Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2013, Article ID 705313, 8 pages
http://dx.doi.org/10.1155/2013/705313

Research Article

A Numerical Method for Partial Differential Algebraic Equations Based on Differential Transform Method

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Received 14 January 2013; Accepted 28 February 2013

Academic Editor: Adem Kiliçman

We have considered linear partial differential algebraic equations (LPDAEs) of the form $Au_t(t,x)+Bu_{xx}(t,x)+Cu(t,x)=f(t,x)$, which has at least one singular matrix of $A,B\in\mathbb{R}^{n\times n}$. We have first introduced a uniform differential time index and a differential space index. The initial conditions and boundary conditions of the given system cannot be prescribed for all components of the solution vector u here. To overcome this, we introduced these indexes. Furthermore, differential transform method has been given to solve LPDAEs. We have applied this method to a test problem, and numerical solution of the problem has been compared with analytical solution.

1. Introduction

The partial differential algebraic equation was first studied by Marszalek. He also studied the analysis of the partial differential algebraic equations [1]. Lucht et al. [2–4] studied the numerical solution and indexes of the linear partial differential equations with constant coefficients. A study about characteristics analysis and differential index of the partial differential algebraic equations was given by Martinson and Barton [5, 6]. Debrabant and Strehmel investigated the convergence of Runge-Kutta method for linear partial differential algebraic equations [7].

There are numerous LPDAEs applications in scientific areas given, for instance, in the field of Navier-Stokes equations, in chemical engineering, in magnetohydrodynamics, and in the theory of elastic multibody systems [4, 8–12].

On the other hand, the differential transform method was used by Zhou [13] to solve linear and nonlinear initial value problems in electric circuit analysis. Analysis of nonlinear circuits by using differential Taylor transform was given by Köksal and Herdem [14]. Using one-dimensional differential transform, Abdel-Halim Hassan [15] proposed a method to solve eigenvalue problems. The two-dimensional differential transform methods have been applied to the partial differential equations [16–19]. The differential transform method extended to solve differential-difference equations by Arikoglu and Ozkol [20]. Jang et al. have used differential transform method to solve initial value problems [21]. The numerical solution of the differential-algebraic equation systems has been studied by using differential transform method [22, 23].

In this paper, we have considered linear partial differential equations with constant coefficients of the form

$$Au_t(t,x)+Bu_{xx}(t,x)+Cu(t,x)=f(t,x),\quad (t,x)\in J\times\Omega, \tag{1}$$

where $J=[0,\infty)$, $\Omega=[-l,l]$, $l>0$, and $A,B,C\in\mathbb{R}^{n\times n}$. In (1) at least one of the matrices $A,B\in\mathbb{R}^{n\times n}$ should be singular. If $A=0$ or $B=0$, then (1) becomes ordinary differential equation or differential algebraic equation, so we assume that none of the matrices A or B is the zero matrix.

2. Indexes of Partial Differential Algebraic Equation

Let us consider (1), with initial values and boundary conditions given as follows:

$$u_j(t, \pm l) = 0 \quad \text{for } t \in J,$$
$$u_i(0, x) = g(x) \quad \text{for } x \in \Omega, \tag{2}$$

where $j \in \mathfrak{M}_{BC} \subseteq \{1, 2, \dots, n\}$, $\mathfrak{M}_{BC}$ is the set of indices of components of u for which boundary conditions can be prescribed arbitrarily, and $i \in \mathfrak{M}_{IC} \subseteq \{1, 2, \dots, n\}$, $\mathfrak{M}_{IC}$ is the set of indices of components of u for which initial conditions can be prescribed arbitrarily. The initial boundary value problem (IBVP) (1) has only one solution where a function u is a solution of the problem, if it is sufficiently smooth, uniquely determined by its initial values (IVs) and boundary values (BVs), and if it solves the LPDAE point wise.

Definition of the indexes can be given using the following assumptions.

(i) Each component of the vectors u, u_t, and f satisfy the following condition:
$$|y(t, x)| \le Me^{\alpha t}, \quad \alpha \ge 0, \ t \ge 0, \tag{3}$$
where M and α are independent of t and x.

(ii) $(B, \xi A + C), \text{Re}(\xi) > \alpha$, called as the matrix pencil, is regular.

(iii) $(A, \mu_k B + C)$ is regular for all k, where μ_k is an eigenvalue of the operator $\partial^2/\partial x^2$ together with prescribed BCs.

(iv) The vector $f(t, x)$ and the initial vector $g(x)$ are sufficiently smooth.

