Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2013, Article ID 707686, 8 pages
http://dx.doi.org/10.1155/2013/707686

Research Article

Estimate of Number of Periodic Solutions of Second-Order Asymptotically Linear Difference System

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Received 1 April 2013; Accepted 5 June 2013

Academic Editor: Zidong Wang

We investigate the number of periodic solutions of second-order asymptotically linear difference system. The main tools are Morse theory and twist number, and the discussion in this paper is divided into three cases. As the system is resonant at infinity, we use perturbation method to study the compactness condition of functional. We obtain some new results concerning the lower bounds of the nonconstant periodic solutions for discrete system.

1. Introduction

In this paper we are interested in the lower bound of the number of periodic solutions for second-order autonomous difference system

$$\Delta^2 x_{n-1} + f(x_n) = 0, \quad n \in \mathbb{Z}, \tag{1}$$

where $x_n \in \mathbb{R}^N$, $f = (f_1, f_2, \ldots, f_N)^T \in C^1(\mathbb{R}^N, \mathbb{R}^N)$, $\Delta x_n = x_{n+1} - x_n$, $\Delta^2 x_n = \Delta(\Delta x_n)$, and N is a fixed positive integer.

Discrete systems have been investigated by many authors using various methods, and many interesting results have obtained; see [1–7] and references therein. The critical point theory [8, 9] is a useful tool to investigate differential equations, which is developed to study difference equations. Using minimax methods in critical point theory, Guo and Yu [10, 11] investigated the existence of periodic and subharmonic solutions of system (1), where nonlinearity f is either sublinear or superlinear. In this paper, we assume that

(P1) there exist a function $g \in C^1(\mathbb{R}^N, \mathbb{R}^N)$ and a $N \times N$ symmetric matrix A_∞ such that $f(x_n) = A_\infty x_n + g(x_n)$, $x_n \in \mathbb{R}^N$, and

$$|g(x_n)| = o(|x_n|) \quad \text{as } |x_n| \longrightarrow \infty, \tag{2}$$

where $|\cdot|$ denotes the usual norm in $\mathbb{R}^N$. Moreover there exist functions F, G such that $F'(x_n) = (\partial F/\partial x_{n1}, \ldots, \partial F/\partial x_{nN})^T = f(x_n) = (f_1, f_2, \ldots, f_N)^T$, $G'(x_n) = g(x_n)$, where $'$ denotes the gradient of function.

System (1) can be regarded as discrete analogous of the following differential system:

$$-\Delta u = f(u). \tag{3}$$

A great deal of research has been devoted to (3). For example, by using minimax theory, Rabinowitz [12] has given some interesting results, and Mawhin and Willem [9] obtained some results using the critical point theory. Moreover, there is a vast literature on the problems concerning periodic solutions, BVP, asymptotically behavior of solutions, and so forth.

Morse theory [8, 9, 13–16] has been used to solve the asymptotically linear problem. Chang [17], Amann and Zehnder [18] obtained the existence of three distinct solutions via Morse theory, where (3) was nonresonant at infinity. Moreover, the resonant case has been considered in [19–23]. The estimate of number of periodic solutions of (3) was established in [24]. Motivated by [24], we will use Morse theory to consider the lower bound of number of periodic solutions for system (1).

Throughout this paper we employ some standard notations. Denote by $\mathbb{R}, \mathbb{Z}$ the real number and the integer sets, respectively. $\mathbb{R}^N$ is the real space with dimension N. $Z[a, b] = \{a, a+1, \ldots, b\}$ if $a \le b$ and $a, b \in \mathbb{Z}$. A^T or x^T denotes the transpose of matrix A or vector x.

If $g(t)$ and $G(t)$ are bounded on $\mathbb{R}^N$, and system (1) is p-resonant at ∞, then functional J does not satisfy the compactness condition of the Palais-Smale type. Therefore our discussion will be divided into three cases. Moreover, we assume that

(P2) J has a finite number of nondegenerated critical points;

(P3) all p-periodic solutions of system (1) are not p-resonant;

(P4) for $m \in Z[0,r]$, $\sigma(A_\infty) \subset (\lambda_m, \lambda_{m+1}]$, where $\lambda_m = 4\sin^2(m\pi/p)$ and $r = [p/2]$.

Now we state the main results as follows.

Theorem 1. *Assume that (P1)–(P4) hold, and system* (1) *is not p-resonant at ∞. Then*

$$n(p) \geq \frac{1}{2}\Theta p - h\left(pN + \frac{1}{2}\right) + \frac{1}{2}, \tag{4}$$

where $n(p)$ is the number of the nonconstant p-periodic solutions of system (1)*, Θ is the global twist number (see* (32)*), and h will be defined in Section 3.*

Theorem 2. *Assume that (P1)–(P4) hold, system* (1) *is p-resonant at ∞, and $g(t)$ is bounded in $\mathbb{R}^N$, $\lim_{|t|\to+\infty} G(t) = -\infty$. Then* (4) *is valid.*

Theorem 3. *Assume that (P1)–(P4) hold, system* (1) *is p-resonant at ∞, and $g(t), G(t)$ are bounded in $\mathbb{R}^N$. Then*

$$n(p) \geq \frac{1}{2}\Theta p - h(pN + 1). \tag{5}$$

Remark 4. Benci and Fortunato [24] studied asymptotically linear equation (3). Theorem 1 extends and generalizes the analogous results in [24], and Theorems 2-3 are new results.

The organization of this paper is organized as follows. In Section 2 we study the compactness condition for functional J. Some facts about Morse theory and necessary preliminaries are given in Section 3. In Section 4 the main results are proved.

2. (PS) Condition

We say that a C^1-functional ϕ on Hilbert space X satisfies the Palais-Smale (PS) condition, if every sequence $\{x^{(j)}\}$ in X such that $\{\phi(x^{(j)})\}$ is bounded and $\phi'(x^{(j)}) \to 0$ as $j \to \infty$, contains a convergent subsequence.

