

Research Article

Existence Results for Solutions of Integral Boundary Value Problems on Time Scales

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This paper deals with the existence of solutions for integral boundary value problems (IBVPs) on time scales. We provide sufficient conditions for the existence of solutions by using Schauder fixed point theorem in a cone. Existence result for this problem is also given by the method of upper and lower solutions.

1. Introduction

The study of dynamic equations on time scales goes back to its founder Hilger [1]. The main motive of the subject of dynamic equations on time scales is that they build bridges between continuous and discrete cases. We begin by presenting some basic definitions on time scale calculus.

A time scale \mathbb{T} is a nonempty closed subset of \mathbb{R} . It follows that the jump operators $\sigma, \rho : \mathbb{T} \rightarrow \mathbb{T}$,

$$\begin{aligned} \sigma(t) &= \inf \{s \in \mathbb{T} : s > t\}, \\ \rho(t) &= \sup \{s \in \mathbb{T} : s < t\} \end{aligned} \quad (1)$$

are well defined. The point $t \in \mathbb{T}$ is left-dense, left-scattered, right-dense, and right-scattered if $\rho(t) = t$, $\rho(t) < t$, $\sigma(t) = t$, and $\sigma(t) > t$, respectively. If \mathbb{T} has a right-scattered minimum m , define $\mathbb{T}_\kappa = \mathbb{T} - m$; otherwise, set $\mathbb{T}_\kappa = \mathbb{T}$. If \mathbb{T} has a left-scattered maximum M , define $\mathbb{T}^\kappa = \mathbb{T} - M$; otherwise, set $\mathbb{T}^\kappa = \mathbb{T}$. A function $f : \mathbb{T} \rightarrow \mathbb{R}$ is ld-continuous provided it is continuous at left dense points in \mathbb{T} , and its right-sided limit exists at right dense points in \mathbb{T} . For $f : \mathbb{T} \rightarrow \mathbb{R}$ and $t \in \mathbb{T}^\kappa$, the delta derivative of f at t , denoted by $f^\Delta(t)$, is the number with the property that given any $\epsilon > 0$, there is a neighborhood $U \subset \mathbb{T}$ of t such that

$$|f(\sigma(t)) - f(s) - f^\Delta(t)[\sigma(t) - s]| \leq \epsilon |\sigma(t) - s| \quad (2)$$

for all $s \in U$. For $f : \mathbb{T} \rightarrow \mathbb{R}$ and $t \in \mathbb{T}_\kappa$, the nabla derivative of f at t , denoted by $f^\nabla(t)$, is the number with the property

that given any $\epsilon > 0$, there is a neighborhood $U \subset \mathbb{T}$ of t such that

$$|f(\rho(t)) - f(s) - f^\nabla(t)[\rho(t) - s]| \leq \epsilon |\rho(t) - s| \quad (3)$$

for all $s \in U$.

A function $F : \mathbb{T} \rightarrow \mathbb{R}$ is called a nabla antiderivative of $f : \mathbb{T} \rightarrow \mathbb{R}$ provided that $F^\nabla(t) = f(t)$ holds for all $t \in \mathbb{T}_\kappa$. We then define the nabla integral of f by

$$\int_a^t f(s) \nabla s = F(t) - F(a) \quad \forall a, t \in \mathbb{T}. \quad (4)$$

For the details of basic notions connected to time scales, we refer the readers to the books [2, 3] and the papers [4, 5], which are useful references for calculus on time scales. Hereafter, we use the notation $[a, b]_\mathbb{T}$ to indicate the time scale interval $[a, b] \cap \mathbb{T}$. The intervals $[a, b]_\mathbb{T}$, $(a, b]_\mathbb{T}$, and $(a, b)_\mathbb{T}$ are similarly defined.

Let \mathbb{T} be a time scale such that $a \in ([a, b]_\mathbb{T})_\kappa$ and $b \in ([a, b]_\mathbb{T})^\kappa$. We are concerned with existence of solutions of the following integral boundary value problem (IBVP):

$$- [p(t)x^\Delta(t)]^\nabla + q(t)x(t) = f(t, x(t)), \quad t \in [a, b]_\mathbb{T}, \quad (5)$$

$$\alpha x(\rho(a)) - \beta x^{[\Delta]}(\rho(a)) = \int_{\rho(a)}^b h_1(x(s)) \nabla s, \quad (6)$$

$$\gamma x(b) + \delta x^{[\Delta]}(b) = \int_{\rho(a)}^b h_2(x(s)) \nabla s, \quad (7)$$

where $f : [\rho(a), \sigma(b)] \times \mathbb{R} \rightarrow \mathbb{R}$ and $h_i : \mathbb{R} \rightarrow \mathbb{R}$ are continuous, $p(t) > 0$, $q(t) > 0$, $\alpha, \beta, \gamma, \delta \geq 0$, $\alpha + \beta > 0$, and $\gamma + \delta > 0$, and $x^{[\Delta]}(t) = p(t)x^\Delta(t)$.