If we use Laplace transform, from assumption (ii), (1) can be transformed into

$$Bu''_\xi(x) + (\xi A + C) u_\xi(x) = f_\xi(x) + Ag(x), \quad \text{Re}(\xi) > \alpha, \tag{4}$$

if B is a singular matrix, then (4) is a DAE depending on the parameter ξ. To characterize $\mathfrak{M}_{BC}$, we introduce $j \in \mathfrak{M}_{BC}^{(\xi)} \subseteq \{1, 2, \dots, n\}$ as the set of indices of components of u_ξ for which boundary conditions can be prescribed arbitrarily.

In order to define a spatial index, we need the Kronecker normal form of the DAE (4). Assumption (iii) guarantees that there are nonsingular matrices $P_{L,\xi}, Q_{L,\xi} \in \mathbb{C}^{n\times n}$ such that

$$P_{L,\xi} B Q_{L,\xi} = \begin{pmatrix} I_{m_1} & 0 \\ 0 & N_{L,\xi} \end{pmatrix},$$
$$P_{L,\xi}(\xi A + C) Q_{L,\xi} = \begin{pmatrix} R_{L,\xi} & 0 \\ 0 & I_{m_2} \end{pmatrix}, \tag{5}$$

where $R_{L,\xi} \in \mathbb{C}^{m_1\times m_2}$ and $N_{L,\xi} \in \mathbb{R}^{m_2\times m_2}$ is a nilpotent Jordan chain matrix with $m_1 + m_2 = n$. I_k is the unit matrix of order k. The Riesz index (or nilpotency) of $N_{L,\xi}$ is denoted by $\nu_{L,\xi}$ (i.e. $N_{L,\xi}^{\nu_{L,\xi}} = 0, N_{L,\xi}^{\nu_{L,\xi}-1} \ne 0$).

Here, we will assume that there is a real number $\alpha^* \ge \alpha$ such that the index set $\mathfrak{M}_{BC}^{(\xi)}$ is independent of the Laplace parameter ξ, provided $\text{Re}(\xi) \ge \alpha^*$.

Definition 1. Let $\alpha^* \in \mathbb{R}^+$ be a number with $\alpha^* \ge \alpha$, such that for all $\xi \in \mathbb{C}$ with $\text{Re}(\xi) \ge \alpha^*$

(1) the matrix pencil $(B, \xi A + C)$ is regular,

(2) $\mathfrak{M}_{BC}^{(\xi)}$ is independent of ξ, i.e., $\mathfrak{M}_{BC}^{(\xi)} = \mathfrak{M}_{BC}$,

(3) the nilpotency of $N_{L,\xi}$ is $\nu_L \ge 1$.

Then $\nu_{d,x} = 2\nu_L - 1$ is called the "*differential spatial index*" of the LPDAE. If $\nu_L = 0$, then the differential spatial index of LPDAE is defined to be zero.

If we use Fourier transform, (1) can be transformed into

$$A\hat{u}'_k(t) + (\mu_k B + C)\hat{u}_k(t) = \hat{f}_k(t) + B\rho_k(t) \tag{6}$$

with $\rho_k(t) = (\rho_{k1}(t), \dots, \rho_{kn}(t))^T$ and

$$\rho_{ki}(t) = 0 \quad \text{for } i \in \mathfrak{M}_{BC},$$
$$\rho_{kj}(t) = \frac{1}{l}\left[\phi'_k(x) u_j(t, x) - \phi_k(x) u_{x,j}(t, x)\right]_{x=-l}^{x=l} \tag{7}$$

for $j \notin \mathfrak{M}_{BC}$, which results from partial integration of the term $\int_{-l}^{l} u_{xx}(t, x)\phi_k(x)dx$.

If A is a singular matrix, then (6) is a DAE depending on the parameter μ_k which can be solved uniquely with suitable ICs under the assumptions (iv) and (v). Analogous to the case of the Laplace transform, the above assumption (iv) implies that there exist regular matrices $P_{F,k}, Q_{F,k}$ such that

$$P_{F,k} A Q_{F,k} = \begin{pmatrix} I_{n_1} & 0 \\ 0 & N_{F,k} \end{pmatrix},$$
$$P_{F,k}(\mu_k B + C) Q_{F,k} = \begin{pmatrix} R_{F,k} & 0 \\ 0 & I_{n_2} \end{pmatrix}. \tag{8}$$

With $R_{F,k} \in \mathbb{R}^{n_1\times n_1}$. $N_{F,k} \in \mathbb{R}^{n_2\times n_2}$ is again a nilpotent Jordan chain matrix with Riesz index $\nu_{F,k}$, where $n_1 + n_2 = n$.