Here we first introduce space E_p.

Let $E_p = \{x = \{x_n\} \in S \mid x_{n+p} = x_n, n \in \mathbb{Z}\}$, where $S = \{x = \{x_n\} \mid x_n \in \mathbb{R}^N, n \in \mathbb{Z}\}$. For any $x, y \in S$, $a, b \in \mathbb{R}$, $ax + by = \{ax_n + by_n\}_{n\in\mathbb{Z}}$. Then S is a linear space. Let E_p equip with inner product and norm as follows:

$$\langle x, y\rangle = \sum_{n=1}^{p} (x_n, y_n), \qquad \|x\| = \left(\sum_{n=1}^{p} |x_n|^2\right)^{1/2}, \quad \forall x, y \in E_p, \tag{6}$$

where $(\cdot,\cdot)$ and $|\cdot|$ are the usual inner product and norm in $\mathbb{R}^N$, respectively. Obviously, E_p is a Hilbert space with dimension pN and homeomorphism to $\mathbb{R}^{pN}$.

By the variational method, the p-periodic solutions of (1) are same as the critical points of the C^2-functional

$$J(x) = \sum_{n=1}^{p} \left[\frac{1}{2}|\Delta x_n|^2 - F(x_n)\right], \quad x \in E_p. \tag{7}$$

By assumption (P1), the functional J can be rewritten as

$$J(x) = \frac{1}{2}\sum_{n=1}^{p} \left[|\Delta x_n|^2 - (A_\infty x_n, x_n)\right] - \sum_{n=1}^{p} G(x_n), \tag{8}$$

and we write $I(x) = \sum_{n=1}^{p} [|\Delta x_n|^2 - (A_\infty x_n, x_n)]$.

Consider eigenvalue problem

$$-\Delta^2 x_{n-1} = \lambda x_n, \quad x_{n+p} = x_n, \quad x_n \in \mathbb{R}^N, \tag{9}$$

that is, $x_{n+1} + (\lambda-2)x_n + x_{n-1} = 0$, $x_{n+p} = x_n$. By the periodicity, the difference system has complexity solution $x_n = e^{in\theta}c$ for $c \in \mathbb{C}^N$, where $\theta = 2k\pi/p$, $k \in \mathbb{Z}$. Moreover, $\lambda = 2 - e^{-i\theta} - e^{i\theta} = 2(1 - \cos\theta) = 4\sin^2(k\pi/p)$.

Let η_k denote the real eigenvector corresponding to the eigenvalues $\lambda_k = 4\sin^2(k\pi/p)$, $k \in Z[0,r]$, and $r = [p/2]$, where $[\cdot]$ stands for the greatest-integer function. In terms of eigenvalue $\lambda_m = 4\sin^2(m\pi/p)$ for some $m \in Z[0,r]$, we can split space E_p as follows:

$$E_p = W^- \oplus W^0 \oplus W^+, \tag{10}$$

where

$$W^- = \text{span}\{\eta_k \mid k \in Z[0, m-1]\}, \qquad W^0 = \text{span}\{\eta_m\},$$
$$W^+ = \text{span}\{\eta_k \mid k \in Z[m+1, r]\}. \tag{11}$$

Moreover, there exists $\delta > 0$ such that

$$I(u) \geq \delta\|u\|^2 \quad \text{for } u \in W^+,$$
$$I(v) \leq -\delta\|v\|^2 \quad \text{for } v \in W^-, \quad I(w) = 0 \quad \text{for } w \in W^0. \tag{12}$$

Let us recall the definition of resonance (see [24]).

A p-periodic solution $\{x_n\}$ of (1) is called p-resonance, if there exists $\lambda_k = 4\sin^2(k\pi/p) \in \sigma(F''(x_n))$, where F'' denotes the Hessian matrix of F and $\sigma(\cdot)$ is the spectrum of matrix. We say that (1) is p-resonant at ∞, if there exists $\lambda_k = 4\sin^2(k\pi/p) \in \sigma(A_\infty)$.

Lemma 5. *Assume that (P1) and (P4) hold, and system* (1) *is not p-resonant at ∞. Then functional J (see* (8)*) satisfies the (PS) condition.*

Proof. Let $\{x^{(j)}\} \subset E_p$ be the (PS) sequence for functional J; that is, $\{J(x^{(j)})\}$ is bounded, and $J'(x^{(j)}) \to 0$ as $j \to \infty$. Therefore, for any $\varphi \in E_p$, we have

$$\left\langle J'\left(x^{(j)}\right), \varphi\right\rangle = o(\|\varphi\|) \quad \text{as } j \longrightarrow \infty. \tag{13}$$

By $W^0 = \{0\}$, we write $x^{(j)} = u^{(j)} + v^{(j)}$ with $u^{(j)} \in W^+$, $v^{(j)} \in W^-$. To show that J satisfies (PS) condition, it is enough to prove that $\{x^{(j)}\}$ is bounded in E_p. That is, we need only to prove that $\{u^{(j)}\}$ and $\{v^{(j)}\}$ are bounded in E_p. By contradiction, without loss of generality, there exists $k \in Z[1, p]$ such that

$$\begin{gathered} \left|x_n^{(j)}\right| \longrightarrow \infty \quad \text{as } j \longrightarrow \infty \text{ for } n \in Z[1,k], \\ x_n^{(j)} \text{ are bounded for } n \in Z[k+1,p]. \end{gathered} \tag{14}$$