We would like to mention some results of Khan [6], Yang [7], Ahmad et al. [8], and Atici and Guseinov [9] which motivate us to consider the problem (5)–(7). In [6], Khan considered the method of quasilinearization for the nonlinear boundary value problem with integral boundary conditions

$$\begin{aligned} x''(t) &= f(t, x), \quad t \in [0, 1], \\ x(0) - k_1 x'(0) &= \int_0^1 h_1(x(s)) ds, \\ x(1) + k_2 x'(1) &= \int_0^1 h_2(x(s)) ds, \end{aligned} \tag{8}$$

where $f : [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$ and $h_i : \mathbb{R} \rightarrow \mathbb{R}$ ($i = 1, 2$) are continuous functions and k_i are nonnegative constants. He obtained some results for the existence of solutions in an ordered interval generated by the lower and upper solutions of the boundary value problem. Our work will extend some known results which Khan obtained in [6] for integral boundary value problems to any time scales.

Boundary value problems with integral boundary conditions constitute a very interesting and important class of problems. Various problems in heat conduction, chemical engineering, underground water flow, thermoelasticity, population dynamics, and plasmaphysics [8, 10–12] can be reduced to the problems with integral boundary conditions. For more details of boundary value problems involving integral boundary conditions, see, for instance, [6, 13–19] and references therein. Also this type of problems includes two-point, three-point, and multipoint boundary value problems as special cases [4, 5, 7, 20] and the references therein.

In this section, we obtain some inequalities needed later for certain Green's function. In Section 2, the main tool used in the proof of existence of solutions for the IBVP (5)–(7), is a fixed point theorem in a cone, result due to Schauder [21]. Besides this, in this section, we prove the existence of solutions which will lie between the lower and upper solutions when the lower solution is under the upper solution.

To obtain a solution for the IBVP (5)–(7), we need a mapping whose kernel $G(t, s)$ is the Green's function of the equation

$$-\left[p(t)x^\Delta(t)\right]^\nabla + q(t)x(t) = g(t), \quad t \in [a, b]_{\mathbb{T}} \tag{9}$$

with the integral boundary conditions (6)–(7).

In [9], Atici and Guseinov have shown that the solution $x(t)$ of the nonhomogeneous equation (9) with the nonhomogeneous boundary condition

$$\begin{aligned} \alpha x(\rho(a)) - \beta x^{[\Delta]}(\rho(a)) &= d_1, \\ \gamma x(b) + \delta x^{[\Delta]}(b) &= d_2 \end{aligned} \tag{10}$$

is given by

$$x(t) = w(t) + \frac{1}{D} \int_{\rho(a)}^b G(t, s) g(s) \nabla s, \tag{11}$$

where

$$w(t) = \frac{1}{D} \{\theta(t)d_2 - \varphi(t)d_1\}, \tag{12}$$

$$\begin{aligned} D &= -W_t[\theta, \varphi] \\ &= \varphi(t)\theta^{[\Delta]}(t) - \varphi^{[\Delta]}(t)\theta(t). \end{aligned} \tag{13}$$

Since the Wronskian of two solutions of the corresponding homogeneous equation

$$-\left[p(t)x^\Delta(t)\right]^\nabla + q(t)x(t) = 0, \tag{14}$$

under the initial conditions

$$\begin{aligned} \theta(\rho(a)) &= \beta, & \theta^{[\Delta]}(\rho(a)) &= \alpha, \\ \varphi(b) &= \delta, & \varphi^{[\Delta]}(b) &= -\gamma \end{aligned} \tag{15}$$

is independent of $t \in [\rho(a), \sigma(b)]_{\mathbb{T}}$, taking $t = \rho(a)$ and $t = b$ in (13), we find $D = \alpha\varphi(\rho(a)) - \beta\varphi^{[\Delta]}(\rho(a)) = \delta\theta^{[\Delta]}(b) + \gamma\theta(b) \neq 0$, where $\theta(t)$ and $\varphi(t)$ are the linearly independent solutions of (14) subject to conditions (15).