To characterize $\mathfrak{M}_{IC}$, we introduce $\mathfrak{M}_{IC}^{(k)} \subseteq \{1, 2, \dots, n\}$ as the set of indices of components of $\hat{u}_k$ for which initial conditions can be prescribed arbitrarily. Therefore, we always assume in the context of a Fourier analysis of u that $\mathfrak{M}_{IC}^{(k)}$ is independent of $k \in \mathbb{N}_+$, i.e., $\mathfrak{M}_{IC}^{(k)} = \mathfrak{M}_{IC}$.

Definition 2. Assume for $k = 1, 2, \dots$ that

(1) the matrix pencil $(A, \mu_k B + C)$ is regular,

(2) $\mathfrak{M}_{IC}^{(k)}$ is independent of k, i.e., $\mathfrak{M}_{IC}^{(k)} = \mathfrak{M}_{IC}$,

(3) the nilpotency of $N_{F,k}$ is $\nu_{F,k} = \nu_F$.

Then the PDAE (1) is said to have uniform differential *time index* $\nu_{d,t} = \nu_F$.

The differential spatial and time indexes are used to decide which initial and boundary values can be taken to solve the problem.

3. Two-Dimensional Differential Transform Method

The two-dimensional differential transform of function $w(x,y)$ is defined as

$$W(k,h)=\frac{1}{k!h!}\left[\frac{\partial^{k+h}w(x,y)}{\partial x^k\partial y^h}\right]_{\substack{x=0\\y=0}}, \tag{9}$$

where it is noted that upper case symbol $W(k,h)$ is used to denote the two-dimensional differential transform of a function represented by a corresponding lower case symbol $w(x,y)$. The differential inverse transform of $W(k,h)$ is defined as

$$w(x,y)=\sum_{k=0}^{\infty}\sum_{h=0}^{\infty}W(k,h)\,x^k y^h. \tag{10}$$

From (9) and (10), we obtain

$$w(x,y)=\sum_{k=0}^{\infty}\sum_{h=0}^{\infty}\frac{x^k y^h}{k!h!}\left[\frac{\partial^{k+h}w(x,y)}{\partial x^k\partial y^h}\right]_{\substack{x=0\\y=0}}. \tag{11}$$

The concept of two-dimensional differential transform is derived from two-dimensional Taylor series expansion, but the method doesn't evaluate the derivatives symbolically.

Theorem 3. *Differential transform of the function $w(x,y)=u(x,y)\pm v(x,y)$ is*

$$W(k,h)=U(k,h)\pm V(k,h), \tag{12}$$

see [17].

Theorem 4. *Differential transform of the function $w(x,y)=\lambda u(x,y)$ is*

$$W(k,h)=\lambda U(k,h), \tag{13}$$

see [17].

Theorem 5. *Differential transform of the function $w(x,y)=\partial u(x,y)/\partial x$ is*

$$W(k,h)=(k+1)\,U(k+1,h), \tag{14}$$

see [17].

Theorem 6. *Differential transform of the function $w(x,y)=\partial u(x,y)/\partial y$ is*

$$W(k,h)=(h+1)\,U(k,h+1), \tag{15}$$

see [17].

Theorem 7. *Differential transform of the function $w(x,y)=\partial^{r+s}u(x,y)/\partial x^r\partial y^s$ is*

$$\begin{aligned}W(k,h)=&\ (k+1)(k+2)\cdots(k+r)(h+1)\\&\times(h+2)\cdots(h+s)\,U(k+r,h+s),\end{aligned} \tag{16}$$

see [17].

Theorem 8. *Differential transform of the function $w(x,y)=u(x,y)\cdot v(x,y)$ is*

$$W(k,h)=\sum_{r=0}^{k}\sum_{s=0}^{h}U(r,h-s)\,V(k-r,s), \tag{17}$$

see [17].

Theorem 9. *Differential transform of the function $w(x,y)=x^m y^n$ is*

$$W(k,h)=\delta(k-m,h-n)=\delta(k-m)\,\delta(h-n), \tag{18}$$

see [17], where

$$\begin{aligned}\delta(k-m)&=\begin{cases}1, & k=m\\0, & k\neq m,\end{cases}\\ \delta(h-n)&=\begin{cases}1, & h=n\\0, & h\neq n.\end{cases}\end{aligned} \tag{19}$$