Therefore, for all $n \in Z[1, p]$, by assumption (P1), there exist $\varepsilon > 0$ and $c_1 > 0$ such that

$$\left|G\left(x_n^{(j)}\right)\right| \le \varepsilon\left|x_n^{(j)}\right|^2 + c_1, \qquad \left|g\left(x_n^{(j)}\right)\right| \le \varepsilon\left|x_n^{(j)}\right| + c_1 \tag{15}$$

for large j. Thus there is $c > 0$, $|\sum_{n=1}^{p}(g(x_n^{(j)}), x_n^{(j)})| \le \sum_{n=1}^{p}|g(x_n^{(j)})|\,|x_n^{(j)}| \le \varepsilon\|x^{(j)}\|^2 + c\|x^{(j)}\|$. Taking $\varphi = u^{(j)} - v^{(j)}$ in (13), by previous argument,

$$\begin{aligned} & o\left(\left\|u^{(j)}\right\| + \left\|v^{(j)}\right\|\right) \\ &\quad = \left\langle J'\left(x^{(j)}\right), u^{(j)} - v^{(j)}\right\rangle \\ &\quad = I\left(u^{(j)}\right) - I\left(v^{(j)}\right) - \sum_{n=1}^{p}\left(g\left(x_n^{(j)}\right), u_n^{(j)} - v_n^{(j)}\right) \\ &\quad \ge \delta\left(\left\|u^{(j)}\right\|^2 + \left\|v^{(j)}\right\|^2\right) - \varepsilon\left(\left\|u^{(j)}\right\|^2 + \left\|v^{(j)}\right\|^2\right) \\ &\qquad - c\left(\left\|u^{(j)}\right\| + \left\|v^{(j)}\right\|\right), \end{aligned} \tag{16}$$

it follows a contradiction. Therefore $\{u^{(j)}\}$ and $\{v^{(j)}\}$ are bounded in E_p. This completes the proof. □

Here and in the sequel, the letter δ will be indiscriminately used to denote various positive constants whose exact values are irrelevant, and $\varepsilon \in (0, 1)$ is arbitrarily small. Moreover we also denote by c the various positive constants in this paper.

Lemma 6. *Assume that (P1) and (P4) hold. System* (1) *is p-resonant at ∞, $g(t)$ is bounded in $\mathbb{R}^N$, and* $\lim_{|t|\to+\infty} G(t) = -\infty$. *Then J satisfies the (PS) condition.*

Proof. Let $\{x^{(j)}\} \subset E_p$ be the (PS) sequence for functional J; that is, $\{J(x^{(j)})\}$ is bounded, and $J'(x^{(j)}) \to 0$ as $j \to \infty$.

Since system (1) is p-resonant at ∞, $W^0 \ne \{0\}$. Similarly, let $x^{(j)} = u^{(j)} + v^{(j)} + w^{(j)}$ with $u^{(j)} \in W^+$, $v^{(j)} \in W^-$, and $w^{(j)} \in W^0$. By the same method as proof of Lemma 5, it also follows that $\{u^{(j)}\}$ and $\{v^{(j)}\}$ are bounded in E_p. Next we prove that $\{w^{(j)}\}$ is bounded in E_p.

$J(x^{(j)}) = (1/2)I(u^{(j)}) + (1/2)I(v^{(j)}) - \sum_{n=1}^{p} G(x_n^{(j)})$, by $\{u^{(j)}\}$, $\{v^{(j)}\}$, and $J(x^{(j)})$ are bounded in E_p, and it follows that $\sum_{n=1}^{p} G(x_n^{(j)})$ is bounded. On the other hand, $|\sum_{n=1}^{p} G(x_n^{(j)}) - \sum_{n=1}^{p} G(w_n^{(j)})| \le \sup_{t\in E_p}\|g(t)\|(\|u^{(j)}\| + \|v^{(j)}\|)$, so $\sum_{n=1}^{p} G(w_n^{(j)})$ is bounded. It is easy to see from assumption $\lim_{|t|\to+\infty} G(t) = -\infty$ that $\{w^{(j)}\}$ is bounded. The proof is completed. □

If we assume that $G(t)$, $g(t)$ are bounded and system (1) is p-resonant at ∞, then functional J does not satisfy the (PS) condition. In order to overcome the difficult arising from the lack of compactness condition, we use a suitable penalization technique (one can refer to [20, 24]) and add a perturbation term to the functional J. Define

$$\varphi_R(t) = \begin{cases} (t-R)^4, & \text{if } t > R, \\ 0, & \text{if } t \le R, \end{cases} \tag{17}$$

where R is a positive real number and the penalized functional is given by

$$J_R(x) = J(x) + \varphi_R\left(\|w\|^2\right), \tag{18}$$

where $x = u + v + w \in W^+ \oplus W^- \oplus W^0$. Obviously, if $x \in E_p$ is a critical point of J_R with $\|w\|^2 \le R$, then x is also the critical point of J.

Lemma 7. *Assume that (P1) and (P4) hold, $G(t)$, $g(t)$ are bounded in $\mathbb{R}^N$, and system* (1) *is p-resonant at ∞. Then J_R satisfies the (PS) condition. Moreover, for any critical point x of J_R, there exists $M > 0$ such that $\|u + v\| \le M$, where $x = u + v + w \in E_p$, $u \in W^+$, $v \in W^-$, and $w \in W^0$.*

Proof. Let $\{x^{(j)}\} \subset E_p$ be the (PS) sequence for functional J_R; that is, $\{J_R(x^{(j)})\}$ is bounded in E_p, and for any $\varphi \in E_p$,

$$\left\langle J_R'\left(x^{(j)}\right), \varphi\right\rangle = o(\|\varphi\|) \quad \text{as } j \longrightarrow \infty. \tag{19}$$

Similarly to the proof of Lemma 5, we need only to prove that $\{w^{(j)}\}$ is bounded in E_p.

Taking $\varphi = w^{(j)}$ in (19), it follows that $o(\|w^{(j)}\|) = \langle J_R'(x^{(j)}), w^{(j)}\rangle \ge -c\|w\| + 2\|w\|^2\varphi_R'(\|w\|^2)$. By the definition of φ_R, it follows that $\{w^{(j)}\}$ is bounded. Therefore the penalized functional J_R satisfies the (PS) condition.

