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Research Article

Approximate Controllability for Impulsive Riemann-Liouville Fractional Differential Inclusions

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We study the control systems governed by impulsive Riemann-Liouville fractional differential inclusions and their approximate controllability in Banach space. Firstly, we introduce the $PC_{1-\alpha}$-mild solutions for the impulsive Riemann-Liouville fractional differential inclusions in Banach spaces. Secondly, by using the fractional power of operators and a fixed point theorem for multivalued maps, we establish sufficient conditions for the approximate controllability for a class of Riemann-Liouville fractional impulsive differential inclusions, which is a generalization and continuation of the recent results on this issue. At the end, we give an example to illustrate the application of the abstract results.

1. Introduction

The concept of controllability plays an important part in the analysis and design of control systems. Since Kalman [1] first introduced its definition in 1963, controllability of the deterministic and stochastic dynamical control systems in finite-dimensional and infinite-dimensional spaces is well developed in different classes of approaches, and more details can be found in papers [2–4]. Some authors [5–7] have studied the exact controllability for nonlinear evolution systems by using the fixed point theorems. In [5–7], to prove the controllability results for fractional-order semilinear systems, the authors made an assumption that the semigroup associated with the linear part is compact. But if C_0-semigroup $T(t)$ is compact or the operator B is compact, then the controllability operator is also compact and hence the inverse of it does not exist if the state space V is infinite dimensional [8]. Thus, it is shown that the concept of exact controllability is difficult to be satisfied in infinite-dimensional space. Therefore, it is important to study the weaker concept of controllability, namely, approximate controllability for differential equations. In these years, several researchers [9–17] have studied it for control systems.

In [13], Sakthivel et al. studied on the approximate controllability of semilinear fractional differential systems:

$$\begin{gathered} {}^{c}D^{\alpha}x(t) = Ax(t) + Bu(t) + f(t, x(t)), \quad t \in J = [0, T], \\ x(0) = x_0, \end{gathered} \tag{1}$$

where ${}^{c}D_t^{\alpha}$ is Caputo's fractional derivative of $0 < \alpha < 1$ and A is the infinitesimal generator of a C_0-semigroup $T(t)$ of bounded operators on the Hilbert space X; the control function $u(\cdot)$ is given in $L^2(J, U)$; U is a Hilbert space; B is a bounded linear operator from U to X; $f : J \times X \to X$ is a given function satisfying some assumptions and x_0 is an element of the Hilbert space X.

In [16], Sukavanam and Kumar researched approximate controllability of fractional-order semilinear delay systems:

$$\begin{gathered} \frac{d^{\alpha}x(t)}{t^{\alpha}} = Ax(t) + Bu(t) + f(t, x_t, u(t)), \quad t \in [0, \tau], \\ x_0(\theta) = \phi(\theta), \quad \theta \in [-h, 0], \end{gathered} \tag{2}$$

where $1/2 < \alpha < 1$; $A : D(A) \subseteq V \to V$ is a closed linear operator with dense domain $D(A)$ generating a C_0-semigroup $S(t)$; the state $x(\cdot)$ takes values in the Banach space V; the control function $u(\cdot)$ takes values in $\widehat{V}$; B is a bounded linear operator from $L^2([0,\tau];\widehat{V})$ to $L^2([0,\tau];V)$; the operator $f:[0,\tau]\times C([-h,0];V)\times\widehat{V}\to V$ is nonlinear. If $x:[-h,\tau]\to V$ is a continuous function, then $x_t:[-h,0]\to V$ is defined as $x_t(\theta)=x(t+\theta)$ for $\theta\in[-h,0]$ and $\phi\in C([-h,0];V)$.

In [18], Rykaczewski studied the approximate controllability of an inclusion of the form

$$\dot{x}(t)\in Ax(t)+F(t,x(t))+Bu(t),\quad t\in J=[0,b],$$
$$x(0)=x_0, \tag{3}$$

where A is a linear operator which generates a compact semigroup, F is u.h.c. multivalued perturbation with weakly compact values, and the state $x(\cdot)$ takes values in the Hilbert space H. U is a Hilbert space of all admissible controls. $B:U\to H$ is a continuous linear operator.

Fractional differential equations have recently proved to be valuable tools in the modeling of many phenomena in various fields of engineering, physics, and economics. Indeed, we can find numerous applications in viscoelasticity, electrochemistry, control, porous media, electromagnetic, and so forth; see [19–27] for example. As a consequence there was an intensive development of the theory of differential equations of fractional order. One can see the monographs of Kilbas et al. [28] and Podlubny [29] and the references therein. The definitions of Riemann-Liouville fractional derivatives or integrals with initial conditions are strong tools to resolve some fractional differential problems in the real world. Heymans and Podlubny [30] have verified that it was possible to attribute physical meaning to initial conditions expressed in terms of Riemann-Liouville fractional derivatives or integrals, and such initial conditions are more appropriate than physically interpretable initial conditions. Furthermore, they have investigated that the impulse response with Riemann-Liouville fractional derivatives was seldom used in the fields of physics, such as viscoelasticity. In recent years, many authors [18, 27, 31] were devoted to mild solutions to fractional evolution equations with Caputo fractional derivative, and there have been a lot of interesting works. As for the study of the fractional differential systems with Caputo fractional derivative, we can refer to [27, 31, 32] for the existence results. Its approximate controllability was considered in [9, 13–16]. The approximate controllability of Caputo fractional inclusion systems has been investigated by [10]. We know that differential inclusions are strong tools to solve some problems in various fields of engineering, physics, and optimal control; see [10, 32–35]. However, the approximate controllability for the impulsive fractional differential evolution inclusion with Riemann-Liouville fractional derivatives is still an untreated topic in the literature.

Motivated by the above work, in this paper, we consider the following system:

$$D_t^{\alpha}x(t)\in Ax(t)+F(t,x(t))+Bu(t),$$
$$t\in J'=(0,b]\setminus\{t_1,t_2,\ldots,t_m\},$$
$$\Delta I_{0^+}^{1-\alpha}x(t)\big|_{t=t_k}=I_k\left(x\left(t_k^-\right)\right),\quad k=1,2,\ldots,m,$$
$$I_t^{1-\alpha}x(t)\big|_{t=0}=x_0\in X, \tag{4}$$

where $1/2<\alpha\le 1$ and D_t^{α} denotes the Riemann-Liouville fractional derivative of order α with the lower limit zero. $F:J\times X\to\mathscr{P}(X):=2^X\setminus\{\emptyset\}$ is a nonempty, bounded, closed, and convex multivalued map. $A:D(A)\subseteq X\to X$ is the infinitesimal generator of a C_0-semigroup $T(t)$ $(t\ge 0)$ on a Banach space X. $0=t_0<t_1<\cdots<t_m<t_{m+1}=b$, $I_k:X\mapsto X$, $\Delta I_{0^+}^{1-\alpha}x(t_k)=I_{0^+}^{1-\alpha}x(t_k^+)-I_{0^+}^{1-\alpha}x(t_k^-)$, $I_{0^+}^{1-\alpha}x(t_k^+)$ and $I_{0^+}^{1-\alpha}x(t_k^-)$ denote the right and the left limits of $I_{0^+}^{1-\alpha}x(t)$ at $t=t_k$, $k=1,2,\ldots,m$. The control function $u(t)$ takes value in $V=L^p([0,b];U)$, $p>1/\alpha$, and U is a Banach space; B is a linear operator from V to $L^p([0,b];X)$.

The purpose of this paper is to provide some suitable sufficient conditions for the existence of mild solutions and approximate controllability results for the impulsive fractional abstract Cauchy problems with Riemann-Liouville fractional derivatives. The main tools used in our study are fixed point theorem, semigroup theory for multivalued maps, and the theory from fractional differential equations. The rest of this paper is organized as follows. In Section 2, we present some preliminaries to prove our main results. In Section 3, by applying some standard fixed point principles, we prove the existence of the mild solutions for semilinear fractional differential equations, and the approximate controllability of the system (4) is proved. In Section 4, we give an example to illustrate our main results.

2. Preliminaries

In this section, we introduce some basic definitions and preliminaries which are used throughout this paper. The norm of a Banach space X will be denoted by $\|\cdot\|_X$. $L_b(X,Y)$ denotes the space of bounded linear operators from X to Y. For the uniformly bounded C_0-semigroup $T(t)$ $(t\ge 0)$, we set $M:=\sup_{t\in[0,\infty)}\|T(t)\|_{L_b(X)}<\infty$. Let $C(J,X)$ denote the Banach space of all X value continuous functions from $J=[0,b]$ to X with the norm $\|x\|_C=\sup_{t\in J}\|x(t)\|_X$. Let $C_{1-\alpha}(J,X)=\{x:t^{1-\alpha}x(t)\in C(J,X)\}$ with the norm

$$\|x\|_{C_{1-\alpha}}=\sup\left\{t^{1-\alpha}\|x(t)\|_X:t\in J\right\}. \tag{5}$$

Obviously, the space $C_{1-\alpha}(J,X)$ is a Banach space.

To define the mild solutions of (4), we also consider the Banach space $PC_{1-\alpha}(J,X)=\{x:(t-t_k)^{1-\alpha}x(t)\in C((t_k,t_{k+1}],X),\ \lim_{t\to t_k^+}(t-t_k)^{1-\alpha}x(t)$ exist, and $x(t_k^-)$ with $x(t_k^-)=x(t_k)$, $k=0,1,2,\ldots,m\}$ with the norm

$$\|x\|_{PC_{1-\alpha}}=\max\left\{\sup\left(t-t_k\right)^{1-\alpha}\|x(t)\|_X:t\in\left(t_k,t_{k+1}\right],\right.$$
$$\left.k=0,1,2,\ldots,m\right\}. \tag{6}$$

It is easily known that the space $PC_{1-\alpha}(J,X)$ is a Banach space.

Let (X, d) be a metric space. We use the notations

$$\begin{aligned} P_{cl}(X) &= \{Y \in \mathscr{P}(X) : Y \text{ is closed}\}, \\ P_{b}(X) &= \{Y \in \mathscr{P}(X) : Y \text{ is bounded}\}, \\ P_{cv}(X) &= \{Y \in \mathscr{P}(X) : Y \text{ is convex}\}, \\ P_{cp}(X) &= \{Y \in \mathscr{P}(X) : Y \text{ is compact}\}. \end{aligned} \tag{7}$$

Firstly, let us recall the following basic definitions from fractional calculus. For more details, one can see [28, 29].

Definition 1. The integral

$$I_t^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)\, ds, \quad \alpha > 0, \tag{8}$$

is called Riemann-Liouville fractional integral of order α, where Γ is the gamma function.

Definition 2. For a function $f(t)$ given in the interval $[0, \infty)$, the expression

$$D_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_0^t (t-s)^{n-\alpha-1} f(s)\, dt, \tag{9}$$

where $n = [\alpha] + 1$ and $[\alpha]$ denotes the integer part of number α, is called the Riemann-Liouville fractional derivative of order α.