Theorem 10. *Differential transform of the function $w(x,y)=g(x+a,y)$ is*

$$W(k,h)=\sum_{p=k}^{N}\binom{p}{k}a^{p-k}G(p,h). \tag{20}$$

Proof. From Definition 1, we can write

$$\begin{aligned}w(x,y)=&\sum_{h=0}^{\infty}\sum_{k=0}^{\infty}G(k,h)(x+a)^k y^h\\
=&\ G(0,0)+G(0,1)\,y+aG(1,0)+G(1,0)\,x\\
&+G(0,2)\,y^2+G(1,1)\,xy+aG(1,1)\,y\\
&+a^2G(2,0)+2aG(2,0)\,x+G(2,0)\,x^2\\
&+G(0,3)\,y^3+G(1,2)\,xy^2+aG(1,2)\,y^2\\
&+G(2,1)\,x^2y+2aG(2,1)\,xy+a^2G(2,1)\,y\\
&+a^3G(3,0)+3a^2G(3,0)\,x+3aG(3,0)\,x^2\\
&+G(3,0)\,x^3+\cdots,\\
w(x,y)=&\ \big[G(0,0)+aG(1,0)\\
&\quad+a^2G(2,0)+a^3G(3,0)+\cdots\big]\\
&+x\left[G(1,0)+2aG(2,0)+3a^2G(3,0)+\cdots\right]\\
&+y\left[G(0,1)+aG(1,1)+a^2G(2,1)+\cdots\right]\\
&+x^2\left[G(2,0)+3aG(3,0)+\cdots\right]\\
&+xy\left[G(1,1)+2aG(2,1)+\cdots\right]\\
&+y^2\left[G(0,2)+aG(1,2)+\cdots\right]+\cdots\end{aligned}$$

$$= \sum_{p=0}^{\infty} a^p G(p,0) + x \sum_{p=1}^{\infty} p a^{p-1} G(p,0) + y \sum_{p=0}^{\infty} a^p G(p,1) + x^2 \sum_{p=2}^{\infty} \frac{p!}{(p-2)!2!} \times a^{p-2} G(p,0) + xy \sum_{p=1}^{\infty} p a^{p-1} G(p,1) + \cdots, \tag{21}$$

where

$$w(x,y) = \sum_{h=0}^{\infty} \sum_{k=0}^{\infty} \sum_{p=k}^{\infty} \binom{p}{k} a^{p-k} G(p,h) x^k y^h \tag{22}$$

hence,

$$W(k,h) = \sum_{p=k}^{N} \binom{p}{k} a^{p-k} G(p,h). \tag{23}$$

□

Theorem 11. *Differential transform of the function* $w(x,y) = g(x+a, y+b)$ *is*

$$W(k,h) = \sum_{p=k}^{N} \sum_{q=h}^{N} \binom{q}{h} \binom{p}{k} a^{p-k} b^{q-h} G(p,q). \tag{24}$$

Proof. From Definition 2, we can write

$$\begin{aligned} w(x,y) &= \sum_{h=0}^{\infty} \sum_{k=0}^{\infty} G(k,h)(x+a)^k (y+b)^h \\ &= G(0,0) + G(1,0)x + aG(1,0) + G(0,1)y + bG(0,1) + G(2,0)x^2 \\ &\quad + 2aG(2,0)x + a^2 G(2,0) + G(1,1)xy + bG(1,1)x + aG(1,1)y + abG(1,1) \\ &\quad + G(0,2)y^2 + 2bG(0,2)y + b^2 G(0,2) + G(3,0)x^3 + 3aG(3,0)x^2 + 3a^2 G(3,0)x \\ &\quad + a^3 G(3,0) + G(2,1)x^2 y + \cdots \\ w(x,y) &= \left[G(0,0) + aG(1,0) + bG(0,1) + a^2 G(2,0) + abG(1,1) + \cdots\right] \\ &\quad + \left[G(1,0) + 2aG(2,0) + bG(1,1) + 3a^2 G(3,0) + 2abG(2,1) + \cdots\right] x \\ &\quad + \left[G(0,1) + aG(1,1) + 2bG(0,2) + a^2 G(2,1) + \cdots\right] y + \cdots \\ &= \sum_{p=0}^{N} \sum_{q=0}^{N} a^p b^q G(p,q) + x \sum_{p=1}^{N} \sum_{q=0}^{N} p a^{p-1} b^q G(p,q) \\ &\quad + y \sum_{p=0}^{N} \sum_{q=1}^{N} q a^p b^{q-1} G(p,q) + x^2 \sum_{p=2}^{N} \sum_{q=0}^{N} \frac{p!}{(p-2)!2!} a^{p-2} b^q G(p,q) \\ &\quad + xy \sum_{p=1}^{N} \sum_{q=1}^{N} pq a^{p-1} b^{q-1} G(p,q) + \cdots. \end{aligned} \tag{25}$$

Hence, we can write

$$w(x,y) = \sum_{h=0}^{\infty} \sum_{k=0}^{\infty} \sum_{p=k}^{N} \sum_{q=h}^{N} \binom{q}{h} \binom{p}{k} \times a^{p-k} b^{q-h} G(p,q) x^k y^h. \tag{26}$$

Using Definition 2, we obtain

$$W(k,h) = \sum_{p=k}^{N} \sum_{q=h}^{N} \binom{q}{h} \binom{p}{k} a^{p-k} b^{q-h} G(p,q). \tag{27}$$