Let x be the critical point of J_R, then

$$\begin{aligned} 0 = \left\langle J_R'(x), u - v\right\rangle &= I(u) - I(v) \\ &\quad - \sum_{n=1}^{p}\left(g(x_n), u_n - v_n\right) \ge \delta\|u+v\|^2 \\ &\quad - \varepsilon\|u+v\|^2 - c\|u+v\|. \end{aligned} \tag{20}$$

So there is a $M > 0$ such that $\| u + v \| \le M$, and the proof is completed. □

3. Preliminaries

Let E be a real Hilbert space, and let ϕ be a C^2-functional on E. We denote by $\operatorname{crit}(\phi) = \{x \in E \mid \phi'(x) = 0\}$ the set of critical points of ϕ, $\phi^c = \{x \in E \mid \phi(x) \le c\}$ the level set of ϕ, and $\phi_a^b = \{x \in E \mid a \le \phi(x) \le b\}$. In the following we suppose that ϕ is a C^2-functional on E which satisfies the (PS) condition.

Definition 8 (see [9, 14]). Let x be a critical point of ϕ. The Morse index of x by $m(x,\phi)$ is defined as the supremum of the dimensions of the vector subspace of E on which $\phi''(x)$ is negative definite. The nullity of x by $\nu(x,\phi)$ is defined as the dimension of $\operatorname{Ker}\phi''(x)$. A critical point x will be said to be nondegenerate if $\phi''(x)$ is invertible.

Denote by m_∞, ν_∞ the Morse index and nullity of ∞ for functional J. By (10), $m_\infty = \dim W^-$, $\nu_\infty = \dim W^0$.

A set $K \subset E$ is called critical set if $K \subset \phi^{-1}(c) \cap \operatorname{crit}(\phi)$ for some $c \in \mathbb{R}$. A critical set K is called discrete nondegenerate critical manifold, if K is connected and $m(x,\phi)$ does not depend on $x \in K$.

Definition 9. The Poincare polynomial of the pair (ϕ^b,ϕ^a) is defined by $P_\lambda(\phi^b,\phi^a) = \sum_{n=0}^{\infty} \dim H_n(\phi^b,\phi^a;\Gamma)\lambda^n$, where $H_n(\phi^b,\phi^a;\Gamma)$ denotes the nth singular relative homology of the pair (ϕ^b,ϕ^a) with coefficients in field Γ. Define the topological Morse index of critical set K as $i_\lambda(K) = \sum_{n=0}^{\infty} \dim H_n(\phi^c,\phi^c \setminus K;\Gamma)\lambda^n$.

For simplicity, we write $m(x)$ and $m(K)$ instead of $m(x,\phi)$ and $m(K,\phi)$, respectively. It is well known that if x is a nondegenerate critical point and $m(x)$ is finite, then $i_\lambda(x) = \lambda^{m(x)}$. If K is a nondegenerate critical manifold and $m(K)$ is finite, then $i_\lambda(K) = \lambda^{m(K)}Q(\lambda)$, where $Q(\lambda)$ is a polynomial with nonnegative integer coefficients (see [13, 15]).

Next we investigate $P_\lambda(E,\phi^a)$ and use functional J (see (8)) or J_R (see (18)) instead of ϕ, E_p instead of E.

Lemma 10 (see [19, 24]). *Assume that (P1) and (P4) hold, and system* (1) *is not p-resonant at ∞. Then there exists $a \in \mathbb{R}$, $a < J(crit\,(J))$ such that*

$$P_\lambda\left(E_p, J^a\right) = \lambda^{m(\infty)}. \quad (21)$$

Lemma 11. *Assume that (P1) and (P4) hold, system* (1) *is p-resonant at ∞, $\lim_{|t|\to+\infty} G(t) = -\infty$, and $g(t)$ is bounded in $\mathbb{R}^N$. Then there exists $a \in \mathbb{R}$, and* (21) *is valid.*

Proof. Write $x = u + v + w \in E_p$ with $u \in W^+$, $v \in W^-$, $w \in W^0$. Then there exist $M_1 > 0$, $M_2 > 0$ such that $\langle J'(x),u\rangle \ge \delta\|u\|^2 - c\|u\| > 0$ as $\|u\| > M_1$, $\langle J'(x),v\rangle \le -\delta\|v\|^2 + c\|v\| < 0$ as $\|v\| > M_2$. Let $B_{M_1} = \{x \in E_p \mid \|u\| \le M_1\}$, $B_{M_2} = \{x \in E_p \mid \|v\| \le M_2\}$. By previous argument, it follows that J has no critical points in $E_p \setminus (B_{M_1} \cup B_{M_2})$.

On the other hand, for all $x \in B_{M_1} \cup B_{M_2}$,

$$J(x) \ge c - \sum_{n=1}^{p} G\left(u_n + v_n + \|w_n\| \cdot \frac{w_n}{\|w_n\|}\right) \longrightarrow +\infty, \quad \text{as } \|w\| \longrightarrow \infty. \quad (22)$$

Therefore there exists $a_1 \in \mathbb{R}$, such that $a_1 < J(\operatorname{crit}(J))$. For $x \in B_{M_2}$, we have

$$J(x) \ge \frac{1}{2}\delta\|u\|^2 - c - \sum_{n=1}^{p} G\left(\|u_n\| \cdot \frac{u_n}{\|u_n\|} + v_n + \|w_n\| \cdot \frac{w_n}{\|w_n\|}\right), \quad (23)$$

hence $J(x) \to +\infty$ as $\|u + w\| \to \infty$, which implies that J is bounded from the following in B_{M_2}. Let $a < \min\{a_1, \inf_{x\in B_{M_2}} J(x)\}$, then $J^a \subset E_p \setminus B_{M_2}$, and J^a is a strong deformation retraction of $E_p \setminus B_{M_2}$. By Lemma 6, J satisfies (PS) condition, and we have

$$\begin{aligned} H_n\left(E_p, J^a\right) &\cong H_n\left(E_p, E_p \setminus B_{M_2}\right) \\ &\cong H_n\left(W^-, W^- \setminus B_{M_2}\right) \cong \delta_{n,m(\infty)}\Gamma. \end{aligned} \quad (24)$$

