta}\right)^{\Delta}\right)(\sigma(t))} \\ &\leq -Mq(t)\alpha^{\sigma}(t) \end{aligned}$$

$$\times \int_{t_3}^{\sigma(t)} \left(\int_{t_2}^{s} \left(\frac{\int_{t_1}^{v} (1/a(u)) \Delta u}{b(v)} \Delta v \right) \times (c(s))^{-1} \right) \Delta s$$
$$\times \left(\int_{t_1}^{\sigma(t)} \frac{1}{a(s)} \Delta s \right)^{-1}$$
$$+ \frac{(\alpha^{\Delta}(t))_+}{\alpha(t)} R(t) - \alpha^{\sigma}(t) \frac{R^2(t)}{a(t)\alpha^2(t)}$$
$$\times \frac{\int_{t_1}^{t} (1/a(s)) \Delta s}{\int_{t_1}^{\sigma(t)} (1/a(s)) \Delta s} \le -Mq(t)\alpha^{\sigma}(t)$$
$$\times \int_{t_3}^{\sigma(t)} \left(\int_{t_2}^{s} \left(\frac{\int_{t_1}^{v} (1/a(u)) \Delta u}{b(v)} \Delta v \right) \times (c(s))^{-1} \right) \Delta s$$
$$\times \left(\int_{t_1}^{\sigma(t)} \frac{1}{a(s)} \Delta s \right)^{-1}$$
$$+ \frac{\left[(\alpha^{\Delta}(t))_+ \right]^2 a(t) \int_{t_1}^{\sigma(t)} (1/a(s)) \Delta s}{4\alpha^{\sigma}(t) \int_{t_1}^{t} (1/a(s)) \Delta s}. \tag{68}$$

Integrating the last inequality from t_4 ($t_4 \in [t_3, \infty)_{\mathbf{T}}$) to t, we get

$$\int_{t_4}^{t} \Bigg[Mq(s)\alpha^{\sigma}(s) \times \int_{t_3}^{\sigma(s)} \left(\int_{t_2}^{z} \left(\frac{\int_{t_1}^{v} (1/a(u)) \Delta u}{b(v)} \Delta v \right) \times (c(z))^{-1} \right) \Delta z \tag{69}$$
$$\times \left(\int_{t_1}^{\sigma(s)} \frac{1}{a(z)} \Delta z \right)^{-1} - \frac{\left[(\alpha^{\Delta}(s))_+ \right]^2 a(s) \int_{t_1}^{\sigma(s)} (1/a(z)) \Delta z}{4\alpha^{\sigma}(s) \int_{t_1}^{s} (1/a(z)) \Delta z} \Bigg] \Delta s$$
$$\le R(t_4),$$

which contradicts assumption (21).

If case (4) holds, then

$$(A(t))^{\Delta} \le Mq(t)x(\sigma(t)) < 0. \tag{70}$$

Integrating the previous inequality from t to z, we get

$$A(z) - A(t) \le \int_{t}^{z} Mq(s)x(\sigma(s)) \Delta s. \tag{71}$$

Letting $z \to \infty$ in this inequality, we obtain

$$-\left(b(t)\left(c(t)x^{\Delta}(t)\right)^{\Delta}\right)^{\Delta} + \frac{M\int_{t}^{\infty} q(s) \Delta s}{a(t)} x(\sigma(t)) \le 0. \tag{72}$$

Integrating the previous inequality from t to z, we get

$$b(t)\left(c(t)x^{\Delta}(t)\right)^{\Delta} - b(z)\left(c(z)x^{\Delta}(z)\right)^{\Delta} + Mx(\sigma(t)) \int_{t}^{z} \frac{\int_{s}^{\infty} q(u) \Delta u}{a(s)} \Delta s \le 0. \tag{73}$$

Letting $z \to \infty$ in this inequality, we obtain

$$b(t)\left(c(t)x^{\Delta}(t)\right)^{\Delta} + Mx(\sigma(t)) \int_{t}^{\infty} \frac{\int_{s}^{\infty} q(u) \Delta u}{a(s)} \Delta s \le 0. \tag{74}$$

Now we set

$$R(t) = \beta(t) \frac{c(t)x^{\Delta}(t)}{x(t)} \quad \text{for } t \in [t_1, \infty)_{\mathbf{T}}. \tag{75}$$

Thus, $R(t) > 0$ and for any $t \in [t_1, \infty)_{\mathbf{T}}$,

$$R^{\Delta}(t) = \beta^{\Delta}(t) \frac{c(t)x^{\Delta}(t)}{x(t)} + \beta^{\sigma}(t) \frac{\left(c(t)x^{\Delta}(t)\right)^{\Delta} x(t) - c(t)x^{\Delta}(t)x^{\Delta}(t)}{x(t)x(\sigma(t))}$$
$$\le \frac{\beta^{\Delta}(t)}{\beta(t)} R(t) + \beta^{\sigma}(t) \frac{\left(c(t)x^{\Delta}(t)\right)^{\Delta}}{x(\sigma(t))} - \frac{\beta^{\sigma}(t)}{\beta^2(t)c(t)} \frac{x(t)}{x^{\sigma}(t)} R^2(t). \tag{76}$$