In our problem, we can easily see that the solution of the IBVP (5)–(7) is

$$x(t) = P(t) + \frac{1}{D} \int_{\rho(a)}^b G(t, s) g(s) \nabla s, \tag{16}$$

where

$$\begin{aligned} P(t) &= \frac{1}{D} \left\{ \theta(t) \int_{\rho(a)}^b h_2(x(s)) \nabla s \right. \\ &\quad \left. - \varphi(t) \int_{\rho(a)}^b h_1(x(s)) \nabla s \right\}. \end{aligned} \tag{17}$$

The Green's function in this formula is

$$G(t, s) = \begin{cases} \theta(s)\varphi(t), & s \leq t, \\ \theta(t)\varphi(s), & s \geq t. \end{cases} \tag{18}$$

Lemma 1. *Let $\theta(t)$ and $\varphi(t)$ be the solutions of (14) under conditions (15). Then, $\theta(t)$ is strictly increasing and positive on $[\rho(a), \sigma(b)]_{\mathbb{T}}$, and $\varphi(t)$ is strictly decreasing and positive on $[\rho(a), b]_{\mathbb{T}}$.*

Proof. In [9], it is shown by Lemma 5.1 that the solutions θ and φ of the BVP (14)–(15) possess the following properties:

$$\theta(t) > 0, \quad \forall t \in (\rho(a), \sigma(b)]_{\mathbb{T}}, \tag{19}$$

$$\varphi(t) > 0, \quad \forall t \in [\rho(a), b]_{\mathbb{T}}. \tag{20}$$

Suppose that there exists at least one $r \in (\rho(a), \sigma(b)]_{\mathbb{T}}$ such $\theta^\Delta(r) \leq 0$. From (19) and (14), we obtain

$$\left[p(t)\theta^\Delta(t)\right]^\nabla = q(t)\theta(t) > 0, \tag{21}$$

and integrating over $[\rho(a), r]_{\mathbb{T}}$, we get

$$p(r)\theta^\Delta(r) = p(\rho(a))\theta^\Delta(\rho(a)) + \int_{\rho(a)}^r q(t)\theta(t)\nabla t. \quad (22)$$

Since $p(\rho(a)) > 0$, $\theta^\Delta(\rho(a)) \geq 0$, $q(t) > 0$, and $\theta(t) > 0$, we obtain $p(r)\theta^\Delta(r) > 0$. Thus, we determine $\theta^\Delta(r) > 0$. This contradiction shows that the solution $\theta(t)$ is strictly increasing and positive on $[\rho(a), \sigma(b)]_{\mathbb{T}}$ as desired. Similar arguments can be applied for the proof of $\varphi^\Delta(t) < 0$ and $\varphi(t) > 0$ on $[\rho(a), b]_{\mathbb{T}}$. \square

Lemma 2. *The Green's function $G(t, s)$ defined by (18) satisfies the inequality $G(t, s) = G(s, t) > 0$, for all $(t, s) \in [\rho(a), b]_{\mathbb{T}} \times [\rho(a), b]_{\mathbb{T}}$ and $G(t, s) \leq \theta(\sigma(b))\varphi(\rho(a))$.*

Lemma 3. *Let $G(t, s)$ be defined by (14). Then, there results are*

$$G(t, s) \geq h(t)G(t_0, s), \quad t, t_0, s \in (\rho(a), b)_{\mathbb{T}}, \quad (23)$$

where

$$h(t) = \frac{1}{M} \min \{ \theta(t), \varphi(t) \}, \quad (24)$$

$$M = \max \{ \theta(\sigma(b)), \varphi(\rho(a)) \}.$$

Proof. For this purpose, we have four cases which are $t, t_0 \leq s$, $t, t_0 \geq s$, $t \leq s \leq t_0$, and $t_0 \leq s \leq t$. We consider only two cases; the others can be shown similarly.