So we obtain (21). □

Lemma 12. *Under the assumption of Theorem 3, there exists $a \in \mathbb{R}$ such that $P_\lambda(E_p, J_R^a) = \lambda^{m(\infty)}$.*

Proof. Let $x = u + v + w \in E_p$ with $u \in W^+$, $v \in W^-$, and $w \in W^0$. Then there exist $R_1 > R + 1$ such that all critical points of J_R are in $B_{M_1} \cap B_{M_2} \cap B_{M_3}$, where B_{M_1} and B_{M_2} are the same as in proof of Lemma 11, and $B_{M_3} = \{x \in E_p \mid \|w\|^2 \le R_1\}$. In fact,

$$\begin{aligned} \left\langle J_R'(x), u\right\rangle &= \left\langle J'(x), u\right\rangle > 0, \quad x \notin B_{M_1}, \\ \left\langle J_R'(x), v\right\rangle &= \left\langle J'(x), v\right\rangle < 0, \quad x \notin B_{M_2}, \\ \left\langle J_R'(x), w\right\rangle &\ge -c\|w\| + 8\|w\|^2\left(\|w\|^2 - R\right)^3 \\ &\ge 8\|w\|^2 - c\|w\| > 0, \quad x \notin B_{M_3}. \end{aligned} \quad (25)$$

Similarly, for $x \in B_{M_2}$, $J_R(x) \ge (1/2)\delta\|u\|^2 + \varphi_R(\|w\|^2) - c - \sum_{n=1}^{p} G(x_n)$, and $J_R(x) \to +\infty$ as $\|u + w\| \to \infty$, which implies that $J_R(x)$ is bounded from the following in B_{M_2}. Let $a_0 = \inf_{x\in B_{M_2}} J_R(x)$. If $a < \min\{a_0, J_R(\operatorname{crit}(J_R))\}$, by J_R satisfies (PS) condition, and methods of strong deformation retract, we have $P_\lambda(E_p, J_R^a) = \lambda^{m(\infty)}$. The proof is completed. □

Assume that on Hilbert space E there is an action of discrete group G, and denote by $\operatorname{fix}(G)$ the fixed points set for the G action; that is, $\operatorname{fix}(G) = \{x \in E \mid gx = x, \forall g \in G\}$. The functional ϕ is called G invariant, if $\phi(gx) = \phi(x)$, $\forall x \in E$, and $\forall g \in G$. In the following, Z_p denotes a cyclic group of p order. In terms of Proposition 8.2 and Proposition 8.5 in [13], we have following lemma.

Lemma 13. *Assume that ϕ is a C^2-functional on an Hilbert space E and satisfies (PS) condition. Let a, b (b possible ∞) be two regular values of ϕ. Assume that $crit(\phi_a^b) = crit(\phi) \cap \phi^{-1}(a,b)$ consists only of critical sets, and then the following Morse relation holds:*

$$\sum_{K\subset crit(\phi_a^b)} i_\lambda(K) = P_\lambda\left(\phi^b,\phi^a\right) + (1+\lambda)Q(\lambda), \quad (26)$$

where $Q(\lambda)$ is a polynomial with nonnegative integer coefficients. If all the critical points of ϕ in ϕ_a^b are nondegenerate and have finite Morse index, then (26) *can be written as*

$$\sum_{x\in crit(\phi_a^b)} \lambda^{m(x)} = P_\lambda\left(\phi^b,\phi^a\right) + (1+\lambda)Q(\lambda). \quad (27)$$

Now if ϕ is Z_p invariant, and $crit(\phi_a^b) \cap fix(Z_p)$ consists only of nondegenerate critical points having finite Morse index, then (26) *becomes*

$$\sum_{x \in crit(\phi_a^b) \cap fix(Z_p)} \lambda^{m(x)} + (1+\lambda) Z(\lambda) = P_\lambda \left(\phi^b, \phi^a\right) + (1+\lambda) Q(\lambda), \tag{28}$$

where $Z(\lambda)$ is a formal series with nonnegative integer coefficients. Moreover if $crit(\phi_a^b) - fix(Z_p)$ consists only of nondegenerate critical manifolds having finite Morse index, then

$$Z(\lambda) = \sum_{K \subset crit(\phi_a^b) - fix(Z_p)} \lambda^{m(K)}. \tag{29}$$

Remark 14. By (29), our main goal in this paper is to estimate $Z(1)$ which gives a lower bound of the number of the nonconstant critical points of J in E_p.

Lemma 15. *Let $z = \{z_n\}$ be a critical point of functional J. Denote by $\tau_1^2, \tau_2^2, \ldots, \tau_l^2$ the positive eigenvalues (repeated according to their multiplicity) of $F''(z_n)$, where $\tau_j > 0$, $j \in Z[1,l]$. Under the assumption (P2), we have $\sharp(z,J) = l + 2\sum_{j=1}^{l}[(p/\pi)\arcsin(\tau_j/2)]$, where $[\cdot]$ denotes the greatest-integer function and $\sharp(z,J)$ is the number of eigenvalues $\lambda < 0$ such that $\langle J''(z)u, u\rangle = \lambda \|u\|^2$.*

Proof. By $\langle J''(z)u, u\rangle = \sum_{n=1}^{p}[|\Delta u_n|^2 - (F''(z_n)u_n, u_n)] = -\sum_{n=1}^{p}[(\Delta^2 u_{n-1} + F''(z_n)u_n, u_n)]$, we consider the equation $\Delta^2 y_{n-1} + F''(z_n)y_n = -\lambda y_n$, $y_{n+p} = y_n$, where $z = \{z_n\}$ is the critical point of J. It is easy to see that $\lambda_{k,j} = 4\sin^2(k\pi/p) - \tau_j^2$ are eigenvalues of $\Delta^2 y_{n-1} + F''(z_n)y_n$ on $\mathbb{R}^N$, where $n, k \in Z[1,p]$, $j \in Z[1,l]$. Therefore the number of negative eigenvalues $\lambda_{k,j}$ is just what we are looking for; the proof is completed. □