Lemma 3 (see [28]). *Let $\alpha > 0$, $m = [\alpha] + 1$, and let $x_{m-\alpha}(t) = I_t^{m-\alpha} x(t)$ be the fractional integral of order $m - \alpha$. If $x(t) \in L^1(J, X)$ and $x_{m-\alpha}(t) \in AC^m(J, X)$, then one has the following equality:*

$$I_t^{\alpha} D_t^{\alpha} x(t) = x(t) - \sum_{k=1}^{m} \frac{x_{m-\alpha}^{(m-k)}(0)}{\Gamma(\alpha - k + 1)} t^{\alpha-k}. \tag{10}$$

In order to study the PC-mild solutions of (4) in Banach space $PC_{1-\alpha}(J, X)$, we give the following results which will be used throughout this paper.

Lemma 4. *Let $0 < \alpha \leq 1$, and let $x_{1-\alpha}(t) = I_t^{1-\alpha} x(t)$ be the fractional integral of order $1 - \alpha$. If $x(t) \in PC_{1-\alpha}(J, X)$ and $x_{1-\alpha}(t) \in PC(J, X)$, then one has the following equality:*

$$I_t^{\alpha} D_t^{\alpha} x(t) = \begin{cases} x(t) - x_{1-\alpha}(t)\big|_{t=0} \dfrac{t^{\alpha-1}}{\Gamma(\alpha)}, & t \in [0, t_1], \\ x(t) - \displaystyle\sum_{k=1}^{m} \frac{\Delta x_{1-\alpha}(t_i)}{\Gamma(\alpha)} (t - t_i)^{\alpha-1} \\ \quad - x_{1-\alpha}(t)\big|_{t=0} \dfrac{t^{\alpha-1}}{\Gamma(\alpha)}, & t \in (t_k, t_{k+1}], \end{cases} \tag{11}$$

where $\Delta x_{1-\alpha}(t_k) = x_{1-\alpha}(t_k^+) - x_{1-\alpha}(t_k^-)$, $k = 1, 2, \ldots, m$.

Proof. If $t \in [0, t_1]$, then, by Lemma 3, we easily get

$$I_{0^+}^{\alpha} D_{0^+}^{\alpha} x(t) = x(t) - x_{1-\alpha}(t)\big|_{t=0} \frac{t^{\alpha-1}}{\Gamma(\alpha)}. \tag{12}$$

If $t \in (t_1, t_2]$, since

$$\begin{aligned} I_t^{\alpha} D_t^{\alpha} x(t) &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}\, {}^{L}D_t^{\alpha} x(s)\, ds \\ &= \frac{d}{dt} \left\{ \frac{1}{\Gamma(\alpha+1)} \int_0^t (t-s)^{\alpha}\, {}^{L}D_t^{\alpha} x(s)\, ds \right\}, \end{aligned} \tag{13}$$

then, by (13), we have

$$\begin{aligned} &\frac{1}{\Gamma(\alpha+1)} \int_0^t (t-s)^{\alpha}\, {}^{L}D_t^{\alpha} x(s)\, ds \\ &\quad = \frac{1}{\Gamma(\alpha+1)} \int_0^t (t-s)^{\alpha} \frac{d}{ds} \left\{ I_{0^+}^{1-\alpha} x(s) \right\} ds \\ &\quad = \frac{1}{\Gamma(\alpha+1)\Gamma(1-\alpha)} \\ &\qquad \times \int_0^t (t-s)^{\alpha} \frac{d}{ds} \int_0^s (s-\tau)^{1-\alpha-1} x(\tau)\, d\tau\, ds \\ &\quad = \frac{1}{\Gamma(\alpha+1)\Gamma(1-\alpha)} \\ &\qquad \times \int_0^{t_1} (t-s)^{\alpha} \frac{d}{ds} \int_0^s (s-\tau)^{1-\alpha-1} x(\tau)\, d\tau\, ds \\ &\qquad + \frac{1}{\Gamma(\alpha+1)\Gamma(1-\alpha)} \\ &\qquad \times \int_{t_1}^t (t-s)^{\alpha} \frac{d}{ds} \int_0^{t_1} (s-\tau)^{1-\alpha-1} x(\tau)\, d\tau\, ds \\ &\qquad + \frac{1}{\Gamma(\alpha+1)\Gamma(1-\alpha)} \\ &\qquad \times \int_{t_1}^t (t-s)^{\alpha} \frac{d}{ds} \int_{t_1}^s (s-\tau)^{1-\alpha-1} x(\tau)\, d\tau\, ds \\ &\quad = \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)} \\ &\qquad \times \int_0^{t_1} (t-s)^{\alpha-1} \int_0^s (s-\tau)^{1-\alpha-1} x(\tau)\, d\tau\, ds \\ &\qquad + \frac{1}{\Gamma(\alpha+1)\Gamma(1-\alpha)} \\ &\qquad \times \left[(t-s)^{\alpha} \int_0^s (s-\tau)^{1-\alpha-1} x(\tau)\, d\tau \right] \Bigg|_{s=0}^{s=t_1} \\ &\qquad + \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)} \\ &\qquad \times \int_{t_1}^t (t-s)^{\alpha-1} \int_0^{t_1} (s-\tau)^{1-\alpha-1} x(\tau)\, d\tau\, ds \\ &\qquad + \frac{1}{\Gamma(\alpha+1)\Gamma(1-\alpha)} \end{aligned}$$

$$
\begin{aligned}
&\times \left[(t-s)^{\alpha} \int_0^{t_1} (s-\tau)^{1-\alpha-1} x(\tau)\, d\tau \right] \Bigg|_{s=t_1}^{s=t} \\
&+ \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)} \\
&\times \int_{t_1}^{t} (t-s)^{\alpha-1} \int_{t_1}^{s} (s-\tau)^{1-\alpha-1} x(\tau)\, d\tau\, ds \\
&+ \frac{1}{\Gamma(\alpha+1)\Gamma(1-\alpha)} \\
&\times \left[(t-s)^{\alpha} \int_{t_1}^{s} (s-\tau)^{1-\alpha-1} x(\tau)\, d\tau \right] \Bigg|_{s=t_1}^{s=t} \\
= {} & \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)} \\
&\times \int_0^{t_1} x(\tau)\, d\tau \int_{\tau}^{t_1} (t-s)^{\alpha-1} (s-\tau)^{1-\alpha-1} ds \\
&+ \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)} \\
&\times \int_0^{t_1} x(\tau)\, d\tau \int_{t_1}^{t} (t-s)^{\alpha-1} (s-\tau)^{1-\alpha-1} ds \\
&+ \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)} \\
&\times \int_{t_1}^{t} x(\tau)\, d\tau \int_{\tau}^{t} (t-s)^{\alpha-1} (s-\tau)^{1-\alpha-1} ds \\
&- \frac{x_{1-\alpha}(0)}{\Gamma(\alpha+1)} t^{\alpha} - \frac{\Delta x_{1-\alpha}(t_1)}{\Gamma(\alpha+1)} (t-t_1)^{\alpha} \\
= {} & \int_0^{t} x(\tau)\, d\tau - \frac{x_{1-\alpha}(0)}{\Gamma(\alpha+1)} t^{\alpha} - \frac{\Delta x_{1-\alpha}(t_1)}{\Gamma(\alpha+1)} (t-t_1)^{\alpha}.
\end{aligned}
\tag{14}
$$

Thus, by (13) and (14), we get

$$
I_t^{\alpha} D_t^{\alpha} x(t) = x(t) - \frac{x_{1-\alpha}(0)}{\Gamma(\alpha)} t^{\alpha-1} - \frac{\Delta x_{1-\alpha}(t_1)}{\Gamma(\alpha)} (t-t_1)^{\alpha-1}. \tag{15}
$$

Similarly, if $t \in (t_k, t_{k+1}]$, $k = 2, \ldots, m$, we can get

$$
\begin{aligned}
&I_t^{\alpha} D_t^{\alpha} x(t) \\
&= x(t) - \sum_{k=1}^{m} \frac{\Delta x_{1-\alpha}(t_k)}{\Gamma(\alpha)} (t-t_k)^{\alpha-1} - x_{1-\alpha}(t)\big|_{t=0} \frac{t^{\alpha-1}}{\Gamma(\alpha)}.
\end{aligned}
\tag{16}
$$

The proof is completed. □

The Laplace transform formula for the Riemann-Liouville fractional integral is defined by

$$
L\{I_t^{\alpha} x(t); \lambda\} = \frac{1}{\lambda^{\alpha}} \widehat{x}(\lambda), \tag{17}
$$

where $\widehat{x}(\lambda)$ is the Laplace of x defined by

$$
\begin{gathered}
\widehat{x}(\lambda) = \int_0^{\infty} e^{-\lambda t} x(t)\, dt, \\
Re\lambda > \omega, \qquad |x(t)| \leq c e^{\omega t}, \quad c \text{ is a constant.}
\end{gathered}
\tag{18}
$$

Lemma 5. *Let $\alpha \in (0,1]$ and $h \in L^p(J, X)$, $p > 1/\alpha$; if $x(t) \in PC_{1-\alpha}(J, X)$, $x_{1-\alpha}(t) \in PC(J, X)$, and x is a solution of the following problem:*

$$
\begin{gathered}
D_t^{\alpha} x(t) = Ax(t) + h(t), \\
t \in (0, b], \quad t \neq t_k, \quad k = 1, 2, \ldots, m, \\
\Delta I_t^{1-\alpha} x(t)\big|_{t=t_k} = I_k(x(t_k^-)), \quad k = 1, 2, \ldots, m, \\
I_t^{1-\alpha} x(t)\big|_{t=0} = x_0 \in X,
\end{gathered}
\tag{19}
$$

then, x satisfies the following equation:

$$
\begin{aligned}
&x(t) \\
&= \begin{cases}
t^{\alpha-1} T_{\alpha}(t) x_0 + \displaystyle\int_0^{t} (t-s)^{\alpha-1} T_{\alpha}(t-s) h(s)\, ds, \\
\qquad\qquad t \in [0, t_1], \\
t^{\alpha-1} T_{\alpha}(t) x_0 + \displaystyle\sum_{k=1}^{m} T_{\alpha}(t-t_k)(t-t_k)^{\alpha-1} \Delta I_{0^+}^{1-\alpha} x(t_k) \\
\quad + \displaystyle\int_0^{t} (t-s)^{\alpha-1} T_{\alpha}(t-s) h(s)\, ds, \\
\qquad\qquad t \in (t_k, t_{k+1}],
\end{cases}
\end{aligned}
\tag{20}
$$

where $k = 1, 2, \ldots, m$,

$$
\begin{gathered}
T_{\alpha}(t) = \alpha \int_0^{\infty} \theta \xi_{\alpha}(\theta) T(t^{\alpha}\theta)\, d\theta, \\
\xi_{\alpha}(\theta) = \frac{1}{\alpha} \theta^{-1-(1/\alpha)} \varpi_{\alpha}(\theta^{-1/\alpha}), \\
\varpi_{\alpha}(\theta) = \frac{1}{\pi} \sum_{n=1}^{\infty} (-1)^{n-1} \theta^{-n\alpha-1} \frac{\Gamma(n\alpha+1)}{n!} \sin(n\pi\alpha), \\
\theta \in (0, \infty),
\end{gathered}
\tag{21}
$$

where ξ_{α} is a probability density function defined on $(0, \infty)$; that is,

$$
\xi_{\alpha}(\theta) \geq 0, \quad \theta \in (0, \infty), \qquad \int_0^{\infty} \xi_{\alpha}(\theta)\, d\theta = 1. \tag{22}
$$

Proof. We observe that $x(\cdot)$ can be decomposed to $p(\cdot) + q(\cdot)$, where p is the continuous mild solution for

$$
\begin{gathered}
D_t^{\alpha} p(t) = Ap(t) + h(t), \quad t \in (0, b], \\
I_t^{1-\alpha} p(t)\big|_{t=0} = x_0 \in X,
\end{gathered}
\tag{23}
$$

and q is the $PC_{1-\alpha}$-mild solution for

$$\begin{gathered} D_t^{\alpha} q(t) = Aq(t), \\ t \in (0,b], \quad t \neq t_k, \quad k = 1,2,\ldots,m, \\ \Delta I_t^{1-\alpha} q(t)\big|_{t=t_k} = y_k, \quad k = 1,2,\ldots,m, \\ I_t^{1-\alpha} q(t)\big|_{t=0} = 0 \in X. \end{gathered} \tag{24}$$

Indeed, by adding together (23) and (24), it follows by (19). Since p is continuous, then $p(t_k^+) = p(t_k^-)$, $k = 1,2,\ldots,m$. On the other hand, any solution of (19) can be decomposed to (23) and (24). So we show the results by the following.