Case 1. Let $t_0 \leq s \leq t$. By using Lemma 1, we get

$$\begin{aligned} \frac{G(t, s)}{G(t_0, s)} &\geq \frac{\varphi(t)\theta(s)}{\theta(s)\varphi(t_0)} \geq \frac{\varphi(t)}{\varphi(t_0)} \\ &\geq \frac{\varphi(t)}{\varphi(\rho(a))} \geq \frac{\min \{ \theta(t), \varphi(t) \}}{M} = h(t). \end{aligned} \quad (25)$$

Case 2. Let $t_0 \leq t \leq s$ and $t \leq t_0 \leq s$. By using Lemma 1, we get

$$\begin{aligned} \frac{G(t, s)}{G(t_0, s)} &= \frac{\theta(t)\varphi(s)}{\theta(t_0)\theta(s)} \geq \frac{\theta(t)\varphi(t_0)}{\theta(\sigma(b))\varphi(t_0)} \\ &= \frac{\theta(t)}{\theta(\sigma(b))} \geq \frac{\min \{ \theta(t), \varphi(t) \}}{M} = h(t). \end{aligned} \quad (26)$$

From these cases, we hold $G(t, s) \geq h(t)G(t_0, s)$. \square

Also, we get

$$\begin{aligned} |P(t)| &\leq \left| \frac{1}{D} \left\{ \theta(t) \int_{\rho(a)}^b h_2(x(s)) \nabla s \right. \right. \\ &\quad \left. \left. + \varphi(t) \int_{\rho(a)}^b h_1(x(s)) \nabla s \right\} \right| \\ &\leq \frac{M}{D} \int_{\rho(a)}^b |h_1(x(s))| + |h_2(x(s))| \nabla s := R. \end{aligned} \quad (27)$$

2. Existence of Solutions

We will consider the Banach space $B = \{x : x \in C[\rho(a), \sigma(b)]_{\mathbb{T}}\}$, with the norm $\|x\| = \max_{t \in [\rho(a), \sigma(b)]_{\mathbb{T}}} |x(t)|$.

Theorem 4. *Assume that the function $f(t, \xi)$ is continuous with respect to $\xi \in \mathbb{R}$. If $N > 0$ satisfies*

$$DR + M^2Q(b - \rho(a)) \leq ND, \quad (28)$$

where $Q > 0$ satisfies

$$Q \geq \max_{\|x\| \leq N} |f(t, x)|, \quad \text{for } t \in [\rho(a), b]_{\mathbb{T}}, \quad (29)$$

then the IBVP (5)–(7) has a solution $x(t)$.

Proof. Let $K := \{x \in B : \|x\| \leq N\}$. Note that K is closed, bounded, and convex subset of B to which the Schauder fixed point theorem is applicable. Define $A : K \rightarrow B$ by

$$Ax(t) := P(t) + \frac{1}{D} \int_{\rho(a)}^b G(t, s) f(s, x(s)) \nabla s, \quad (30)$$

for $t \in [\rho(a), b]_{\mathbb{T}}$. Obviously, the solutions of problem (5)–(7) are the fixed points of operator A . In view of the continuity of the function $f(t, x(t))$, it follows that $A : K \rightarrow B$ is continuous.

Now, we show that $A : K \rightarrow K$. Let $x \in K$. Consider

$$\begin{aligned} |Ax(t)| &= \left| P(t) + \frac{1}{D} \int_{\rho(a)}^b G(t, s) f(s, x(s)) \nabla s \right| \\ &\leq |P(t)| + \frac{1}{D} \int_{\rho(a)}^b |G(t, s)| |f(s, x(s))| \nabla s \\ &\leq R + \frac{M^2}{D} \int_{\rho(a)}^b |f(s, x(s))| \nabla s \\ &\leq R + \frac{M^2Q(b - \rho(a))}{D} \leq N \end{aligned} \quad (31)$$

for every $t \in [\rho(a), b]_{\mathbb{T}}$. This implies that $\|Ax\| \leq N$.

Thus, all functions which belong to $A(K)$ are equibounded and $A(K) \subset K$. The uniform continuity of the $G(t, s)$ and $f(t, x)$ implies that all functions in $A(K)$ are equicontinuous. So, by Arzela-Ascoli theorem, the operator $A : K \rightarrow K$ is compact. Hence, A has a fixed point in K by Schauder fixed point theorem. \square

Corollary 5. *If f is continuous and bounded on $[\rho(a), b]_{\mathbb{T}} \times \mathbb{R}$, then the IBVP (5)–(7) has a solution.*

Let us define the set S by

$$\begin{aligned} S := \{x : x^\Delta \text{ is continuous on } [\rho(a), b]_{\mathbb{T}} \text{ and } px^\Delta \\ \text{is nabla differentiable on } [a, b]_{\mathbb{T}} \text{ and } (px^\Delta)^\nabla \\ \text{is ld-continuous on } [a, b]_{\mathbb{T}}\}. \end{aligned} \quad (32)$$