Definition 16. For any critical point z of J, there are l positive eigenvalues (repeated according to their multiplicity) of $F''(z_n)$, which will be denoted by $\tau_1^2, \ldots, \tau_l^2$. The number $\rho(z) = (2p/\pi)\sum_{j=1}^{l}\arcsin(\tau_j/2)$ is called twist number of z. Moreover the twist number of ∞ is defined by $\rho(\infty) = (2p/\pi)\sum_{j=1}^{l(\infty)}\arcsin(\tau_j/2)$, where $l(\infty)$ is the number of the positive eigenvalues (repeated according to their multiplicity) of A_∞.

Let $z = \{z_n\}$ be a constant critical point of functional J; that is, $z_1 = z_2 = \cdots = z_p$. By Lemma 15 and Definition 16, it is easy to deduce the following relation between the Morse index and the twist number as follows:

$$\rho(z)p - pN \le m(z,J) \le \rho(z)p + pN. \tag{30}$$

In view of the number l or $l(\infty)$ of the positive eigenvalues (repeated according to their multiplicity) of $F''(z)$ or A_∞, the constant critical point z is called τ-positive (resp., τ-negative) if l is even (resp., odd). On the contrary, the virtual critical point ∞ is called τ-positive (resp., τ-negative) if $l(\infty)$ is odd (resp., even), see [24].

We denote by h_1 and h_2 the number of τ-positive and τ-negative critical points of J. If A_∞ is invertible, then $h_1 - h_2 = (-1)^{l(\infty)}$. Thus, if we consider ∞ as a virtual critical point, we have that the number of τ-positive critical points equals the number of τ-negative critical points. However, if A_∞ is singular, the result is not hold in general. If we introduce $|h_1 - h_2|$ virtual critical points having twist number zero, where they are considered as τ-positive if $h_1 < h_2$ and as τ-negative if $h_1 > h_2$, then the number of τ-positive critical points is also equal to the number of τ-negative critical points.

Let $h = \max\{h_1, h_2\}$, which has been used in (4) and (5). We denote by $x_1, \ldots, x_h$ the τ-positive critical points and by $y_1, \ldots, y_h$ the τ-negative critical points such that

$$\rho(x_1) \le \cdots \le \rho(x_h), \qquad \rho(y_1) \le \cdots \le \rho(y_h). \tag{31}$$

Then the global twist number Θ of the system (1) is defined by

$$\Theta = \sum_{i=1}^{h} \left|\rho(x_i) - \rho(y_i)\right|. \tag{32}$$

4. Proof of Main Results

Proof of Theorem 1. The argument is analogous to one used by Benci and Fortunato in [24]. Set $m(z) = m(z,J)$. Under the assumption (P2), let $z_1, \ldots, z_n$ be the nondegenerate constant critical points of J.

By Lemmas 5 and 10, functional J satisfies (PS) condition, and there exists sufficiently small $a \in \mathbb{R}$ such that $P_\lambda(J^b, J^a) = P_\lambda(E_p, J^a) = \lambda^{m(\infty)}$, where $b = \infty$. Since J is C^2 and Z_p invariant functional on E_p, then by assumption (P3), we have $\sum_{i=1}^{n}\lambda^{m(z_i)} + (1+\lambda)Z(\lambda) = \lambda^{m(\infty)} + (1+\lambda)Q(\lambda)$; that is,

$$\sum_{i=1}^{n}\lambda^{m(z_i)} - \lambda^{m(\infty)} = (1+\lambda)(Q(\lambda) - Z(\lambda)). \tag{33}$$

Let m_i, f_i $(i \in Z[1,h])$ denote the Morse indices of the τ-positive and τ-negative critical points (including ∞) of J, and without loss of generalities, assume that ∞ is τ-negative. So $m(\infty) = f_j$ for some $j \in Z[1,h]$, where h is referred to (31). Then (33) becomes

$$\sum_{i=1, i\ne j}^{h} \frac{\lambda^{m_i} + \lambda^{f_i}}{1+\lambda} + \frac{\lambda^{m_j} - \lambda^{m(\infty)}}{1+\lambda} = Q(\lambda) - Z(\lambda). \tag{34}$$

Set $Q(\lambda) = \sum_s q_s\lambda^s$, $Z(\lambda) = \sum_s z_s\lambda^s$, and $B(\lambda) = Q(\lambda) - Z(\lambda) = \sum_s b_s\lambda^s$, where q_s, z_s are nonnegative integer and $b_s = q_s - z_s$.

By Remark 14, the lower bound of the number of nonconstant p-periodic solutions for system (1) is to estimate $Z(1)$. Since $q_s \ge 0$, $z_s \ge 0$, then

$$n(p) = \sum_s z_s \ge \sum_{b_s<0} z_s = \sum_{b_s<0}(q_s - b_s) \ge -\sum_{b_s<0} b_s. \tag{35}$$

Let $B^- = -\sum_{b_s<0} b_s$. By (35), we turn our attention to estimate B^-.

If l is even (resp., odd), by Lemma 15, $m(z, J)$ is also even (resp., odd). Therefore by the definition of τ-positive and τ-negative critical points of J, m_i $(i \in Z[1,h])$ are even numbers, f_i are odd numbers for $i \neq j$, $i \in Z[1,h]$, and $f_j = m(\infty)$ is a even number.