At first, we calculate the mild solution of (23).

Apply Riemann-Liouville fractional integral operator on both sides of (23); then, by Lemma 3, we get

$$\begin{aligned} p(t) &= \frac{I_t^{1-\alpha} p(t)\big|_{t=0}}{\Gamma(\alpha)} t^{\alpha-1} + I_t^{\alpha} Ap(t) + I_t^{\alpha} h(t) \\ &= \frac{t^{\alpha-1}}{\Gamma(\alpha)} x_0 + I_t^{\alpha} Ap(t) + I_t^{\alpha} h(t). \end{aligned} \tag{25}$$

That is,

$$p(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)} x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left[Ap(s) + h(s)\right] ds. \tag{26}$$

Let $\lambda > 0$; taking the Laplace transformations

$$\hat{p}(\lambda) = \int_0^{\infty} e^{-\lambda t} p(t)\, dt, \qquad \hat{h}(\lambda) = \int_0^{\infty} e^{-\lambda t} h(t)\, dt \tag{27}$$

to (26), we obtain

$$\begin{aligned} \hat{p}(\lambda) &= \frac{1}{\lambda^{\alpha}} x_0 + \frac{1}{\lambda^{\alpha}} A\hat{p}(\lambda) + \frac{1}{\lambda^{\alpha}} \hat{h}(\lambda) \\ &= \left(\lambda^{\alpha} I - A\right)^{-1} x_0 + \left(\lambda^{\alpha} I - A\right)^{-1} \hat{h}(\lambda) \\ &= \int_0^{\infty} e^{-\lambda^{\alpha} t} T(t) x_0 dt + \int_0^{\infty} e^{-\lambda^{\alpha} t} T(t) \hat{h}(\lambda)\, dt. \end{aligned} \tag{28}$$

Consider the one-sided stable probability density

$$\varpi_{\alpha}(\theta) = \frac{1}{\pi} \sum_{n=1}^{\infty} (-1)^{n-1} \theta^{-n\alpha-1} \frac{\Gamma(n\alpha+1)}{n!} \sin(n\pi\alpha), \quad \theta \in (0,\infty), \tag{29}$$

whose Laplace transformation is given by

$$\int_0^{\infty} e^{-\lambda\theta} \varpi_{\alpha}(\theta)\, d\theta = e^{-\lambda^{\alpha}}, \quad \alpha \in (0,1). \tag{30}$$

Hence, it follows from (28) and (30) that

$$\begin{aligned} &\int_0^{\infty} e^{-\lambda^{\alpha} t} T(t) x_0 dt \\ &= \int_0^{\infty} e^{-(\lambda s)^{\alpha}} T(s^{\alpha}) x_0 ds^{\alpha} \quad (t = s^{\alpha}) \\ &= \alpha \int_0^{\infty} e^{-(\lambda s)^{\alpha}} T(s^{\alpha}) s^{\alpha-1} x_0 ds \\ &= \alpha \iint_0^{\infty} \varpi_{\alpha}(\theta) e^{-\lambda s\theta} T(s^{\alpha}) s^{\alpha-1} x_0 d\theta\, ds \\ &= \alpha \iint_0^{\infty} \varpi_{\alpha}(\theta) e^{-\lambda u} T\left(\frac{u^{\alpha}}{\theta^{\alpha}}\right) \frac{s^{\alpha-1}}{\theta^{\alpha}} x_0 du\, d\theta \quad (u = \theta s) \\ &= \int_0^{\infty} e^{-\lambda u} \left[\alpha \int_0^{\infty} \varpi_{\alpha}(\theta) T\left(\frac{u^{\alpha}}{\theta^{\alpha}}\right) \frac{s^{\alpha-1}}{\theta^{\alpha}} x_0 d\theta\right] du, \\ &\int_0^{\infty} e^{-\lambda^{\alpha} t} T(t) \hat{h}(\lambda)\, dt \\ &= \int_0^{\infty} e^{-\lambda^{\alpha} t} T(t) \left[\int_0^{\infty} e^{-\lambda s} h(s)\, ds\right] dt \\ &= \iint_0^{\infty} e^{-\lambda^{\alpha} t} T(t) e^{-\lambda s} h(s)\, ds\, dt \\ &= \iint_0^{\infty} e^{-(\lambda\mu)^{\alpha}} T(\mu^{\alpha}) q\mu^{\alpha-1} e^{-\lambda s} h(s)\, ds\, d\mu \quad (t = \mu^{\alpha}) \\ &= \iiint_0^{\infty} \alpha\varpi_{\alpha}(\theta) e^{-\lambda\mu\theta} T(\mu^{\alpha}) \mu^{\alpha-1} e^{-\lambda s} h(s)\, d\theta\, ds\, d\mu \\ &= \iiint_0^{\infty} \alpha\varpi_{\alpha}(\theta) e^{-\lambda\nu} T\left(\frac{\nu^{\alpha}}{\theta^{\alpha}}\right) \\ &\qquad \times \frac{\nu^{\alpha-1}}{\theta^{\alpha}} e^{-\lambda s} h(s)\, d\nu\, d\theta\, ds \quad (\nu = \mu\theta) \\ &= \iiint_0^{\infty} \alpha\varpi_{\alpha}(\theta) e^{-\lambda(\nu+s)} T\left(\frac{\nu^{\alpha}}{\theta^{\alpha}}\right) \frac{\nu^{\alpha-1}}{\theta^{\alpha}} h(s)\, d\nu\, d\theta\, ds \\ &= \iint_0^{\infty} \int_s^{\infty} \alpha\varpi_{\alpha}(\theta) e^{-\lambda\tau} T\left(\frac{(\tau-s)^{\alpha}}{\theta^{\alpha}}\right) \\ &\qquad \times \frac{(\tau-s)^{\alpha-1}}{\theta^{\alpha}} h(s)\, d\tau\, d\theta\, ds \quad (\tau = \nu + s) \\ &= \int_0^{\infty} e^{-\lambda\tau} \left[\alpha \int_0^{\tau} \int_0^{\infty} \varpi_{\alpha}(\theta) T\left(\frac{(\tau-s)^{\alpha}}{\theta^{\alpha}}\right) \right. \\ &\qquad \left. \times \frac{(\tau-s)^{\alpha-1}}{\theta^{\alpha}} h(s)\, d\theta\, ds\right] d\tau. \end{aligned} \tag{31}$$

According to the above work, we get

$$\widehat{p}(\lambda) = \int_0^\infty e^{-\lambda t}\left[\alpha \int_0^\infty \varpi_\alpha(\theta) T\left(\frac{t^\alpha}{\theta^\alpha}\right)\frac{t^{\alpha-1}}{\theta^\alpha}x_0 d\theta + \alpha\int_0^t\int_0^\infty \varpi_\alpha(\theta) T\left(\frac{(t-s)^\alpha}{\theta^\alpha}\right) \times \frac{(t-s)^{\alpha-1}}{\theta^\alpha}h(s)\,d\theta\,ds\right]dt. \tag{32}$$

Now, we can invert the Laplace transform to (20) and obtain

$$\begin{aligned} p(t) &= \alpha\int_0^\infty \varpi_\alpha(\theta) T\left(\frac{t^\alpha}{\theta^\alpha}\right)\frac{t^{\alpha-1}}{\theta^\alpha}x_0 d\theta \\ &\quad + \alpha\int_0^t\int_0^\infty \varpi_\alpha(\theta) T\left(\frac{(t-s)^\alpha}{\theta^\alpha}\right)\frac{(t-s)^{\alpha-1}}{\theta^\alpha}h(s)\,d\theta\,dt \\ &= \alpha\int_0^\infty \frac{1}{\alpha}\theta^{-1-(1/\alpha)}\varpi_\alpha\left(\theta^{-1/\alpha}\right)\theta T(t^\alpha\theta)\,t^{\alpha-1}x_0 d\theta \\ &\quad + \alpha\int_0^t\int_0^\infty (t-s)^{\alpha-1}\frac{1}{\alpha}\theta^{-1-(1/\alpha)}\varpi_\alpha\left(\theta^{-1/\alpha}\right)\theta T \\ &\quad \times ((t-s)^\alpha\theta)\,h(s)\,d\theta\,dt. \end{aligned} \tag{33}$$

Let

$$\begin{aligned} \xi_\alpha(\theta) &= \frac{1}{\alpha}\theta^{-1-(1/\alpha)}\varpi_\alpha\left(\theta^{-1/\alpha}\right), \\ T_\alpha(t) &= \alpha\int_0^\infty \theta\xi_\alpha(\theta)\,T(t^\alpha\theta)\,d\theta. \end{aligned} \tag{34}$$

Then, we get

$$p(t) = t^{\alpha-1}T_\alpha(t)\,x_0 + \int_0^t (t-s)^{\alpha-1}T_\alpha(t-s)\,h(s)\,ds. \tag{35}$$

Now we calculate the $PC_{1-\alpha}$-mild solution of (24).