For any $u, v \in S$, we define the sector $[u, v]$ by

$$[u, v] := \{w \in E : u \leq w \leq v\}. \quad (33)$$

Definition 6. A real valued function $u(t) \in S$ on $[\rho(a), b]_{\mathbb{T}}$ is a lower solution for IBVP (5)–(7) if

$$\begin{aligned}
 -[p(t)u^\Delta(t)]^\nabla + q(t)u(t) &\leq f(t, u(t)), \quad t \in [a, b]_{\mathbb{T}}, \\
 \alpha u(\rho(a)) - \beta u^{[\Delta]}(\rho(a)) &\leq \int_{\rho(a)}^b h_1(u(s)) \nabla s, \\
 \gamma u(b) + \delta u^{[\Delta]}(b) &\leq \int_{\rho(a)}^b h_2(u(s)) \nabla s.
 \end{aligned} \tag{34}$$

Similarly, real valued function $v(t) \in S$ on $[\rho(a), b]_{\mathbb{T}}$ is an upper solution for IBVP (5)–(7) if the inequalities in (34) are satisfied in the reverse direction for $v(t)$.

Theorem 7. Assume that u and v are, respectively, lower and upper solutions of (5)–(7). If $f : [\rho(a), \sigma(b)]_{\mathbb{T}} \times \mathbb{R} \rightarrow \mathbb{R}$ is continuous, $h_i : \mathbb{R} \rightarrow \mathbb{R}$ ($i = 1, 2$) are continuously differentiable, $f(t, x)$ is decreasing in x for $t \in [\rho(a), b]_{\mathbb{T}}$, and $0 \leq h'_1(x) < \alpha/(b - \rho(a))$, then

$$u(t) \leq v(t), \quad t \in [\rho(a), b]_{\mathbb{T}}. \tag{35}$$

Proof. Define $g := u - v$. For the sake of contradiction, assume that the result is not true on $[\rho(a), b]_{\mathbb{T}}$. Then, the function $g(t)$ has a positive maximum at $t_0 \in (\rho(a), b)_{\mathbb{T}}$ and $(u - v)(t) < (u - v)(t_0)$ for $t \in (t_0, b)_{\mathbb{T}}$. So, we have $g(t_0) > 0$, $g^\Delta(t_0) \leq 0$, and $(pg^\Delta)^\nabla(t_0) \leq 0$; hence,

$$\begin{aligned}
 -[p(t_0)g^\Delta(t_0)]^\nabla &\leq -[p(t_0)g^\Delta(t_0)]^\nabla + q(t_0)g(t_0) \\
 &= -[p(t_0)u^\Delta(t_0)]^\nabla + q(t_0)u(t_0) \\
 &\quad - \left\{ -[p(t_0)v^\Delta(t_0)]^\nabla + q(t_0)v(t_0) \right\} \\
 &\leq f(t_0, u(t_0)) - f(t_0, v(t_0)) < 0,
 \end{aligned} \tag{36}$$

a contradiction. If $t_0 = \rho(a)$, then $g(\rho(a)) > 0$ and $g^\Delta(\rho(a)) \leq 0$. Using the boundary conditions, we have

$$\begin{aligned}
 \alpha g(\rho(a)) &\leq \beta p(\rho(a))g^\Delta(\rho(a)) \\
 &\quad + \int_{\rho(a)}^b (h_1(u(s)) - h_1(v(s))) \nabla s \\
 &\leq \int_{\rho(a)}^b (h_1(u(s)) - h_1(v(s))) \nabla s.
 \end{aligned} \tag{37}$$

On the other hand, using the mean value theorem and the assumption on $h_1(x)$, we obtain

$$\begin{aligned}
 \alpha g(\rho(a)) &\leq \int_{\rho(a)}^b h'_1(c)g(s) \nabla s \\
 &< \frac{\alpha}{b - \rho(a)} \int_{\rho(a)}^b g(s) \nabla s \\
 &< \frac{\alpha}{b - \rho(a)} \max_{t \in [\rho(a), b]_{\mathbb{T}}} g(t)(b - \rho(a)) \\
 &< \alpha g(\rho(a)),
 \end{aligned} \tag{38}$$

where $c \in (u(s), v(s))$, a contradiction. Hence,

$$u(t) \leq v(t), \quad t \in [\rho(a), b]_{\mathbb{T}}. \tag{39}$$

□

Now, we state and prove the existence and uniqueness of solutions in an ordered interval generated by the lower and upper solutions of the boundary value problem.