Set $M_1 = \{r \mid m_r > f_r, r \in Z[1,h], r \neq j\}$, $M_2 = \{r \mid m_r < f_r, r \in Z[1,h], r \neq j\}$, and

$$C(\lambda) = \sum_{r\in M_1} \frac{\lambda^{m_r} + \lambda^{f_r}}{1+\lambda}, \qquad D(\lambda) = \sum_{r\in M_2} \frac{\lambda^{m_r} + \lambda^{f_r}}{1+\lambda}, \qquad E(\lambda) = \frac{\lambda^{m_j} - \lambda^{f_j}}{1+\lambda}. \tag{36}$$

By (34), we have $B(\lambda) = Q(\lambda) - Z(\lambda) = \sum_s b_s\lambda^s = C(\lambda) + D(\lambda) + E(\lambda)$, and

$$C(\lambda) = \sum_{r\in M_1} \sum_{i=f_r}^{m_r-1} c_{r,i}\lambda^i, \qquad D(\lambda) = \sum_{r\in M_2} \sum_{i=m_r}^{f_r-1} d_{r,i}\lambda^i, \tag{37}$$

where $c_{r,i} = (-1)^{i+1}$, $d_{r,i} = (-1)^i$. Meanwhile, if $m_j > f_j$, $E(\lambda) = \sum_{i=f_j}^{m_j-1} e_{i,1}\lambda^i$, $e_{i,1} = (-1)^{i+1}$. If $m_j < f_j$, $E(\lambda) = \sum_{i=m_j}^{f_j-1} e_{i,2}\lambda^i$, $e_{i,2} = (-1)^i$. Clearly $E(\lambda) = 0$ if $m_j = f_j$.

A straight analysis shows that $B^- = (1/2)\sum_{r=1}^{h} |m_r - f_r| - ((h-1)/2)$. By (30) and the definition of global twist number that refer to (32), we have $n(p) \geq (1/2)\Theta p - h(pN + (1/2)) + (1/2)$. It completes the proof of Theorem 1. □

Proof of Theorem 2. Under the assumptions of Theorem 2, by Lemmas 6 and 11, functional J satisfies (PS) condition, and there exists $a \in \mathbb{R}$ such that $P_\lambda(J^b, J^a) = P_\lambda(E_p, J^a) = \lambda^{m(\infty)}$, for $b = \infty$.

Similarly, we have $\sum_{i=1}^{n} \lambda^{m(z_i)} + (1+\lambda)Z(\lambda) = \lambda^{m(\infty)} + (1+\lambda)Q(\lambda)$, where z_i $(i \in Z[1,n])$ are nondegenerate critical points of J. The remainder is the same as that of Theorem 1. □

The following lemma is needed to prove Theorem 3.

Lemma 17. *If all assumptions in Theorem 3 hold, then there exists $Q > 0$ (independent of R) such that*

$$m(x, J_R) + \nu(x, J_R) \leq m(\infty) + \nu(\infty), \tag{38}$$

where $x = u+v+w$ with $u \in W^+$, $v \in W^-$, $w \in W^0$, $\|w\| \geq Q$, and $m(x, J_R)$, $\nu(x, J_R)$ denote the Morse index and nullity of critical point x for functional J_R, respectively.

Proof. Let $x = u + v + w \in E_p$ be a critical point of J_R. By Lemma 7, we have $\|u + v\| \leq M$. Therefore $\|x\| \to \infty$ if and only if $\|w\| \to \infty$. Since

$$\begin{aligned}\left\langle J_R''(x)u,u\right\rangle &= \sum_{n=1}^{p} \left[|\Delta u_n|^2 - (A_\infty u_n, u_n) - \left(g'(x_n)u_n,u_n\right)\right] \geq \delta\|u\|^2 \\ &\quad - \sum_{n=1}^{p}\left(g'(x_n)u_n,u_n\right),\end{aligned} \tag{39}$$

by assumption (P1), there exists $\varepsilon \in (0,\delta)$ such that

$$\begin{aligned}&\sum_{n=1}^{p}\left(g'(x_n)u_n,u_n\right) \\ &\quad = \sum_{n=1}^{p}\left(g'\left(\|w\|\cdot\frac{w_n}{\|w\|} + u_n + v_n\right)u_n,u_n\right) \\ &\quad \leq \varepsilon\|u\|^2,\end{aligned} \tag{40}$$

as $\|w\| \to \infty$. Therefore there exists $Q > 0$ such that $\langle J_R''(x)u,u\rangle > 0$ as $\|w\| \geq Q$. It follows that relation (38) is valid, and the proof is completed. □

Proof of Theorem 3. Let $L = m(\infty) + \nu(\infty)$, and denote by $z_1, \ldots, z_n$ the constant critical points of J. We assume, without loss of generalities, $m(z_i, J) \geq L+1$ as $i \in Z[1,r]$, $m(z_i, J) < L$ as $i \in Z[r+1,n]$. Clearly $L \leq pN$.

Set $H_1 = \{x \in \mathrm{crit}(J_R) \mid \|w\| > Q\}$, $H_2 = \{x \in \mathrm{crit}(J_R) \cap \mathrm{fix}(Z_p) \mid \|w\| \leq Q\}$, and $H_3 = \mathrm{crit}(J_R) - (H_1 \cup H_2)$, where $x = u + v + w$ is the decomposition of $x \in E_p$ with $w \in W^0$ and Q is large enough.

By Lemma 17, H_1 contains only critical points of J_R which have $m(x, J_R) + \nu(x, J_R) \leq L$. $H_2 = \{z_i \mid i \in Z[1,n]\}$, since $J_R(x) = J(x)$ as $\|w\| \leq Q$. $H_3 \subset \mathrm{crit}(J) - \mathrm{fix}(Z_p)$, since $\varphi_R(x) = 0$ when $\|w\| \leq Q$. Moreover by assumption (P3), H_3 contains only nondegenerate critical manifolds.