Applying Riemann-Liouville fractional integral operator on both sides of (24), then by Lemma 4, we get

$$q(t) = \begin{cases} \dfrac{1}{\Gamma(\alpha)}\displaystyle\int_0^t (t-s)^{\alpha-1}Aq(s)\,ds, & t\in[0,t_1], \\ \displaystyle\sum_{k=1}^m \frac{I_k(x(t_k^-))}{\Gamma(\alpha)}(t-t_i)^{\alpha-1} + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}Aq(s)\,ds, & t\in(t_k,t_{k+1}]. \end{cases} \tag{36}$$

The above equation (36) can be rewritten as

$$q(t) = \sum_{k=1}^m \frac{I_k(x(t_k^-))}{\Gamma(\alpha)}(t-t_k)^{\alpha-1}\chi_k(t) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}Aq(s)\,ds, \tag{37}$$

where

$$\chi_k(t) = \begin{cases} 0, & t\le t_k, \\ 1, & t>t_k. \end{cases} \tag{38}$$

Let $\lambda > 0$; taking the Laplace transformation to (37), we obtain

$$\widehat{q}(\lambda) = \sum_{k=1}^m \frac{I_k(x(t_k^-))\,e^{-\lambda t_k}}{\lambda^\alpha} + \frac{1}{\lambda^\alpha}A\widehat{q}(\lambda). \tag{39}$$

That is,

$$\widehat{q}(\lambda) = \sum_{k=1}^m (\lambda^\alpha I - A)^{-1}I_k(x(t_k^-))\,e^{-\lambda t_k}. \tag{40}$$

Notice that the Laplace transform of $t^{\alpha-1}T_\alpha(t)y_k$ is $(\lambda^\alpha I - A)^{-1}y_k$. Thus one can calculate the mild solution of (24) as

$$q(t) = \sum_{k=1}^m \chi_k(t)\,(t-t_i)^{\alpha-1}T_\alpha(t-t_i)\,I_k(x(t_k^-)). \tag{41}$$

By the above work, the $PC_{1-\alpha}$-mild solution of (19) is given by

$$\begin{aligned} x(t) &= t^{\alpha-1}T_\alpha(t)\,x_0 \\ &\quad + \sum_{k=1}^m \chi_k(t)\,(t-t_k)^{\alpha-1}T_\alpha(t-t_k)\,I_k(x(t_k^-)) \\ &\quad + \int_0^t (t-s)^{\alpha-1}T_\alpha(t-s)\,h(s)\,ds. \end{aligned} \tag{42}$$

That is,

$$x(t) = \begin{cases} t^{\alpha-1}T_\alpha(t)\,x_0 + \displaystyle\int_0^t (t-s)^{\alpha-1}T_\alpha(t-s)\,h(s)\,ds, & t\in[0,t_1], \\ t^{\alpha-1}T_\alpha(t)\,x_0 + \displaystyle\sum_{k=1}^m T_\alpha(t-t_k)\,(t-t_k)^{\alpha-1}I_k(x(t_k^-)) + \int_0^t (t-s)^{\alpha-1}T_\alpha(t-s)\,h(s)\,ds, & t\in(t_k,t_{k+1}], \end{cases} \tag{43}$$

where $k = 1, 2, \ldots, m$,

$$\begin{aligned} T_\alpha(t) &= \alpha\int_0^\infty \theta\xi_\alpha(\theta)\,T(t^\alpha\theta)\,d\theta, \\ \xi_\alpha(\theta) &= \frac{1}{\alpha}\theta^{-1-(1/\alpha)}\varpi_\alpha\left(\theta^{-1/\alpha}\right), \\ \varpi_\alpha(\theta) &= \frac{1}{\pi}\sum_{n=1}^\infty (-1)^{n-1}\theta^{-n\alpha-1}\frac{\Gamma(n\alpha+1)}{n!}\sin(n\pi\alpha), \\ &\qquad \theta\in(0,\infty), \end{aligned} \tag{44}$$

where ξ_α is a probability density function defined on $(0,\infty)$; that is,

$$\xi_\alpha(\theta) \geq 0, \quad \theta \in (0,\infty), \quad \int_0^\infty \xi_\alpha(\theta)\,d\theta = 1. \tag{45}$$

This completes the proof of the lemma. □

According to Lemma 5, we give the following definition.

Definition 6. A function $x \in PC_{1-\alpha}(J,X)$ is called a mild solution of (4) if $I_t^{1-\alpha}x(t)|_{t=0} = x_0$ and there exits $f \in L^1(J < X)$ such that $f(t) \in F(t,x(t))$ a.e. on $t \in J$ and

$$x(t) = \begin{cases} t^{\alpha-1}T_\alpha(t)x_0 \\ \quad + \int_0^t (t-s)^{\alpha-1}T_\alpha(t-s)\left[Bu(s) + f(s,x(s))\right]ds, \\ \qquad\qquad t \in [0,t_1], \\ t^{\alpha-1}T_\alpha(t)x_0 + \sum_{k=1}^m T_\alpha(t-t_k)(t-t_k)^{\alpha-1}I_k(x(t_k^-)) \\ \quad + \int_0^t (t-s)^{\alpha-1}T_\alpha(t-s)\left[Bu(s) + f(s,x(s))\right]ds, \\ \qquad\qquad t \in (t_k,t_{k+1}], \end{cases} \tag{46}$$

where $k = 1,2,\ldots,m$,

$$\begin{gathered} T_\alpha(t) = \alpha\int_0^\infty \theta\xi_\alpha(\theta)T(t^\alpha\theta)\,d\theta, \\ \xi_\alpha(\theta) = \frac{1}{\alpha}\theta^{-1-(1/\alpha)}\varpi_\alpha\left(\theta^{-1/\alpha}\right), \\ \varpi_\alpha(\theta) = \frac{1}{\pi}\sum_{n=1}^\infty(-1)^{n-1}\theta^{-n\alpha-1}\frac{\Gamma(n\alpha+1)}{n!}\sin(n\pi\alpha), \\ \theta \in (0,\infty), \end{gathered} \tag{47}$$

where ξ_α is a probability density function defined on $(0,\infty)$; that is,

$$\xi_\alpha(\theta) \geq 0, \quad \theta \in (0,\infty), \quad \int_0^\infty \xi_\alpha(\theta)\,d\theta = 1. \tag{48}$$

Due to the work of the paper [31], we have the following result.

Lemma 7. *The operator $T_\alpha(t)$ has the following properties.*

(i) *For any fixed $t \geq 0$, $T_\alpha(t)$ is linear and bounded operator; that is, for any $x \in X$,*

$$\|T_\alpha(t)x\| \leq \frac{M}{\Gamma(\alpha)}\|x\|. \tag{49}$$

(ii) *$T_\alpha(t)$ $(t \geq 0)$ is strongly continuous.*

Now, we also introduce some basic definitions on multivalued maps. For more details, see [36–38].

A multivalued map $G : X \to \mathscr{P}(X)$ is convex (closed) valued if $G(x)$ is convex (closed) for all $x \in X$. G is bounded on bounded sets if $G(B) = \bigcup_{x\in B}. G(x)$ is bounded on X for any bounded set B of X; that is, $\sup_{x\in B}\{\sup\{\|y\| : y \in G(x)\}\} < \infty$.

G is called upper semicontinuous (u.s.c.) on X if, for each $x_0 \in X$, the set $G(x_0)$ is a nonempty closed subset of X, and if, for each open set U of X containing $G(x_0)$, there exists an open neighborhood V of x_0 such that $G(V) \subseteq U$.

G is said to be completely continuous if $G(B)$ is relatively compact for every $B \in P_b(X)$.

If the multivalued map G is completely continuous with nonempty compact values, then G is u.s.c. if and only if G has a closed graph (i.e., $x_n \to x_*$, $y_n \to y_*$, $y_n \in G(x_n)$ imply $y_* \in G(x_*)$).

We say that G has a fixed point if there is a $x \in X$ such that $x \in G(x)$.

A multivalued map $G : J \to P_{cl}(X)$ is said to be measurable if for each $x \in X$ the function $Y : J \to R^+$ defined by $Y(t) = d(x,G(t)) = \inf\{\|x - z\| : z \in G(t)\}$ is measurable.

Definition 8. The system (4) is said to be exactly controllable on J, if, for all $x_0, x_1 \in X$, there exists a control $u \in L^p(J,U)$ $(p > 1/\alpha)$ such that the mild solution of (4) satisfies $x(0;u) = x_0$ and $x(b;u) = x_1$.

Definition 9. The system (4) is said to be approximately controllable on the interval J, if, for all $x_0 \in X$, one has $\overline{\mathscr{R}(b,x_0)} = X$, where $\overline{\mathscr{R}(b,x_0)} = \{x(b;u) : u \in L^p(J,U)\,(p > 1/\alpha),\ x(0;u) = x_0\}$ is the reachable set of system (4) with the initial values x_0, x_1 at the terminal time b.

It is convenient at this point to introduce two relevant operators:

$$\begin{gathered} \Gamma_0^b = \int_0^b (b-s)^{2\alpha-2}T_\alpha(b-s)BB^*T_\alpha^*(b-s)\,ds, \\ \frac{1}{2} < \alpha \leq 1, \\ R\left(a,\Gamma_0^b\right) = \left(aI + \Gamma_0^b\right)^{-1}, \quad a > 0, \end{gathered} \tag{50}$$

where B^* denotes the adjoint of B and $T_\alpha^*(t)$ is the adjoint of $T_\alpha(t)$. It is straightforward that the operator Γ_0^b is a linear bounded operator.

We consider the following linear fractional differential system:

$$\begin{gathered} {}^cD^\alpha x(t) = Ax(t) + Bu(t), \quad \frac{1}{2} < \alpha \leq 1,\ t \in J = (0,b], \\ I_t^{1-\alpha}x(t)\big|_{t=0} = x_0 \in X. \end{gathered} \tag{51}$$

Lemma 10. *The linear fractional differential system* (51) *is approximately controllable on J if and only if $aR(a,\Gamma_0^b) \to 0$ as $a \to 0^+$ in the strong operator topology.*

The proof of this lemma is a straightforward adaptation of the proof of [3].

Lemma 11. *Let E be a Banach space and let $\mathscr{W} \subset L^1(J, E)$ be integrably bounded. If, for all $t \in J$, there is a relatively weakly compact set $C(t) \subset E$ such that $w(t) \in C(t)$ for every $w \in \mathscr{W}$, then $\mathscr{W}$ is relatively weakly compact in $L^1(J, E)$.*

Lemma 12 (Lasota and Opial [39]). *Let J be a compact real interval and let X be a Banach space. The multivalued map $F : J \times X \to \mathscr{P}_{b,cl,cv}(X)$ is measurable to t for each fixed $x \in X$, u.s.c. to x for each $t \in J$, and for each $x \in C(J, X)$ the set $S_{F,x} = \{f \in L^1(J, X) : f(t) \in F(t, x(t)), t \in J\}$ is nonempty. Let Γ be a linear continuous mapping from $L^1(J, X)$ to $C(J, X)$; then, the operator*

$$\begin{gathered}\Gamma \circ S_F : C(J, X) \longrightarrow \mathscr{P}_{b,cl,cv}(C(J, X)), \\ x \longmapsto (\Gamma \circ S_F)(x) = \Gamma(S_{F,x})\end{gathered} \tag{52}$$

is a closed graph operator in $C(J, X) \times C(J, X)$.

Lemma 13 (see [37]). *Let D be a bounded, convex, and closed subset in the Banach space E and let $G : D \to 2^X \setminus \{\emptyset\}$ be a u.s.c. condensing multivalued map. If, for every $x \in D$, $G(x)$ is a closed and convex set in D, then G has a fixed point.*

3. Main Results

In this section, we present our main result on approximate controllability of system (4). To do this, we first prove the existence of solutions for fractional control system. Secondly, we show that, under certain assumptions, the approximate controllability of (4) is implied by the approximate controllability of the corresponding linear system.

For convenience, let us introduce some notations:

$$\begin{gathered}M_b = \|B\|, \qquad \beta = \left(\frac{1-\gamma}{\alpha-\gamma} b^{(\alpha-\gamma)/(1-\gamma)}\right)^{1-\gamma}, \\ \Lambda = \frac{M\beta}{\Gamma(\alpha)} + \frac{M^3 M_B^2 b^{2\alpha-1}\beta}{a(2\alpha-1)(\Gamma(\alpha))^3}.\end{gathered} \tag{53}$$

Before stating and proving our main results, we introduce the following assumptions.