□

Theorem 12. *Differential transform of the function* $w(x,y) = \partial^{r+s} g(x+a, y+b)/\partial x^r \partial y^s$ *is*

$$W(k,h) = \frac{(k+r)!}{k!} \frac{(h+s)!}{h!} \times \sum_{p=k+r}^{N} \sum_{q=h+s}^{N} \binom{q}{h+s} \binom{p}{k+r} \times a^{p-k-r} b^{q-h-s} G(p,q). \tag{28}$$

Proof. Let $C(k,h)$ be differential transform of the function $g(x+a, y+b)$. From Theorem 7, we can write that differential transform of the function $w(x,y)$ is

$$W(k,h) = \frac{(k+r)!}{k!} \frac{(h+s)!}{h!} C(k+r, h+s), \tag{29}$$

from Theorem 4, we can write

$$C(k+r, h+s) = \sum_{p=k+r}^{N} \sum_{q=h+s}^{N} \binom{q}{h+s} \binom{p}{k+r} \times a^{p-k-r} b^{q-h-s} G(p,q). \tag{30}$$

If we substitute (30) into (29), we find

$$W(k,h) = \frac{(k+r)!}{k!}\frac{(h+s)!}{h!} \times \sum_{p=k+r}^{N}\sum_{q=h+s}^{N}\binom{q}{h+s}\binom{p}{k+r} \times a^{p-k-r}b^{q-h-s}G(p,q). \quad (31)$$

□

4. Application

We have considered the following PDAE as a test problem:

$$\begin{pmatrix}1 & 1\\ 0 & 0\end{pmatrix}u_t + \begin{pmatrix}-1 & 0\\ -1 & 0\end{pmatrix}u_{xx} + \begin{pmatrix}1 & 1\\ -1 & 0\end{pmatrix}u = f, \quad t \in [0,\infty), \quad x \in [-1,1], \quad (32)$$

with initial values and boundary values

$$\begin{aligned} u_1(0,x) &= x^3 - x, \qquad u_2(0,x) = x^4 - 1,\\ u_1(t,1) &= u_1(t,-1) = 0, \qquad u_2(t,1) = u_2(t,-1) = 0. \end{aligned} \quad (33)$$

The right hand side function f is

$$f = \left(\left(x^4 - 1\right)(\cos t - \sin t) - 6xe^{-t}, -e^{-t}\left(x^3 + 5x\right)\right)^T, \quad (34)$$

and the exact solutions are

$$u_1(t,x) = \left(x^3 - x\right)e^{-t}, \qquad u_2(t,x) = \left(x^4 - 1\right)\cos t. \quad (35)$$

If nonsingular matrices $P_{F,k}$, $Q_{F,k}$, $P_{L,\xi}$, and $Q_{L,\xi}$ are chosen such as

$$\begin{aligned} P_{F,k} &= \begin{pmatrix}1 & \dfrac{-\mu_k}{\mu_k - 1}\\ 0 & \dfrac{1}{\mu_k - 1}\end{pmatrix}, \qquad Q_{F,k} = \begin{pmatrix}0 & -1\\ 1 & -1\end{pmatrix},\\ P_{L,\xi} &= \begin{pmatrix}0 & \dfrac{1}{\xi + 1}\\ \dfrac{1}{\xi + 1} & -\dfrac{1}{\xi + 1}\end{pmatrix}, \qquad Q_{L,\xi} = \begin{pmatrix}-\xi - 1 & 0\\ \xi & 1\end{pmatrix}, \end{aligned} \quad (36)$$

matrices $P_{F,k}AQ_{F,k}$ and $P_{L,\xi}BQ_{L,\xi}$ are found as

$$P_{F,k}AQ_{F,k} = \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}, \qquad P_{L,\xi}BQ_{L,\xi} = \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}. \quad (37)$$

From (38), we have $N_{L,\xi} = 0$ and $N_{F,k} = 0$. Then the PDAE (32) has differential spatial index 1 and differential time index 1. So, it is enough to take $\mathfrak{M}_{BC}^{(\xi)} = \{1\}$ and $\mathfrak{M}_{IC}^{(k)} = \{2\}$ to solve the problem.