Since

$$x(t) \ge x(t) - x(t_1) \ge \int_{t_1}^{t} \frac{c(s)x^{\Delta}(s)}{c(s)} \Delta s \ge c(t)x^{\Delta}(t) \int_{t_1}^{t} \frac{1}{c(s)} \Delta s, \tag{77}$$

we get

$$\left(\frac{x(t)}{\int_{t_1}^{t} (1/c(s)) \Delta s} \right)^{\Delta} \le 0. \tag{78}$$

Hence, by (74) and (78), we get

$$\begin{aligned} R^{\Delta}(t) \le & -\frac{M\beta^{\sigma}(t)}{b(t)} \int_t^{\infty} \frac{\int_s^{\infty} q(u)\,\Delta u}{a(s)} \Delta s \\ & + \frac{(\beta^{\Delta}(t))_+}{\beta(t)} R(t) \\ & - \frac{\beta^{\sigma}(t)}{\beta^2(t)\,c(t)} \frac{\int_{t_1}^{t} (1/c(s))\,\Delta s}{\int_{t_1}^{\sigma(t)} (1/c(s))\,\Delta s} R^2(t) \\ \le & -\frac{M\beta^{\sigma}(t)}{b(t)} \int_t^{\infty} \frac{\int_s^{\infty} q(u)\,\Delta u}{a(s)} \Delta s \\ & + \frac{c(t)\left[(\beta^{\Delta}(t))_+\right]^2 \int_{t_1}^{\sigma(t)} (1/c(s))\,\Delta s}{4\beta^{\sigma}(t) \int_{t_1}^{t} (1/c(s))\,\Delta s}. \end{aligned} \tag{79}$$

Integrating the previous inequality from t_1 to t, we get

$$\begin{aligned} \int_{t_1}^{t} & \left[\frac{M\beta^{\sigma}(s)}{b(s)} \int_s^{\infty} \frac{\int_u^{\infty} q(v)\,\Delta v}{a(u)} \Delta u \right. \\ & \left. - \frac{c(s)\left[(\beta^{\Delta}(s))_+\right]^2 \int_{t_1}^{\sigma(s)} (1/c(u))\,\Delta u}{4\beta^{\sigma}(s) \int_{t_1}^{s} (1/c(u))\,\Delta u} \right] \Delta s \\ & \le R(t_1), \end{aligned} \tag{80}$$

which contradicts assumption (22). The proof is completed. □

4. Example

Finally, we give an example to illustrate our main result.

Example 1. Consider the fourth-order nonlinear dynamic equation

$$\left(t^2\left(t^{1/5}\left(t^{3/5}x^{\Delta}(t)\right)^{\Delta}\right)^{\Delta}\right)^{\Delta} + \frac{\varrho}{t^{6/5}} f(2t) = 0, \quad t \in 2^{\mathbf{Z}}, \tag{81}$$

where $\varrho > 0$ is a constant, and

$$\begin{gathered} a(t) = t^2, \qquad b(t) = t^{1/5}, \qquad c(t) = t^{3/5}, \\ q(t) = \frac{\varrho}{t^{6/5}}, \qquad f(u) = u\ln\left(3 + u^2\right). \end{gathered} \tag{82}$$

So $M = 1$. It is easy to calculate that

$$\begin{gathered} \int_{t_0}^{\infty} \frac{1}{a(s)} \Delta s = \int_{t_0}^{\infty} \frac{1}{s^2} \Delta s = \frac{2}{t_0} < \infty, \\ \int_{t_0}^{\infty} \frac{1}{b(s)} \Delta s = \int_{t_0}^{\infty} \frac{1}{s^{1/5}} \Delta s = \infty, \\ \int_{t_0}^{\infty} \frac{1}{c(s)} \Delta s = \int_{t_0}^{\infty} \frac{1}{s^{3/5}} \Delta s = \infty, \\ Q(t) = \int_{t}^{\infty} \frac{1}{a(s)} \Delta s = \int_{t}^{\infty} \frac{1}{s^2} \Delta s = \frac{2}{t}. \end{gathered} \tag{83}$$

It is obvious that

$$\int_{t_0}^{\infty} \frac{Q(s)}{b(s)} \Delta s = \int_{t_0}^{\infty} \frac{2}{s^{6/5}} \Delta s = \frac{2}{1 - 1/2^{1/5}} \frac{1}{t_0^{1/5}} < \infty. \tag{84}$$

Therefore, we get

$$\int_{t_0}^{\infty} \frac{1}{c(u)} \int_{u}^{\infty} \frac{Q(s)}{b(s)} \Delta s \Delta u = \frac{2}{1 - 1/2^{1/5}} \int_{t_0}^{\infty} \frac{1}{u^{4/5}} \Delta u = \infty. \tag{85}$$