$H(1)$: $T_\alpha(t)$ is compact, $\|aR(a, \Gamma_0^b)\| \le 1$.

$H(2)$: F is a multivalued map satisfying $F : J \times X \to \mathscr{P}_{cp,cv}(X)$ which is measurable to t for each fixed $x \in D$, u.s.c. to x for each $t \in J$, and for each $x \in PC_{1-\alpha}(J, X)$ the set

$$S_{F,x} = \left\{f \in L^1(J, X) : f(t) \in F(t, x(t))\right\} \tag{54}$$

is nonempty.

$H(3)$: There exist a function $P(t) \in L^{1/\gamma}(J, R^+)$, $\gamma \in (0, \alpha)$, and a nondecreasing continuous function $\psi : \to R^+$, such that

$$\begin{aligned}\|F(t, x(t))\|_X &= \sup\left\{\|f(t)\|_X : f(t) \in F(t, x(t))\right\} \\ &\le P(t)\,\psi(\|x\|_D),\end{aligned} \tag{55}$$

for a.e. $t \in J$, for all $x \in D$, and for each $r > 0$, there exists $0 < \rho < 1$, such that

$$\lim_{r\to\infty} \inf \frac{\psi(r)}{r} \|P\|_{L^2} = \rho < 1. \tag{56}$$

$H(4)$: There exist constants $d_k > 0$, $k = 1, 2, \ldots, m$, with $(M/\Gamma(\alpha)) \sum_{k=1}^m d_k < 1$ such that

$$\|I_k(x) - I_k(y)\| \le d_k(t - t_k)^{1-\alpha} \|x - y\|_X, \quad \forall x, y \in X. \tag{57}$$

Theorem 14. *If the conditions $H(1)$–$H(4)$ are held, then the system (4) has a mild solution.*

Proof. We consider a set

$$B_r = \left\{x \in PC_{1-\alpha}(J, X) : \|x\| \le r, r > 0\right\} \tag{58}$$

on the space $PC_{1-\alpha}(J, X)$. We easily know that B_r is a bounded, closed, and convex set in $PC_{1-\alpha}(J, X)$. For $a > 0$, for all $x(\cdot) \in PC_{1-\alpha}(J, X)$, $x_1 \in X$, we take the control function as

$$u(t) = (b-t)^{\alpha-1} B^* T_\alpha^*(b-t)\, R\left(a, \Gamma_0^b\right) P(x(\cdot)), \tag{59}$$

where

$$\begin{aligned}P(x(\cdot)) = x_1 &- t^{\alpha-1} T_\alpha(b)\, x_0 \\ &- \sum_{k=1}^m T_\alpha(t - t_k)(t - t_k)^{\alpha-1} I_k(x(t_k^-)) \\ &- \int_0^b (b-s)^{\alpha-1} T_\alpha(b-s) f(s)\, ds, \quad f \in S_{F,x}.\end{aligned} \tag{60}$$

By this control, we define the operator $\Phi_a : PC_{1-\alpha}(J, X) \to \mathscr{P}(PC_{1-\alpha}(J, X))$ as follows:

$$\begin{aligned}\Phi_a(x) = \Bigg\{\tau \in PC_{1-\alpha}(J, X) \\ &: \tau(t) = t^{\alpha-1} T_\alpha(t)\, x_0 \\ &+ \int_0^t (t-s)^{\alpha-1} T_\alpha(t-s) f(s)\, ds \\ &+ \int_0^t (t-s)^{\alpha-1} T_\alpha(t-s) Bu(s)\, ds, \\ &f \in S_{F,x},\ t \in (0, t_1], \\ &t^{\alpha-1} T_\alpha(t)\, x_0 \\ &+ \sum_{k=1}^m T_\alpha(t - t_k)(t - t_k)^{\alpha-1} I_k(x(t_k^-))\end{aligned}$$

$$+\int_0^t (t-s)^{\alpha-1} T_\alpha (t-s) f(s)\, ds$$
$$+\int_0^t (t-s)^{\alpha-1}$$
$$\times T_\alpha (t-s) Bu(s)\, ds,$$
$$\left. f \in S_{F,x},\ t \in (t_k, t_{k+1}],\ k = 1, 2, \dots, m. \right\}. \tag{61}$$

We will show that, for all $a > 0$, the operator $\Phi_a : PC_{1-\alpha}(J, X) \to \mathscr{P}(PC_{1-\alpha}(J, X))$ has a fixed point. For the sake of convenience, we subdivide the proof into several steps.

Step 1. For each $a > 0$, the operator $\Phi_a(x)$ is convex for each $x \in PC_{1-\alpha}(J, X)$.

In fact, if $\tau_1, \tau_2 \in \Phi_a(x)$, then, for each $t \in (t_k, t_{k+1}]$, $k = 1, 2, \dots, m$, there exist $f_1, f_2 \in S_{F,x}$ such that

$$\begin{aligned}
\tau_i(t) = {} & t^{\alpha-1} T_\alpha(t) x_0 \\
& + \sum_{k=1}^m T_\alpha(t-t_k)(t-t_k)^{\alpha-1} I_k(x(t_k^-)) \\
& + \int_0^t (t-s)^{\alpha-1} T_\alpha(t-s) f_i(s)\, ds \\
& + \int_0^t (t-s)^{\alpha-1} T_\alpha(t-s) BB^* T_\alpha^*(b-t) R\left(a, \Gamma_0^b\right) \\
& \times \Bigg[x_1 - t^{\alpha-1} T_\alpha(t) x_0 \\
& - \sum_{k=1}^m T_\alpha(t-t_k)(t-t_k)^{\alpha-1} I_k(x(t_k^-)) \\
& - \int_0^b (b-\mu)^{\alpha-1} T_\alpha(b-\mu) f_i(\mu)\, d\mu \Bigg] ds, \\
& i = 1, 2.
\end{aligned} \tag{62}$$

Let $0 \le \lambda \le 1$; then, for each $t \in (t_k, t_{k+1}]$, $k = 1, 2, \dots, m$, we have

$$\begin{aligned}
& \lambda\tau_1(t) + (1-\lambda)\tau_2(t) \\
& = t^{\alpha-1} T_\alpha(t) x_0 + \sum_{k=1}^m T_\alpha(t-t_k)(t-t_k)^{\alpha-1} I_k(x(t_k^-)) \\
& + \int_0^t (t-s)^{\alpha-1} T_\alpha(t-s) [\lambda f_1(s) + (1-\lambda) f_2(s)]\, ds \\
& + \int_0^t (t-s)^{\alpha-1} T_\alpha(t-s) BB^* T_\alpha^*(b-t) R\left(a, \Gamma_0^b\right) \\
& \times \Bigg[x_1 - t^{\alpha-1} T_\alpha(t) x_0 \\
& - \sum_{k=1}^m T_\alpha(t-t_k)(t-t_k)^{\alpha-1} I_k(x(t_k^-)) \\
& - \int_0^b (b-\mu)^{\alpha-1} T_\alpha(b-\mu) \\
& \times [\lambda f_1(\mu) + (1-\lambda) f_2(\mu)]\, d\mu \Bigg] ds.
\end{aligned} \tag{63}$$

Since $S_{F,x}$ is convex (because F has convex values), $\lambda f_1 + (1-\lambda) f_2 \in S_{F,x}$; thus, $\lambda\tau_1(t) + (1-\lambda)\tau_2 \in \Phi_a(x)$.

Step 2. For each $a > 0$, there is a positive constant $r_0 = r(a)$, such that $\Phi_a(B_{r_0}) \subset B_{r_0}$.

If this is not true, then there exists $a > 0$ such that, for every $r > 0$, there exists a $\overline{x} \in B_r$ such that $\Phi_a(\overline{x}) \not\subseteq B_r$; that is,

$$\begin{aligned}
& \|\Phi_a(\overline{x})\| \equiv \sup\left\{ \|\tau\|_{PC_{1-\alpha}(J,X)} : \tau \in \Phi_a(\overline{x}) \right\} > r, \\
& \tau(t) = t^{\alpha-1} T_\alpha(t) x_0 \\
& + \sum_{k=1}^m T_\alpha(t-t_k)(t-t_k)^{\alpha-1} I_k(x(t_k^-)) \\
& + \int_0^t (t-s)^{\alpha-1} T_\alpha(t-s) f(s)\, ds \\
& + \int_0^t (t-s)^{\alpha-1} T_\alpha(t-s) BB^* T_\alpha^*(b-t) R\left(a, \Gamma_0^b\right) \\
& \times \Bigg[x_1 - b^{\alpha-1} T_\alpha(b) x_0 \\
& - \sum_{k=1}^m T_\alpha(b-t_k)(b-t_k)^{\alpha-1} I_k(x(t_k^-)) \\
& - \int_0^b (b-\mu)^{\alpha-1} T_\alpha(b-\mu) f(\mu)\, d\mu \Bigg] ds,
\end{aligned} \tag{64}$$

for some $S_{F,\overline{x}}$.

By using Holder's inequality and $H(3)$, we have

$$\begin{aligned}
& \int_0^t (t-s)^{\alpha-1} \|u(s)\|\, ds \\
& \le \frac{M M_B b^{2\alpha-1}}{a\Gamma(\alpha)(2\alpha-1)}
\end{aligned}$$

$$\times \left[\|x_1\| + \frac{M\|x_0\|}{\Gamma(\alpha)} + \sum_{k=1}^{m} \frac{M}{\Gamma(\alpha)} \left(d_k \|x\| + m(b-t_k)^{\alpha-1} \|I_k(0)\| \right) + \frac{M\psi(r)\beta}{\Gamma(\alpha)} \|P\|_{L^{1/\gamma}} \right]. \tag{65}$$

Then, we obtain

$$\begin{aligned}
&(t-t_k)^{1-\alpha} \|\tau(t)\| \\
&\le (t-t_k)^{1-\alpha} \left\| t^{\alpha-1} T_\alpha(t) x_0 \right\| + (t-t_k)^{1-\alpha} \\
&\quad \times \sum_{k=1}^{m} \|T_\alpha(t-t_k)\| (t-t_k)^{\alpha-1} \|I_k(x(t_k^-))\| \\
&\quad + (t-t_k)^{1-\alpha} \times \int_0^t (t-s)^{\alpha-1} \|T_\alpha(t-s)\| \|f(s)\| \, ds \\
&\quad + (t-t_k)^{1-\alpha} \times \int_0^t (t-s)^{\alpha-1} \|T_\alpha(t-s)\| \|B\| \|u(s)\| \, ds \\
&\le \frac{Mb^{\alpha-1}(b-t_k)^{1-\alpha}}{\Gamma(\alpha)} \|x_0\| + (b-t_k)^{1-\alpha} \\
&\quad \times \sum_{k=1}^{m} \frac{M}{\Gamma(\alpha)} \left(d_k \|x\| + m(b-t_k)^{\alpha-1} \|I_k(0)\| \right) \\
&\quad + \frac{M(b-t_k)^{1-\alpha}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} P(s) \psi(r) \, ds \\
&\quad + \frac{MM_B(b-t_k)^{1-\alpha}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(s) \, ds \\
&\le \frac{Mb^{\alpha-1}(b-t_k)^{1-\alpha}}{\Gamma(\alpha)} \|x_0\| + (b-t_k)^{1-\alpha} \\
&\quad \times \sum_{k=1}^{m} \frac{M}{\Gamma(\alpha)} \left(d_k \|x\| + m(b-t_k)^{\alpha-1} \|I_k(0)\| \right) \\
&\quad + \frac{b^{2\alpha-1} M^2 M_B^2 (b-t_k)^{1-\alpha}}{a(2\alpha-1)(\Gamma(\alpha))^2} \\
&\quad \times \left[\|x_1\| + \frac{M\|x_0\|}{\Gamma(\alpha)} + \sum_{k=1}^{m} \frac{M}{\Gamma(\alpha)} \left(d_k \|x\| + m(b-t_k)^{\alpha-1} \|I_k(0)\| \right) \right] \\
&\quad + \psi(r) \Lambda \|P\|_{L^{1/\gamma}}.
\end{aligned} \tag{66}$$