Taking differential transformation of (32), we obtain

$$\begin{aligned} &(k+1)U_1(k+1,h) + (k+1)U_2(k+1,h)\\ &\quad - (h+1)(h+2)U_1(k,h+2) + U_1(k,h)\\ &\quad + U_2(k,h) = F_1(k,h), \end{aligned} \quad (38)$$

$$-(h+1)(h+2)U_1(k,h+2) - U_1(k,h) = F_2(k,h). \quad (39)$$

Table 1: The numerical and exact solution of the test problem(32), where $u_1(t,x)$ is the exact solution and $u_1^*(t,x)$ is the numerical solution, for $x = 0.1$.

t	$u_1(t,x)$	$u_1^*(t,x)$	$\lvert u_1(t,x) - u_1^*(t,x)\rvert$
0.1	−0.0895789043	−0.0895789043	0
0.2	−0.0810543445	−0.0810543422	0.0000000023
0.3	−0.0733410038	−0.0733409887	0.0000000151
0.4	−0.0663616845	−0.0663616355	0.0000000490
0.5	−0.0600465353	−0.0600464409	0.0000000944
0.6	−0.0543323519	−0.0543322800	0.0000000719
0.7	−0.0491619450	−0.0491621943	0.0000002493
0.8	−0.0444835674	−0.0444849422	0.0000013748
0.9	−0.0402503963	−0.0402546487	0.0000042524
1.0	−0.0364200646	−0.0364305555	0.0000104909

Table 2: The numerical and exact solution of the test problem(32), where $u_2(t,x)$ is exact solution and $u_2^*(t,x)$ is numerical solution, for $x = 0.1$.

t	$u_2(t,x)$	$u_2^*(t,x)$	$\lvert u_2(t,x) - u_2^*(t,x)\rvert$
0.1	−0.9949046649	−0.9949046653	0.0000000004
0.2	−0.9799685711	−0.9799685778	0.0000000067
0.3	−0.9552409555	−0.9552409875	0.0000000320
0.4	−0.9209688879	−0.9209689778	0.0000000899
0.5	−0.8774948036	−0.8774949653	0.0000001637
0.6	−0.8252530813	−0.8252532000	0.0000001187
0.7	−0.7647657031	−0.7647652653	0.0000004378
0.8	−0.6966370386	−0.6966345778	0.0000024608
0.9	−0.6215478073	−0.6215398875	0.0000079198
1.0	−0.5402482757	−0.5402277778	0.0000204979

The Taylor series of functions f_1 and f_2 about $x = 0$, $t = 0$ are

$$\begin{aligned} f_1(t,x) = &-1 + t - 6x + \frac{1}{2}t^2 + 6xt\\ &- \frac{1}{6}t^3 - 3xt^2 - \frac{1}{24}t^4 + x^4 + xt^3\\ &- x^4t + \frac{1}{120}t^5 - \frac{1}{4}xt^4 + \frac{1}{720}t^6\\ &- \frac{1}{2}x^4t^2 + \frac{1}{20}xt^5 + \frac{1}{6}x^4t^3\\ &- \frac{1}{5040}t^7 - \frac{1}{120}xt^6 + \cdots, \end{aligned} \quad (40)$$

$$\begin{aligned} f_2(t,x) = &-5x + 5xt - \frac{5}{2}xt^2 - x^3\\ &+ \frac{5}{6}xt^3 + x^3t - \frac{1}{2}x^3t^2 - \frac{5}{24}xt^4\\ &+ \frac{1}{6}x^3t^3 + \frac{1}{24}xt^5 - \frac{1}{144}xt^6\\ &- \frac{1}{24}x^3t^4 + \cdots. \end{aligned} \quad (41)$$

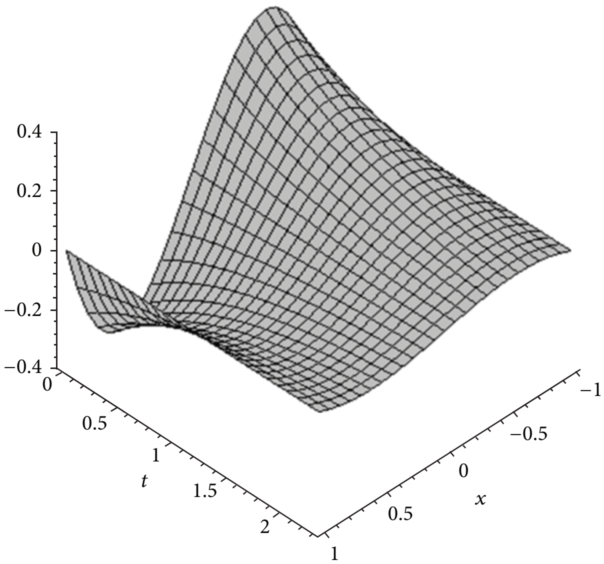

FIGURE 1: The graphic of the function $u_1(t, x)$ in the test problem (32).