Theorem 8. Assume that u and v are, respectively, lower and upper solutions of (5)–(7) such that $u(t) \leq v(t)$, $t \in [\rho(a), \sigma(b)]_{\mathbb{T}}$. If $f : [\rho(a), \sigma(b)]_{\mathbb{T}} \times \mathbb{R} \rightarrow \mathbb{R}$ and $h_i : \mathbb{R} \rightarrow \mathbb{R}$ ($i = 1, 2$) are continuously differentiable and $h'_1(x) \geq 0$, then there exists a solution $x(t)$ of (5)–(7) such that

$$u(t) \leq x(t) \leq v(t), \quad t \in [\rho(a), b]_{\mathbb{T}}. \tag{40}$$

Proof. Define the following modifications of $f(t, x)$ and $h_i(x)$, $i = 1, 2$:

$$F(t, x) = \begin{cases} f(t, v(t)) + \frac{x - v(t)}{1 + |x - v|}, & x \geq v(t), \\ f(t, x(t)), & u(t) \leq x \leq v(t), \\ f(t, u(t)) - \frac{x - u(t)}{1 + |x - u|}, & x \leq u(t) \end{cases} \tag{41}$$

for $t \in [\rho(a), \sigma(b)]_{\mathbb{T}}$, and

$$H_i(x) = \begin{cases} h_i(v(t)), & x \geq v(t), \\ h_i(x(t)), & u(t) \leq x \leq v(t), \\ h_i(u(t)), & x \leq u(t). \end{cases} \tag{42}$$

Consider the modified problem

$$\begin{aligned}
 -[p(t)x^\Delta(t)]^\nabla + q(t)x(t) &= F(t, x(t)), \quad t \in [a, b]_{\mathbb{T}}, \\
 \alpha x(\rho(a)) - \beta x^{[\Delta]}(\rho(a)) &= \int_{\rho(a)}^b H_1(x(s)) \nabla s, \\
 \gamma x(b) + \delta x^{[\Delta]}(b) &= \int_{\rho(a)}^b H_2(x(s)) \nabla s.
 \end{aligned} \tag{43}$$

As $F : [\rho(a), \sigma(b)]_{\mathbb{T}} \times \mathbb{R} \rightarrow \mathbb{R}$ and $H_i : \mathbb{R} \rightarrow \mathbb{R}$ ($i = 1, 2$) are continuous and bounded, it follows that the boundary value problem (43) has a solution. Further, note that

$$\begin{aligned}
 -[p(t)u^\Delta(t)]^\nabla + q(t)u(t) &\leq f(t, u(t)) \\
 &= F(t, u(t)), \quad t \in [a, b]_{\mathbb{T}}, \\
 \alpha u(\rho(a)) - \beta u^{[\Delta]}(\rho(a)) &\leq \int_{\rho(a)}^b h_1(u(s)) \nabla s \\
 &= \int_{\rho(a)}^b H_1(u(s)) \nabla s, \\
 \gamma u(b) + \delta u^{[\Delta]}(b) &\leq \int_{\rho(a)}^b h_2(u(s)) \nabla s \\
 &= \int_{\rho(a)}^b H_2(u(s)) \nabla s,
 \end{aligned} \tag{44}$$

which imply that $u(t)$ is a lower solution of (43). Similarly, $v(t)$ is an upper solution of (43). We need to show that any solution $x(t)$ of (43) is such that $u(t) \leq x(t) \leq v(t)$, $t \in [\rho(a), b]_{\mathbb{T}}$. Assume that $u(t) \leq x(t)$ is not true on $[\rho(a), b]_{\mathbb{T}}$. Then, the function $k(t) = u(t) - x(t)$ has a positive maximum at $t_0 \in (\rho(a), b)_{\mathbb{T}}$ and $(u - x)(t) < (u - x)(t_0)$ for $t \in (t_0, b)_{\mathbb{T}}$. So, we have $k(t_0) > 0$, $k^\Delta(t_0) \leq 0$, and $(pk^\Delta)^\nabla(t_0) \leq 0$ by Lemma 6.17 in [3]; hence,

$$\begin{aligned}
 0 &\leq -[p(t_0)k^\Delta(t_0)]^\nabla \leq -[p(t_0)k^\Delta(t_0)]^\nabla + q(t_0)k(t_0) \\
 &= -[p(t_0)(u - x)^\Delta(t_0)]^\nabla + q(t_0)(u - x)(t_0) \\
 &= -[p(t_0)u^\Delta(t_0)]^\nabla + q(t_0)u(t_0) \\
 &\quad - \left\{ -[p(t_0)x^\Delta(t_0)]^\nabla + q(t_0)x(t_0) \right\} \\
 &\leq f(t_0, u(t_0)) - \left\{ f(t_0, u(t_0)) \right. \\
 &\quad \left. - \frac{x(t_0) - u(t_0)}{1 + |x(t_0) - u(t_0)|} \right\} < 0,
 \end{aligned} \tag{45}$$