Thus,

$$\begin{aligned}
r \le{}& \frac{Mb^{\alpha-1}(b-t_k)^{1-\alpha}}{\Gamma(\alpha)} \|x_0\| + (b-t_k)^{1-\alpha} \\
&\times \sum_{k=1}^{m} \frac{M}{\Gamma(\alpha)} \left(d_k \|x\| + m(b-t_k)^{\alpha-1} \|I_k(0)\| \right) \\
&+ \frac{b^{2\alpha-1} M^2 M_B^2 (b-t_k)^{1-\alpha}}{a(2\alpha-1)(\Gamma(\alpha))^2} \\
&\times \left[\|x_1\| + \frac{M\|x_0\|}{\Gamma(\alpha)} + \sum_{k=1}^{m} \frac{M}{\Gamma(\alpha)} \left(d_k \|x\| + m(b-t_k)^{\alpha-1} \|I_k(0)\| \right) \right] \\
&+ \psi(r) \Lambda \|P\|_{L^{1/\gamma}}.
\end{aligned} \tag{67}$$

Dividing both sides by r and taking the low limit as $r \to \infty$, we get

$$1 \le \lim_{r\to\infty} \inf \frac{\psi(r)}{r} \|P\|_{L^{1/\gamma}}, \tag{68}$$

which is a contradiction to $H(3)$. Thus, for each $a > 0$, there exists r_0 such that $\Phi_a(x)$ maps B_{r_0} into itself.

Step 3. $\Phi_\alpha(x)$ is closed for each $x \in PC_{1-\alpha}(J, X)$.

Indeed, for each given $x \in PC_{1-\alpha}(J, X)$, let $\{\tau_n\}_{n\ge 0} \subset \Phi_a(x)$ such that $\tau_n \to \tau$ in $PC_{1-\alpha}(J, X)$. Then, there exists $f_n \in S_{F,x}$ such that, for each $t \in J$,

$$\begin{aligned}
\tau_n(t) ={}& t^{\alpha-1} T_\alpha(t) x_0 + \sum_{k=1}^{m} T_\alpha(t-t_k)(t-t_k)^{\alpha-1} I_k(x(t_k)) \\
&+ \int_0^t (t-s)^{\alpha-1} T_\alpha(t-s) f_n(s) \, ds \\
&+ \int_0^t (t-s)^{\alpha-1} T_\alpha(t-s) Bu_n(s) \, ds,
\end{aligned} \tag{69}$$

where

$$\begin{aligned}
u_n(t) ={}& B^* T_\alpha^*(b-t) R\left(a, \Gamma_0^b\right) \\
&\times \left[x_1 - b^{\alpha-1} T_\alpha(b) x_0 - \sum_{k=1}^{m} T_\alpha(b-t_k)(b-t_k)^{\alpha-1} I_k(x(t_k^-)) - \int_0^b (b-s)^{\alpha-1} T_\alpha(b-s) f_n(s) \, ds \right].
\end{aligned} \tag{70}$$

Because of [40, Proposition 3.1], $S_{F,x}$ is weakly compact in $L^1(J,X)$ which implies that f_n converges weakly to some $f \in S_{F,x}$ in $L^1(J,X)$. Thus, $u_n \rightharpoonup u$, and

$$u(t) = B^* T_\alpha^* (b-t) R\left(a, \Gamma_0^b\right) \times \left[x_1 - b^{\alpha-1} T_\alpha(b) x_0 - \sum_{k=1}^{m} T_\alpha(b-t_k)(b-t_k)^{\alpha-1} I_k(x(t_k^-)) - \int_0^b (b-s)^{\alpha-1} T_\alpha(b-s) f(s)\, ds \right]. \tag{71}$$

Then, for each $t \in J$,

$$\begin{aligned}\tau_n(t) &\longrightarrow \tau(t) \\ &= t^{\alpha-1} T_\alpha(t) x_0 + \sum_{k=1}^{m} T_\alpha(t-t_k)(t-t_k)^{\alpha-1} I_k(x(t_k^-)) + \int_0^t (t-s)^{\alpha-1} T_\alpha(t-s) f(s)\, ds \\ &\quad + \int_0^t (t-s)^{\alpha-1} T_\alpha(t-s) BB^* T_\alpha^*(b-t) R\left(a, \Gamma_0^b\right) \\ &\quad \times \left[x_1 - b^{\alpha-1} T_\alpha(b) x_0 - \sum_{k=1}^{m} T_\alpha(b-t_k)(b-t_k)^{\alpha-1} I_k(x(t_k^-)) - \int_0^b (b-\mu)^{\alpha-1} T_\alpha(b-\mu) f(\mu)\, d\mu \right] ds.\end{aligned} \tag{72}$$

Thus, we show that $\tau \in \Phi_a(x)$.

Step 4. Φ_a is u.s.c and condensing.

We decompose Φ_a as $\Phi_a = \Phi_a^1 + \Phi_a^2$, where the operators Φ_a^1 and Φ_a^2 are defined by

$$\begin{aligned}\left(\Phi_a^1 x\right)(t) &= \sum_{k=1}^{m} T_\alpha(t-t_k)(t-t_k)^{\alpha-1} I_k(x(t_k^-)), \quad t \in J,\ k = 1, 2, \ldots, m, \\ \Phi_a^2(x) &= \Bigg\{ \tau \in PC_{1-\alpha}(J, X) : \tau(t) = t^{\alpha-1} T_\alpha(t) x_0 + \int_0^t (t-s)^{\alpha-1} T_\alpha(t-s) f(s)\, ds \\ &\quad + \int_0^t (t-s)^{\alpha-1} T_\alpha(t-s) \times BB^* T_\alpha^*(b-t) R\left(a, \Gamma_0^b\right) \\ &\quad \times \left[x_1 - b^{\alpha-1} T_\alpha(b) x_0 - \sum_{k=1}^{m} T_\alpha(b-t_k)(b-t_k)^{\alpha-1} I_k(x(t_k^-)) - \int_0^b (b-\mu)^{\alpha-1} T_\alpha(b-\mu) f(\mu)\, d\mu \right] ds, \\ &\quad f \in S_{F,x},\ t \in J \Bigg\}.\end{aligned} \tag{73}$$

According to [41, Corollary 2.2.1], we will prove that Φ_a^1 is a contraction operator, while Φ_a^2 is a completely continuous operator.

Let us begin proving that Φ_a^1 is a contraction operator. For any $x, y \in X$, we obtain

$$\begin{aligned}\left\| \left(\Phi_a^1 x\right)(t) - \left(\Phi_a^1 y\right)(t) \right\| &\le \left\| \sum_{k=1}^{m} T_\alpha(t-t_k)(t-t_k)^{\alpha-1} I_k(x(t_k^-)) - \sum_{k=1}^{m} T_\alpha(t-t_k)(t-t_k)^{\alpha-1} I_k(y(t_k^-)) \right\| \\ &\le \frac{M}{\Gamma(\alpha)} \sum_{k=1}^{m} d_k \|x - y\|.\end{aligned} \tag{74}$$

Then, Φ_a^1 is a contraction operator, since $(M/\Gamma(\alpha)) \sum_{k=1}^{m} d_k < 1$.

Next, we prove that Φ_a^2 is u.s.c and completely continuous. We subdivide the proof into several claims.

Claim 1. There exists a positive constant r such that $\Phi_a^2(B_r) \subseteq B_r$.

By employing the technique used in Step 2, one can easily show that there exists $r > 0$ such that $\Phi_a^2(B_r) \subseteq B_r$.

Claim 2. $\Phi_a^2(B_r)$ is a family of equicontinuous functions.

Let $0 \le s \le t_1 \le t_2 \le b$. For each $x \in B_r$, $\varphi \in \Phi_a^2(x)$, there exists $f \in S_{F,x}$ such that

$$\tau(t) = t^{\alpha-1} T_\alpha(t) x_0 + \int_0^t (t-s)^{\alpha-1} T_\alpha(t-s) f(s)\, ds + \int_0^t (t-s)^{\alpha-1} T_\alpha(t-s) Bu(s)\, ds, \quad t \in J. \tag{75}$$

Then, we have

$$\begin{aligned}
&\|\tau(t_2)-\tau(t_1)\| \\
&\le \left\|\int_0^{t_1}\left[(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}\right]T_\alpha(t_2-s)f(s)\,ds\right\| \\
&\quad\times\left\|\int_0^{t_1}(t_1-s)^{\alpha-1}\left[T_\alpha(t_2-s)-T_\alpha(t_1-s)\right]f(s)\,ds\right\| \\
&\quad+\left\|\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}T_\alpha(t_2-s)f(s)\,ds\right\| \\
&\quad\times\left\|\int_0^{t_1}\left[(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}\right]T_\alpha(t_2-s)u(s)\,ds\right\| \\
&\quad\times\left\|\int_0^{t_1}(t_1-s)^{\alpha-1}\left[T_\alpha(t_2-s)-T_\alpha(t_1-s)\right]u(s)\,ds\right\| \\
&\quad+\left\|\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}T_\alpha(t_2-s)u(s)\,ds\right\| \\
&\le I_1+I_2+I_3+I_4+I_5+I_6.
\end{aligned} \tag{76}$$

By using Hölder's inequality and assumption $H(3)$, we get

$$\begin{aligned}
I_1&=\frac{M\psi(r)\|P\|_{L^{1/\gamma}}}{\Gamma(\alpha)} \\
&\quad\times\left[\left(\frac{1-\gamma}{\alpha-\gamma}t_1^{(\alpha-\gamma)/(1-\gamma)}\right)^{1-\gamma}\right. \\
&\qquad+\left(\frac{1-\gamma}{\alpha-\gamma}(t_2-t_1)^{(\alpha-\gamma)/(1-\gamma)}\right)^{1-\gamma} \\
&\qquad\left.-\left(\frac{1-\gamma}{\alpha-\gamma}t_2^{(\alpha-\gamma)/(1-\gamma)}\right)^{1-\gamma}\right], \\
I_2&=\psi(r)\|P\|_{L^{1/\gamma}}\left(\frac{1-\gamma}{\alpha-\gamma}t_1^{(\alpha-\gamma)/(1-\gamma)}\right)^{1-\gamma} \\
&\quad\times\sup_{s\in[0,t_1]}\|T_\alpha(t_2-s)-T_\alpha(t_1-s)\|,
\end{aligned} \tag{77}$$

$$\begin{aligned}
I_3&=\frac{M\psi(r)\|P\|_{L^{1/\gamma}}}{\Gamma(\alpha)}\left[\frac{1-\gamma}{\alpha-\gamma}(t_2-t_1)^{(\alpha-\gamma)/(1-\gamma)}\right]^{1-\gamma}, \\
I_4&=\frac{MM_B}{\Gamma(\alpha)}\int_{t_1}^{t_2}\left[(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}\right]\|u(s)\|\,ds, \\
I_5&=M_B\sup_{s\in[0,t_1]}\|T_\alpha(t_2-s)-T_\alpha(t_1-s)\| \\
&\quad\times\int_0^{t_1}(t_1-s)^{\alpha-1}\|u(s)\|\,ds, \\
I_6&=\frac{MM_B}{\Gamma(\alpha)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}\|u(s)\|\,ds.
\end{aligned}$$

It is easy to see that I_2 tends to zero independently of $x \in B_r$ as $t_2 \to t_1$. Note that, from Lemma 10, $T_\alpha(t)$ is continuous in the uniform operator topology for $t > 0$; we can directly obtain I_1 and I_3 tending to zero independently of $x \in B_r$ as $t_2 \to t_1$. Applying the absolute continuity of the Lebesgue integral, we have I_4, I_5, and I_6 tending to zero independently of $x \in B_r$ as $t_2 \to t_1$.