Since J_R satisfies (PS) condition, by Lemma 13, relation (28) reads

$$\begin{aligned}&\sum_{x\in H_1} i_\lambda(x) + \sum_{x\in H_2} i_\lambda(x) + \sum_{K\subset H_3} i_\lambda(K) \\ &\quad = P_\lambda\left(E_p, J_R^a\right) + (1+\lambda)Q(\lambda),\end{aligned} \tag{41}$$

that is,

$$\begin{aligned}&\sum_{x\in H_1} i_\lambda(x) + \sum_{i=1}^{r} i_\lambda(z_i) + \sum_{i=r+1}^{n} i_\lambda(z_i) \\ &\quad + (1+\lambda)Z(\lambda) = \lambda^{m(\infty)} + (1+\lambda)Q(\lambda),\end{aligned} \tag{42}$$

where $Z(\lambda) = \sum_{K\subset H_3} \lambda^{m(K)}$. For $Z(\lambda) = \sum_{i=0}^{\infty} z_i\lambda^i$, we set $Z_l = \sum_{i=l}^{\infty} z_i\lambda^i$, where $l \in \mathbb{N}$. And analogous notation can be introduced for $Q(\lambda)$. Then, considering the terms of degree $\geq L+1$ in (42), we have

$$\sum_{i=1}^{r} \lambda^{m(z_i,J)} + b_L\lambda^{L+1} = (1+\lambda)B(\lambda), \tag{43}$$

where $b_L = z_L - q_L$, $B(\lambda) = Q_{L+1}(\lambda) - Z_{L+1}(\lambda)$. Clearly

$$\begin{aligned}n(p) &= \sum_s z_s \geq \sum_{b_s\leq 0, s\geq L+1} z_s \\ &= \sum_{b_s\leq 0, s\geq L+1} (q_s - b_s) \geq -\sum_{b_s\leq 0, s\geq L+1} b_s = B^-,\end{aligned} \tag{44}$$

that is, B^- is the absolute value of the sum of the negative coefficients of $B(\lambda)$. Next we estimate the number B^-.

Let $x_1,\dots,x_{h_1}$ and $y_1,\dots,y_{h_2}$ $(h_1+h_2=r)$ be the τ-positive and τ-negative critical points of J with nonzero twist numbers, whose order satisfies (31), and $x_i, y_j \in \{z_1,\dots,z_r\}$, $i\in Z[1,h_1]$, $j\in Z[1,h_2]$. Without loss of generalities, assume $h_1\ge h_2$, and introduce $h_3(=h_1-h_2)$ virtual τ-negative critical points $\overline{y}_i$ $(i\in Z[1,h_3])$ having twist number 0 and Morse index 0; that is,

$$\rho(\overline{y}_i)=0,\quad f_i=m(\overline{y}_i,J)=0,\quad i\in Z[1,h_3]. \tag{45}$$

For $i\in Z[1,h_1]$, set $m_i=m(x_i,J)$, $f_i=m(\overline{y}_i,J)$, where $\overline{y}_{j+h_3}=y_j$, $j\in Z[1,h_2]$. Then (43) can be written as $\sum_{i=1}^{h_1}\lambda^{m_i}+\sum_{i=h_3+1}^{h_1}\lambda^{f_i}+b_L\lambda^{L+1}=(1+\lambda)B(\lambda)$. Setting $\lambda=-1$, then $b_L=-h_3$ if L is odd, and $b_L=h_3$ if L is even. So

$$\begin{aligned} B(\lambda)&=\sum_{i=1}^{h_3}\frac{\lambda^{m_i}+\lambda^{L+1}}{1+\lambda}+\sum_{i=h_3+1}^{h_1}\frac{\lambda^{m_i}+\lambda^{f_i}}{1+\lambda},\quad \text{if } L \text{ is even},\\ B(\lambda)&=\sum_{i=1}^{h_3}\frac{\lambda^{m_i}-\lambda^{L+1}}{1+\lambda}+\sum_{i=h_3+1}^{h_1}\frac{\lambda^{m_i}+\lambda^{f_i}}{1+\lambda},\quad \text{if } L \text{ is odd}. \end{aligned} \tag{46}$$

A straight analysis shows that $B^-=\sum_{i=1}^{h_3}[(1/2)(m_i-L)-1]+\sum_{i=h_3+1}^{h}(1/2)|m_i-f_i|$ if L is even, and $B^-=\sum_{i=1}^{h_3}(1/2)(m_i-L)+\sum_{i=h_3+1}^{h}(1/2)(|m_i-f_i|-1)$ if L is odd. Therefore

$$B^-\ge\sum_{i=1}^{h_3}\frac{1}{2}(m_i-L-2)+\sum_{i=h_3+1}^{h}\frac{1}{2}(|m_i-f_i|-1). \tag{47}$$

By (30) and (45), we have

$$\begin{aligned} m_i&=m_i-f_i\ge(\rho(x_i)-\rho(\overline{y}_i))\,p-pN,\quad i\in Z[1,h_3],\\ |m_i-f_i|&\ge|\rho(x_i)-\rho(\overline{y}_i)|\,p-2pN,\quad i\in Z[h_3+1,h]. \end{aligned} \tag{48}$$

In view of (45), (47), and (48), we have

$$\begin{aligned} B^-\ge{}&\frac{1}{2}\sum_{i=1}^{h}p\,|\rho(x_i)-\rho(y_i)|\\ &-\frac{1}{2}pNh_3-\frac{1}{2}Lh_3-pNh_2-h_3\\ &-\frac{1}{2}h_2\ge\frac{1}{2}\Theta p-h(pN+1). \end{aligned} \tag{49}$$

The proof is completed. □

Remark 18. Although A_∞ is invertible under the assumptions of Theorem 3, we do not make use of (42) directly, because we consider only the terms of degree $\ge L+1$ in proof of Theorem 3.

Acknowledgments

This research is supported by the National Natural Science Foundation of China under Grants (11101187), NCETFJ (JA11144), the Excellent Youth Foundation of Fujian Province (2012J06001), and the Foundation of Education of Fujian Province (JA09152).

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