a contradiction. If $t_0 = \rho(a)$, then $k(\rho(a)) > 0$ and $k^\Delta(\rho(a)) \leq 0$, but then the boundary conditions and the nondecreasing property of h_i give

$$\begin{aligned}
 \alpha k(\rho(a)) &= \alpha(u(\rho(a)) - x(\rho(a))) \\
 &\leq \beta u^{[\Delta]}(\rho(a)) + \int_{\rho(a)}^b h_1(u(s)) \nabla s - \beta x^{[\Delta]}(\rho(a)) \\
 &\quad - \int_{\rho(a)}^b H_1(x(s)) \nabla s
 \end{aligned}$$

$$\begin{aligned}
 &= \beta p(\rho(a)) (k^\Delta(\rho(a))) \\
 &\quad + \int_{\rho(a)}^b (h_1(u(s)) - H_1(x(s))) \nabla s \\
 &\leq \int_{\rho(a)}^b (h_1(u(s)) - H_1(x(s))) \nabla s.
 \end{aligned} \tag{46}$$

If $x < u(t)$, then $H_1(x(s)) = h_1(u(s))$, and, hence, $\alpha k(\rho(a)) \leq 0$, a contradiction. If $x > v(t)$, then $H_1(x(s)) = h_1(v(s)) \geq h_1(u(s))$, which implies $\alpha k(\rho(a)) \leq 0$, a contradiction. Hence, $u(t) \leq x(t) \leq v(t)$ and $H_1(x(s)) = h_1(x(s)) \geq h_1(u(s))$ and $\alpha k(\rho(a)) \leq 0$, another contradiction. \square

We can illustrate our results in the following examples.

Example 9. Let \mathbb{T} be any time scales such that $1 \in ([0, 1]_{\mathbb{T}})^{\kappa}$ and $0 \in ([0, 1]_{\mathbb{T}})_{\kappa}$. We consider the following IBVP:

$$\begin{aligned}
 -x^{\Delta\nabla}(t) &= \sin x(t), \quad t \in [0, 1]_{\mathbb{T}}, \\
 2x(\rho(0)) - x^\Delta(\rho(0)) &= \int_{\rho(0)}^1 x(s) \nabla s, \\
 x(1) - 2x^\Delta(1) &= \int_{\rho(0)}^1 x(s) \nabla s.
 \end{aligned} \tag{47}$$

From (15), we have $\theta(t) = 2(t - \rho(0)) + 1$ and $\varphi(t) = -t - 1$. We calculate easily

$$\begin{aligned}
 M &= \max\{\theta(\sigma(1)), \varphi(\rho(0))\} \\
 &= \max\{2(\sigma(1) - \rho(0)) + 1, -\rho(0) - 1\} \\
 &= 2(\sigma(1) - \rho(0)) + 1, \\
 D &= \varphi(t)\theta^\Delta(t) - \varphi^\Delta(t)\theta(t) \\
 &= (-t - 1)2 - (-1)(2(t - \rho(0)) + 1).
 \end{aligned} \tag{48}$$

For $t = 1$ or $t = \rho(0)$, $D = -1 - 2\rho(0)$.

We have

$$\begin{aligned}
 |P(t)| &\leq \frac{M}{D} \int_{\rho(0)}^1 [|h_1(x(s))| + |h_2(x(s))|] \nabla s \\
 &= \frac{2(\sigma(1) - \rho(0)) + 1}{-1 - 2\rho(0)} \int_{\rho(0)}^1 2|x(s)| \nabla s \\
 &\leq \frac{2(\sigma(1) - \rho(0)) + 1}{-1 - 2\rho(0)} 2Q(1 - \rho(0)) := R,
 \end{aligned} \tag{49}$$

where $Q = \max_{t \in [\rho(0), 1]} |x(t)|$. Since $\max_{\|x\| \leq N} |f(t, x)| = \max_{\|x\| \leq N} |\sin x(t)| \leq \max_{\|x\| \leq N} |x(t)| \leq \|x\| \leq N \leq Q$ for $t \in [\rho(0), 1]$, there exists a positive real number N which satisfies $R + M^2Q(1 - \rho(0))/D \leq N$ for the positive Q , and then all condition in Theorem 4 are satisfied. Therefore, the IBVP (5)–(7) has a solution $x(t)$.