Therefore, $\Phi_a^2(B_r) \subset PC_{1-\alpha}(J,X)$ is equicontinuous.

Claim 3. The set $\Pi(t) = \{\tau(t) : \tau \in \Phi_a^2(B_r)\} \subset X$ is relatively compact for each $t \in J$.

Let $0 < t \le b$ be fixed. For $x \in B_r$ and $\tau \in \Phi_a^2(x)$, there exists $f \in S_{F,x}$ such that, for each $t \in J$,

$$\begin{aligned}
\tau(t)&=t^{\alpha-1}T_\alpha(t)x_0+\int_0^t(t-s)^{\alpha-1}T_\alpha(t-s)f(s)\,ds \\
&\quad+\int_0^t(t-s)^{\alpha-1}T_\alpha(t-s)BB^*T_\alpha^*(b-t)R\left(a,\Gamma_0^b\right) \\
&\qquad\times\left[x_1-b^{\alpha-1}T_\alpha(b)x_0\right. \\
&\qquad\quad-\sum_{k=1}^m T_\alpha(b-t_k)(b-t_k)^{\alpha-1}I_k(x(t_k^-)) \\
&\qquad\quad\left.-\int_0^b(b-\mu)^{\alpha-1}T_\alpha(b-\mu)f(\mu)\,d\mu\right]ds, \\
&\qquad t\in J.
\end{aligned} \tag{78}$$

For all $\varepsilon \in (0,t)$ and for all $\delta > 0$, define

$$\begin{aligned}
&\tau^{\varepsilon,\delta}(t) \\
&=t^{\alpha-1}T_\alpha(t)x_0 \\
&\quad+\alpha\int_0^{t-\varepsilon}\int_\delta^\infty\theta(t-s)^{\alpha-1}\xi_\alpha(\theta)T\left((t-s)^\alpha\theta\right)f(s)\,d\theta\,ds \\
&\quad+\alpha\int_0^{t-\varepsilon}\int_\delta^\infty\theta(t-s)^{\alpha-1}\xi_\alpha(\theta)T\left((t-s)^\alpha\theta\right) \\
&\qquad\times BB^*T_\alpha^*(b-s)R\left(a,\Gamma_0^b\right) \\
&\qquad\times\left[x_1-b^{\alpha-1}T_\alpha(b)x_0\right. \\
&\qquad\quad-\sum_{k=1}^m T_\alpha(b-t_k)(b-t_k)^{\alpha-1}I_k(x(t_k^-)) \\
&\qquad\quad\left.-\int_0^b(b-\mu)^{\alpha-1}T_\alpha(b-\mu)f(\mu)\,d\mu\right]d\theta\,ds \\
&=t^{\alpha-1}T_\alpha(t)x_0+T\left(\varepsilon^\alpha\delta\right) \\
&\quad\times\left\{\alpha\int_0^{t-\varepsilon}\int_\delta^\infty\theta(t-s)^{\alpha-1}\xi_\alpha(\theta)T\right. \\
&\qquad\times\left((t-s)^\alpha\theta-\varepsilon^\alpha\delta\right)f(s)\,d\theta\,ds
\end{aligned}$$

$$+\alpha\int_0^{t-\varepsilon}\int_\delta^\infty \theta(t-s)^{\alpha-1}\xi_\alpha(\theta)T\left((t-s)^\alpha\theta-\varepsilon^\alpha\delta\right)$$
$$\times BB^*T_\alpha^*(b-s)R\left(a,\Gamma_0^b\right)$$
$$\times\left[x_1-b^{\alpha-1}T_\alpha(b)x_0\right.$$
$$-\sum_{k=1}^m T_\alpha(b-t_k)(b-t_k)^{\alpha-1}I_k\left(x\left(t_k^-\right)\right)$$
$$\left.-\int_0^b(b-\mu)^{\alpha-1}T_\alpha(b-\mu)f(\mu)\,d\mu\right]d\theta\,ds\Bigg\}. \tag{79}$$

By the compactness of $T(\varepsilon^\alpha\delta)$ $(\varepsilon^\alpha\delta>0)$, we obtain the set

$$\Pi^{\varepsilon,\delta}(t)=\left\{\varphi^{\varepsilon,\delta}(t):\tau\in\Phi_a^2(B_r)\right\} \tag{80}$$

which is relatively compact in X for all $\varepsilon\in(0,t)$ and $\delta>0$. Moreover, we have

$$\left\|\tau(t)-\tau^{\varepsilon,\delta}(t)\right\|$$
$$\le\left\{\frac{\alpha M\psi(r)\beta}{\Gamma(\alpha)}\|P\|_{L^{1/\gamma}}+\frac{\alpha M^2M_B^2b^{2\alpha-1}}{a(2\alpha-1)\Gamma(\alpha)}\right.$$
$$\times\left[\left\|x_1\right\|+\frac{M\left\|x_0\right\|}{\Gamma(\alpha)}\right.$$
$$+\sum_{k=1}^m\frac{M}{\Gamma(\alpha)}\left(d_k\|x\|+m(b-t_k)^{\alpha-1}\left\|I_k(0)\right\|\right)$$
$$\left.\left.+\frac{M\psi(r)\beta}{\Gamma(\alpha)}\|P\|_{L^{1/\gamma}}\right]\right\}\int_0^\delta\theta\xi_\alpha(\theta)\,d\theta$$
$$+\frac{M\psi(r)}{\Gamma(\alpha)}\|P\|_{L^{1/\gamma}}\left(\frac{1-\gamma}{\alpha-\gamma}\varepsilon^{(\alpha-\gamma)/(1-\gamma)}\right)^{1-\gamma} \tag{81}$$
$$+\frac{M^2M_B^2\varepsilon^{2\alpha-1}}{a(2\alpha-1)(\Gamma(\alpha))^2}$$
$$\times\left[\left\|x_1\right\|+\frac{M\left\|x_0\right\|}{\Gamma(\alpha)}\right.$$
$$+\sum_{k=1}^m\frac{M}{\Gamma(\alpha)}\left(d_k\|x\|+m(b-t_k)^{\alpha-1}\left\|I_k(0)\right\|\right)$$
$$\left.+\frac{M\psi(r)\beta}{\Gamma(\alpha)}\|P\|_{L^{1/\gamma}}\right].$$

The right-hand side of the above inequality tends to zero as $\varepsilon\to0$. Therefore, there are relatively compact sets arbitrarily close to the set $\Pi(t)$, $t>0$. Hence the set $\Pi(t)$, $t>0$ is also relatively compact in X. As a consequence of Claims 1–3 together with the Arzola-Ascoli theorem, we can conclude that Φ_a^2 is completely continuous.

Claim 4. Φ_α^2 has a closed graph.

Let $x_n\to x^*$ $(n\to\infty)$, $\tau_n\in\Phi_a^2(x_n)$, $\tau_n\to\tau^*$ $(n\to\infty)$. We will prove that $\tau^*\in\Phi_a^2(x^*)$. Since $\tau_n\in\Phi_a^2(x_n)$, there exists $f_n\in S_{F,x_n}$, such that, for each $t\in J$,

$$\tau_n(t)=t^{\alpha-1}T_\alpha(t)x_0$$
$$+\int_0^t(t-s)^{\alpha-1}T_\alpha(t-s)f_n(s)\,ds$$
$$+\int_0^t(t-s)^{\alpha-1}T_\alpha(t-s)BB^*T_\alpha^*(b-t)R\left(a,\Gamma_0^b\right)$$
$$\times\left[x_1-b^{\alpha-1}T_\alpha(b)x_0\right.$$
$$-\sum_{k=1}^m T_\alpha(b-t_k)(b-t_k)^{\alpha-1}I_k\left(x\left(t_k^-\right)\right)$$
$$\left.-\int_0^b(b-\mu)^{\alpha-1}T_\alpha(b-\mu)f_n(\mu)\,d\mu\right]ds. \tag{82}$$

We must prove that there exists $f^*\in S_{F,x^*}$, such that, for each $t\in J$,

$$\tau^*(t)=t^{\alpha-1}T_\alpha(t)x_0$$
$$+\int_0^t(t-s)^{\alpha-1}T_\alpha(t-s)f^*(s)\,ds$$
$$+\int_0^t(t-s)^{\alpha-1}T_\alpha(t-s)BB^*T_\alpha^*(b-t)R\left(a,\Gamma_0^b\right)$$
$$\times\left[x_1-b^{\alpha-1}T_\alpha(b)x_0\right.$$
$$-\sum_{k=1}^m T_\alpha(b-t_k)(b-t_k)^{\alpha-1}I_k\left(x\left(t_k^-\right)\right)$$
$$\left.-\int_0^b(b-\mu)^{\alpha-1}T_\alpha(b-\mu)f^*(\mu)\,d\mu\right]ds. \tag{83}$$

Since $\tau_n\to\tau^*$ $(n\to\infty)$, we can obtain

$$\left\|\left(\tau_n(t)-t^{\alpha-1}T_\alpha(t)x_0\right.\right.$$
$$-\int_0^t(t-s)^{\alpha-1}T_\alpha(t-s)BB^*T_\alpha^*(b-t)R\left(a,\Gamma_0^b\right)$$
$$\times\left[x_1-b^{\alpha-1}T_\alpha(b)x_0\right.$$
$$\left.\left.-\sum_{k=1}^m T_\alpha(b-t_k)(b-t_k)^{\alpha-1}I_k\left(x\left(t_k^-\right)\right)\right]ds\right)$$

$$-\Bigg(\tau^{*}(t)-t^{\alpha-1}T_{\alpha}(t)x_{0}-\int_{0}^{t}(t-s)^{\alpha-1}T_{\alpha}(t-s)BB^{*}T_{\alpha}^{*}(b-t)R(a,\Gamma_{0}^{b})\times\Big[x_{1}-b^{\alpha-1}T_{\alpha}(b)x_{0}-\sum_{k=1}^{m}T_{\alpha}(b-t_{k})(b-t_{k})^{\alpha-1}\times I_{k}(x(t_{k}^{-}))\Big]ds\Bigg)\Bigg\|\longrightarrow 0,\quad \text{as}\, n\longrightarrow\infty. \tag{84}$$