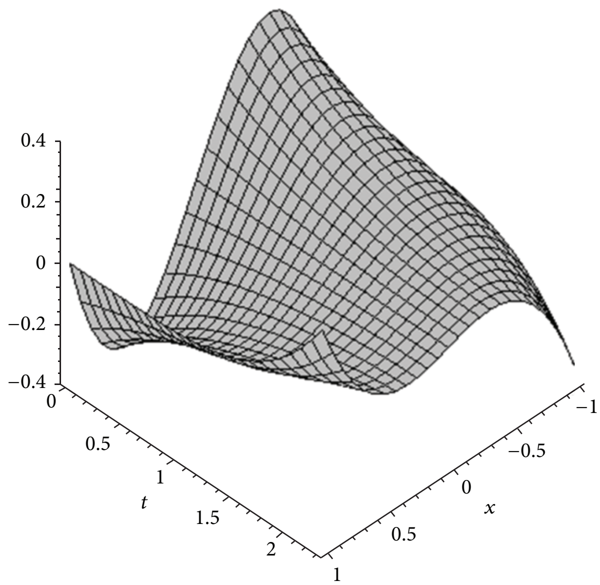

FIGURE 2: The graphic of the function $u_1^*(t, x)$ in the test problem (32).

The values $F_1(k, h)$ and $F_2(k, h)$ in (39) and (40) are coefficients of polynomials (41) and (42). If we use Theorem 3 for boundary values, we obtain

$$\sum_{i=0}^{7} U_1(j, i) = 0, \quad j = 0, 1, \ldots, 7, \tag{42}$$

$$\sum_{i=0}^{7} (-1)^i U_1(j, i) = 0, \quad j = 0, 1, \ldots, 7. \tag{43}$$

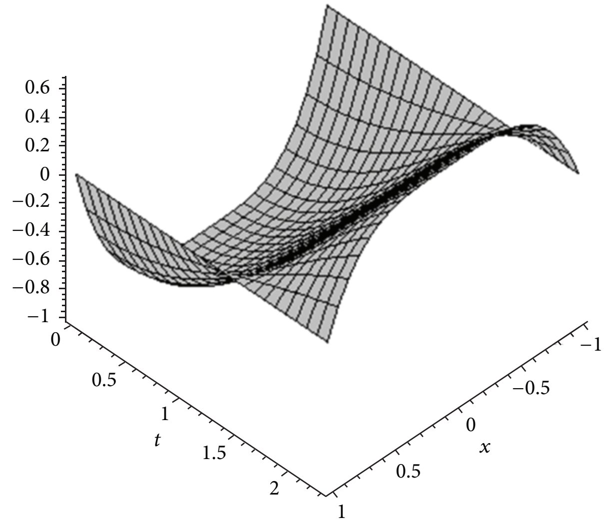

FIGURE 3: The graphic of the function $u_2(t, x)$ in the test problem (32).

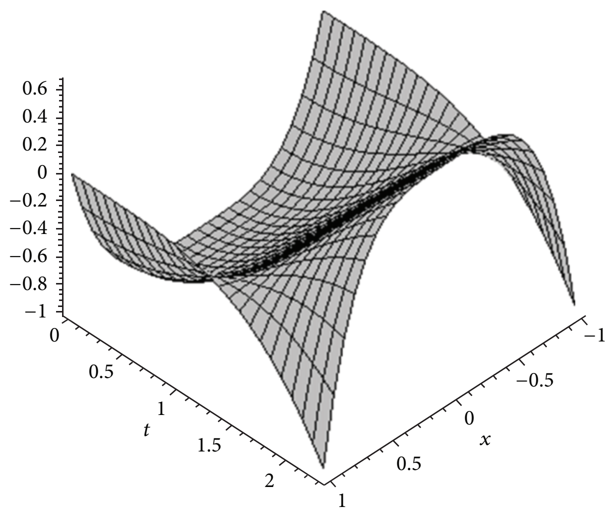

FIGURE 4: The graphic of the function $u_2^*(t, x)$ in the test problem (32).

In order to write $k = 0$ and $h = 0, 1, 2, 3, 4, 5$ in (40), we have

$$\begin{aligned} &2U_1(0,2) + U_1(0,0) = 0, && 20U_1(0,5) + U_1(0,3) = 1, \\ &6U_1(0,3) + U_1(0,1) = 5, && 30U_1(0,6) + U_1(0,4) = 0, \\ &12U_1(0,4) + U_1(0,2) = 0, && 42U_1(0,7) + U_1(0,5) = 0. \end{aligned} \tag{44}$$

If we take $j = 0$ in (43) and (44), we obtain

$$\begin{aligned} &U_1(0,0) + U_1(0,1) + U_1(0,2) + U_1(0,3) \\ &\quad + U_1(0,4) + U_1(0,5) + U_1(0,6) + U_1(0,7) = 0, \\ &U_1(0,0) - U_1(0,1) + U_1(0,2) - U_1(0,3) \\ &\quad + U_1(0,4) - U_1(0,5) + U_1(0,6) - U_1(0,7) = 0. \end{aligned} \tag{45}$$