Then, condition (18) holds. By Lemma 2, we get

$$\begin{gathered} h_2(t, t_0) = \int_{t_0}^{t} (\tau - t_0)\,\Delta\tau = \frac{(t - t_0)(t - 2t_0)}{3}, \\ h_2(\sigma(t), t_0) = h_2(2t, t_0) = \frac{(2t - t_0)(2t - 2t_0)}{3} \ge t^2, \end{gathered} \tag{86}$$

while t sufficiently large. Let $\alpha(t) = 1, \beta(t) = t$. We have that if $\varrho \ge 1/2d$, then

$$\begin{aligned} \limsup_{t\to\infty} & \int_{t_0}^{t} \left[\left(dMq(s)\,Q(\sigma(s))\,h_2(\sigma(s), t_0) \right.\right. \\ & \quad \left. \times \int_{t_1}^{\sigma(s)} (1/b(u))\,\Delta u \right) \\ & \quad \times (\sigma(s)\,c(s))^{-1} \\ & \quad \left. - \frac{1}{4Q(\sigma(s))\,a(s)} \right] \Delta s \\ \ge & \limsup_{t\to\infty} \int_{t_0}^{t} \left[\frac{d\left(\varrho/s^{6/5}\right)(1/s)\,s^2 s^{4/5}}{2ss^{3/5}} - \frac{1}{4s} \right] \Delta s \\ \ge & \left(\frac{d\varrho}{2} - \frac{1}{4} \right) \limsup_{t\to\infty} \int_{t_0}^{t} \frac{1}{s} \Delta s = \infty. \end{aligned} \tag{87}$$

Since

$$\begin{aligned} \int_{t_3}^{\sigma(s)} & \left(\frac{\int_{t_2}^{z} \left(\int_{t_1}^{v} (1/a(u))\,\Delta u / b(v) \right) \Delta v}{c(z)} \right) \Delta z \\ & \ge \frac{1}{t_1} \int_{t_3}^{\sigma(s)} \frac{\int_{t_2}^{z} \left(1/v^{1/5}\right) \Delta v}{c(z)} \Delta z \\ & \ge \frac{1}{t_1} \int_{t_3}^{\sigma(s)} \frac{z^{4/5}}{c(z)} \Delta z \ge \frac{1}{t_1} s^{6/5}, \end{aligned} \tag{88}$$

we get

$$\limsup_{t\to\infty}\int_{t_4}^{t}\Bigg[Mq(s)\,\alpha^{\sigma}(s)\times\int_{t_3}^{\sigma(s)}\left(\frac{\int_{t_2}^{z}\left(\int_{t_1}^{v}(1/a(u))\,\Delta u/b(v)\right)\Delta v}{c(z)}\right)\Delta z\times\left(\int_{t_1}^{\sigma(s)}\left(\frac{1}{a(z)}\right)\Delta z\right)^{-1}-\frac{\left[\left(\alpha^{\Delta}(s)\right)_{+}\right]^{2}a(s)\int_{t_1}^{\sigma(s)}(1/a(z))\,\Delta z}{4\alpha^{\sigma}(s)\int_{t_1}^{s}(1/a(z))\,\Delta z}\Bigg]\Delta s$$
$$\geq\limsup_{t\to\infty}\int_{t_4}^{t}\left[\frac{\left(\varrho/s^{6/5}\right)(1/t_1)\,2^{6/5}s^{6/5}}{1/t_1}\right]\Delta s$$
$$=2^{6/5}\varrho\limsup_{t\to\infty}\int_{t_4}^{t}\Delta s=\infty. \tag{89}$$

Since

$$\int_{s}^{\infty}\frac{\int_{u}^{\infty}q(v)\,\Delta v}{a(u)}\Delta u\geq\varrho\frac{1}{s^{6/5}}, \tag{90}$$

$$\frac{\int_{t_1}^{\sigma(s)}(1/c(u))\,\Delta u}{\int_{t_1}^{s}(1/c(u))\,\Delta u}\leq 2, \tag{91}$$

we obtain

$$\limsup_{t\to\infty}\int_{t_0}^{t}\Bigg[\frac{M\beta^{\sigma}(s)}{b(s)}\int_{s}^{\infty}\frac{\int_{u}^{\infty}q(v)\,\Delta v}{a(u)}\Delta u-\frac{\left[\left(\beta^{\Delta}(s)\right)_{+}\right]^{2}c(s)\int_{t_1}^{\sigma(s)}(1/c(u))\,\Delta u}{4\beta^{\sigma}(s)\int_{t_1}^{s}(1/c(u))\,\Delta u}\Bigg]\Delta s \tag{92}$$
$$\geq\left(2\varrho-\frac{1}{4}\right)\limsup_{t\to\infty}\int_{t_0}^{t}\frac{1}{s^{2/5}}\Delta s=\infty.$$

Hence, conditions (18), (20), (21), and (22) of Theorem 3 are satisfied. By Theorem 3, we see that every solution $x(t)$ of (81) is oscillatory if $\varrho\geq 1/2d$.

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