Example 10. Let $\mathbb{T} = [0, 1/4] \cup \{1/3\} \cup [1/2, 2]$. We consider the following IBVP:

$$\begin{aligned} -x^{\Delta\nabla}(t) + x(t) &= e^{-x(t)}, \quad t \in [0, 1]_{\mathbb{T}}, \\ 2x(\rho(0)) - x^{\Delta}(\rho(0)) &= \int_{\rho(0)}^1 x^2(s) \nabla s, \\ x(1) + 2x^{\Delta}(1) &= \int_{\rho(0)}^1 x(s) \nabla s. \end{aligned} \quad (50)$$

For $u(t) = -1$, we get

$$\begin{aligned} -u^{\Delta\nabla}(t) + u(t) &= -1 \leq e^{-1}, \quad t \in [0, 1]_{\mathbb{T}}, \\ 2(-1) - 0 &\leq \int_{\rho(0)}^1 (-1)^2 \nabla s = 1, \\ (-1) + 0 &\leq \int_{\rho(0)}^1 (-1) \nabla s = -1. \end{aligned} \quad (51)$$

Thus, $u(t) = -1$ is the lower solution.

For $v(t) = 1$, we get

$$\begin{aligned} -v^{\Delta\nabla}(t) + v(t) &= 1 \geq e^{-1}, \quad t \in [0, 1]_{\mathbb{T}}, \\ 2(1) - 0 &\geq \int_{\rho(0)}^1 (1)^2 \nabla s = 1, \\ (1) + 0 &\geq \int_{\rho(0)}^1 (1) \nabla s = 1. \end{aligned} \quad (52)$$

Thus, $v(t) = 1$ is the upper solution.

Theorem 8 implies that IBVP has a solution $x(t)$ such that

$$-1 \leq x(t) \leq 1, \quad t \in [\rho(0), 1]_{\mathbb{T}}. \quad (53)$$

Example 11. Let \mathbb{T} be any time scales such that $1 \in ([0, 1]_{\mathbb{T}})^{\kappa}$ and $0 \in ([0, 1]_{\mathbb{T}})_{\kappa}$. We consider the following IBVP:

$$\begin{aligned} -[(t+1)x^{\Delta}(t)]^{\nabla} + e^t x(t) &= \sin x(t) - x(t), \quad t \in [0, 1]_{\mathbb{T}}, \\ 4(1 - \rho(0))x(\rho(0)) - 3tx^{\Delta}(\rho(0)) &= \int_{\rho(0)}^1 x^3(s) \nabla s, \\ 16(1 - \rho(0))x(1) - 2tx^{\Delta}(1) &= \int_{\rho(0)}^1 x^5(s) \nabla s. \end{aligned} \quad (54)$$

Let $u(t) = 0$. We can easily see that $u(t)$ satisfies all conditions of a lower solution. Hence, $u(t) = 0$ is the lower solution.

Let $v(t) = \pi/2$, and

$$\begin{aligned} -[(t+1)v^{\Delta}(t)]^{\nabla} + e^t v(t) &= e^t \frac{\pi}{2} \geq 1 - \frac{\pi}{2} \\ &= \sin v(t) - v(t), \quad t \in [0, 1]_{\mathbb{T}}. \end{aligned} \quad (55)$$

We get $(e^t + 1)(\pi/2) \geq 1$. From

$$\begin{aligned} 4(1 - \rho(0)) \frac{\pi}{2} - 0 &\geq \int_{\rho(0)}^1 \left(\frac{\pi}{2}\right)^3 \nabla s \\ &= \left(\frac{\pi}{2}\right)^3 (1 - \rho(0)), \end{aligned} \quad (56)$$

we get $16 \geq \pi^2$, and from

$$\begin{aligned} 16(1 - \rho(0)) \frac{\pi}{2} - 0 &\geq \int_{\rho(0)}^1 \left(\frac{\pi}{2}\right)^5 (s) \nabla s \\ &= \left(\frac{\pi}{2}\right)^5 (1 - \rho(0)), \end{aligned} \quad (57)$$

we get $256 \geq \pi^4$. Thus, $v(t) = \pi/2$ is the upper solution.

Theorem 8 implies that IBVP has a solution $x(t)$ such that

$$0 \leq x(t) \leq \frac{\pi}{2}, \quad t \in [\rho(0), 1]_{\mathbb{T}}. \quad (58)$$

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