From (45) and (46), we find

$$\begin{gathered} U_1(0,0)=0, \quad U_1(0,1)=-1, \quad U_1(0,2)=0, \\ U_1(0,3)=1, \quad U_1(0,4)=0, \quad U_1(0,5)=0, \\ U_1(0,6)=0, \quad U_1(0,7)=0. \end{gathered} \tag{46}$$

In this manner, from (40), (44), and (45), the coefficients of the u_1 are obtained as follows:

$$\begin{gathered} U_1(1,0)=0, \quad U_1(1,1)=1, \quad U_1(1,2)=0, \\ U_1(1,3)=-1, \quad U_1(1,4)=0, \quad U_1(1,5)=0, \\ U_1(1,6)=0, \quad U_1(2,0)=0, \quad U_1(2,1)=-\frac{1}{2}, \\ U_1(2,2)=0, \quad U_1(2,3)=\frac{1}{2}, \quad U_1(2,4)=0, \\ U_1(2,5)=0, \quad U_1(3,0)=0, \quad U_1(3,1)=\frac{1}{6}, \\ U_1(3,2)=0, \quad U_1(3,3)=-\frac{1}{6}, \quad U_1(3,4)=0, \\ U_1(4,0)=0, \quad U_1(4,1)=-\frac{1}{24}, \quad U_1(4,2)=0, \\ U_1(4,3)=\frac{1}{24}, \quad U_1(5,0)=0, \quad U_1(5,1)=\frac{1}{120}, \\ U_1(5,2)=0, \quad U_1(6,0)=0, \quad U_1(6,1)=-\frac{1}{720}, \\ U_1(7,0)=0. \end{gathered} \tag{47}$$

Using the initial values for the second component, we obtain the following coefficients:

$$\begin{gathered} U_2(0,0)=-1, \quad U_2(0,1)=0, \quad U_2(0,2)=0, \\ U_2(0,3)=0, \quad U_2(0,4)=1, \quad U_2(0,5)=0, \\ U_2(0,6)=0, \quad U_2(0,7)=0. \end{gathered} \tag{48}$$

The coefficients of the u_2 can be found using (47), (48), (49), and taking $k=0,1,2,\ldots$ and $h=0,1,2,\ldots$ in (39) as follows:

$$\begin{gathered} U_2(1,2)=0, \quad U_2(1,3)=0, \quad U_2(1,4)=0, \\ U_2(1,5)=0, \quad U_2(1,6)=0, \quad U_2(2,1)=0, \\ U_2(2,2)=0, \quad U_2(2,3)=0, \quad U_2(2,4)=-\frac{1}{2}, \\ U_2(2,5)=0, \quad U_2(3,0)=0, \quad U_2(3,1)=0, \\ U_2(3,2)=0, \quad U_2(3,3)=0, \quad U_2(3,4)=0, \\ U_2(4,1)=0, \quad U_2(4,2)=0, \quad U_2(4,3)=0, \\ U_2(4,0)=-\frac{1}{24}, \quad U_2(5,0)=0, \quad U_2(5,1)=0, \\ U_2(5,2)=0, \quad U_2(6,0)=\frac{1}{720}, \quad U_2(6,1)=0, \\ U_2(7,0)=0. \end{gathered} \tag{49}$$

If we write the above values in (39) and (40), then we have

$$\begin{aligned} u_1^*(t,x) = {} & -x+xt-\frac{1}{2}xt^2+x^3+\frac{1}{6}xt^3 \\ & -x^3t+\frac{1}{2}x^3t^2-\frac{1}{24}xt^4-\frac{1}{6}x^3t^3 \\ & +\frac{1}{120}xt^5-\frac{1}{720}xt^6+\frac{1}{24}x^3t^4+\cdots, \end{aligned} \tag{50}$$

$$\begin{aligned} u_2^*(t,x) = {} & -1+\frac{1}{2}t^2-\frac{1}{24}t^4 \\ & +x^4+\frac{1}{720}t^6-\frac{1}{2}x^4t^2+\cdots. \end{aligned} \tag{51}$$

Numerical and exact solution of the given problem has been compared in Tables 1 and 2, and simulations of solutions have been depicted in Figures 1, 2, 3, and 4, respectively.

5. Conclusion

The computations associated with the example discussed above were performed by using Computer Algebra Techniques [24]. We show the results in Tables 1 and 2 for the solution of (32) by numerical method. The numerical values on Tables 1 and 2 obtained above are in full agreement with the exact solutions of (32). This study has shown that the differential transform method often shows superior performance over series approximants, providing a promising tool for using in applied fields.

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