Consider the linear continuous operator

$$\Gamma: L^{1/\gamma}(J,X)\longrightarrow C_{1-\alpha}(J,X),$$
$$(\Gamma f)(t)=\int_{0}^{t}(t-s)^{\alpha-1}T_{\alpha}(t-s)\times\Big[f(s)-B(b-s)^{\alpha-1}B^{*}T_{\alpha}^{*}(b-s)R(a,\Gamma_{0}^{b})\times\int_{0}^{b}T_{\alpha}(b-\eta)f(\eta)d\eta\Big]ds. \tag{85}$$

Clearly it follows from Lemma 13 that $\Gamma\circ S_F$ is a closed graph operator. Moreover, we have

$$\tau_{n}(t)-t^{\alpha-1}T_{\alpha}(t)x_{0}-\int_{0}^{t}(t-s)^{\alpha-1}T_{\alpha}(t-s)BB^{*}T_{\alpha}^{*}(b-s)R(a,\Gamma_{0}^{b})\times\Big[x_{1}-b^{\alpha-1}T_{\alpha}(b)x_{0}-\sum_{k=1}^{m}T_{\alpha}(b-t_{k})(b-t_{k})^{\alpha-1}I_{k}(x(t_{k}^{-}))\Big]ds\in\Gamma(S_{F,x_{n}}). \tag{86}$$

Since $x_n \to x_*$, it follows from Lemma 13 that

$$\tau_{*}(t)-t^{\alpha-1}T_{\alpha}(t)x_{0}-\int_{0}^{t}(t-s)^{\alpha-1}T_{\alpha}(t-s)BB^{*}T_{\alpha}^{*}(b-s)R(a,\Gamma_{0}^{b})\times\Big[x_{1}-b^{\alpha-1}T_{\alpha}(b)x_{0}-\sum_{k=1}^{m}T_{\alpha}(b-t_{k})(b-t_{k})^{\alpha-1}I_{k}(x(t_{k}^{-}))\Big]ds\in\Gamma(S_{F,x_{*}}). \tag{87}$$

Therefore, Φ_a^2 has a closed graph. Since Φ_a^2 is a completely continuous multivalued map with compact value, we have that Φ_a^2 is u.s.c.

Thus $\Phi_a = \Phi_a^1 + \Phi_a^2$ is u.s.c and condensing. Therefore, applying Lemma 13, we conclude that Φ_a has a fixed point $x(\cdot)$ on B_{r_0}. Thus, the fractional control system (4) has a mild solution on J.

The proof is complete. □

The following result concerns the approximate controllability of that problem (4). We assume that the following assumption be held.

> $H'(3)$: There exists a positive constant L such that $\|F(t,x(t))\| \le L$ for all $(t,x)\in J\times X$.

Theorem 15. *Assume that assumptions $H(1)$, $H(2)$, $H'(3)$, and $H(4)$ are satisfied and the linear system (24) is approximately controllable on J. Then system (4) is approximately controllable on J.*

Proof. By employing the technique used in Theorem 14, we can easily show that, for all $0<a<1$, the operator Φ_a has a fixed point in B_{r_0}, where $r_0 = r(a)$. Let $x^a(\cdot)$ be a fixed point of Φ_a in B_{r_0}. Any fixed point of Φ_a is a mild solution of (4); this means that there exists $f^a \in S_{F,x^a}$ such that, for each $t\in J'$,

$$\begin{aligned}x^{a}(t)&=t^{\alpha-1}T_{\alpha}(t)x_{0}+\sum_{k=1}^{m}T_{\alpha}(t-t_{k})(t-t_{k})^{\alpha-1}I_{k}(x(t_{k}^{-}))+\int_{0}^{t}(t-s)^{\alpha-1}T_{\alpha}(t-s)f^{a}(s)ds\\&\quad+\int_{0}^{t}(t-s)^{\alpha-1}T_{\alpha}(t-s)BB^{*}T_{\alpha}^{*}(b-s)R(a,\Gamma_{0}^{b})\times\Big[x_{1}-b^{\alpha-1}T_{\alpha}(b)x_{0}-\sum_{k=1}^{m}T_{\alpha}(b-t_{k})(b-t_{k})^{\alpha-1}I_{k}(x(t_{k}^{-}))\\&\quad-\int_{0}^{b}(b-\mu)^{\alpha-1}T_{\alpha}(b-\mu)f^{a}(\mu)d\mu\Big]ds.\end{aligned} \tag{88}$$

Define

$$P(f^{a})=x_{1}-b^{\alpha-1}T_{\alpha}(b)x_{0}-\sum_{k=1}^{m}T_{\alpha}(b-t_{k})(b-t_{k})^{\alpha-1}I_{k}(x(t_{k}^{-}))$$

$$-\int_0^b (b-s)^{\alpha-1} T_\alpha (b-s) f^a (s)\, ds,$$
$$\text{for some } f^a \in S_{F,x^a}. \tag{89}$$

Noting that $I - \Gamma_0^b R(a, \Gamma_0^b) = aR(a, \Gamma_0^b)$, we get

$$x^a (b) = x_1 - aR\left(a, \Gamma_0^b\right) P\left(f^a\right). \tag{90}$$

By assumption $H'(3)$,

$$\int_0^b \left\| f^a (s)\right\|^2 ds \le L^2 b. \tag{91}$$

Consequently the sequence $\{f^a\}$ is uniformly bounded in $L^{1/\gamma}(J, X)$. Thus, there is a subsequence, still denoted by $\{f^a\}$, that converges weakly to, say, f in $L^{1/\gamma}(J, X)$. Denoting

$$\begin{aligned} h &= x_1 - b^{\alpha-1} T_\alpha (b)\, x_0 \\ &\quad - \sum_{k=1}^{m} T_\alpha (b-t_k)(b-t_k)^{\alpha-1} I_k \left(x\left(t_k^-\right)\right) \\ &\quad - \int_0^b (b-s)^{\alpha-1} T_\alpha (b-s) f (s)\, ds, \end{aligned} \tag{92}$$

we see that

$$\begin{aligned} &\left\|P\left(f^\alpha\right) - h\right\| \\ &\quad = \left\|\int_0^b (b-s)^{\alpha-1} T_\alpha (b-s) \left[f^a (s) - f (s)\right] ds\right\| \\ &\quad \le \sup_{0\le t\le b} \left\|\int_0^b (b-s)^{\alpha-1} T_\alpha (b-s) \left[f^\alpha (s) - f (s)\right] ds\right\|. \end{aligned} \tag{93}$$

By the Ascoli-Arzela theorem, we can show that the linear operator $g \to \int_0^{\cdot} (\cdot - s)^{\alpha-1} T_\alpha(\cdot - s) g(s) ds : L^{1/\gamma}(J, X) \to PC_{1-\alpha}(J, X)$ is compact; consequently the right-hand side of (93) tends to zero as $a \to 0^+$.

This implies that

$$\begin{aligned} &\left\|x^a (b) - x_1\right\| \\ &\quad = \left\|aR\left(a, \Gamma_0^b\right) P\left(f^a\right)\right\| \\ &\quad \le \left\|aR\left(a, \Gamma_0^b\right)(h)\right\| + \left\|aR\left(\alpha, \Gamma_0^b\right)\left(P\left(f^a\right) - h\right)\right\| \\ &\quad \le \left\|aR\left(a, \Gamma_0^b\right)(h)\right\| + \left\|P\left(f^a\right) - h\right\| \longrightarrow 0, \quad \text{as } a \longrightarrow 0^+. \end{aligned} \tag{94}$$

This proves the approximate controllability of system (4). □

Remark 16. In [9], Ganesh et al. have studied the approximate controllability of fractional integrodifferential evolution equations. If our problem (4) can be changed as

$$\begin{aligned} &D_t^\alpha x (t) \in Ax (t) + Bu (t) + F (t, x (t), (Hx) (t)), \\ &\qquad t \in (0, b], \quad \frac{1}{2} < \alpha \le 1, \\ &I_{0^+}^{1-\alpha} x (t)\Big|_{t=0} = x_0 + g (x) \in X, \end{aligned} \tag{95}$$

where D_t^α denotes the Riemann-Liouville fractional derivative, the operator H is defined by $(Hx)(t) = \int_0^b h(t, s, x(s))ds$. By a similar way, we can get the approximate controllability of the system (95).

Remark 17. In [15], Rathinasamy and Yong have researched the approximate controllability of fractional differential equations with state-dependent delay. Our problem (4) can be adopted to the impulsive fractional differential inclusion system with state-dependent delay:

$$\begin{aligned} &D_t^\alpha x (t) \in Ax (t) + Bu (t) + F\left(t, x_{\rho(t,x_t)}\right), \\ &\qquad t \in (0, b], \quad \frac{1}{2} < \alpha \le 1, \\ &\Delta I_{0^+}^{1-\alpha} x (t)\Big|_{t=t_k} = I_k \left(x\left(t_k^-\right)\right), \quad k = 1, 2, \dots, m, \\ &\qquad I_{0^+}^{1-\alpha} x (t)\Big|_{t=0} = \phi \in X, \end{aligned} \tag{96}$$

where D_t^α is the Riemann-Liouville fractional derivative, A is the infinitesimal generator of a C_0-semigroup $\{T(t), t \ge 0\}$ on a Banach space X, the state $x(\cdot)$ takes values in X, the control function $u(\cdot)$ is given in $L^p(J, U)$ $(p > 1/\alpha)$, U is a Banach space of admissible control, and $B : U \to X$ is a bounded linear operator. $F : J \times D \to \mathscr{P}(X) := 2^X \setminus \{\emptyset\}$ with $D := C((-\infty, 0], X)$ is a multivalued map. For a continuous function $x : J^* := (-\infty, b] \to X$, x_t is the element of D defined by $x_t(s) = x(t + s)$, $-\infty \le s \le 0$, and $\phi \in D$. Further, $\rho : [0, b] \times \mathscr{P}(X) \to (-\infty, b]$ are appropriate nonlinear functions.

4. An Example

In this final section, we give an example to illustrate our abstract results.

Example 1. Consider the following fractional partial differential inclusion with control

$$\begin{aligned} &D_t^\alpha x (t, y) \in \frac{\partial^2}{\partial y^2} x (t, y) + F (t, x (t, y)) + Bu (t, y), \\ &\qquad t \in J' = [0, 1] \setminus \left\{\frac{1}{2}\right\}, \quad y \in [0, \pi], \\ &\Delta I_{0^+}^{1-\alpha} x \left(\frac{1}{2}, y\right) = \frac{\left|x\left(y\right)\right|}{2\pi + \left|x\left(y\right)\right|}, \quad y \in [0, \pi], \\ &\qquad x (t, 0) = x (t, \pi) = 0, \quad t \in J = [0, 1], \\ &I_{0^+}^{1-\alpha} x (t, y)\Big|_{t=0} = x_0 \left(y\right), \quad t \in [0, 1], \ y \in [0, \pi], \end{aligned} \tag{97}$$

where D^α is the Riemann-Liouville fractional partial derivative of order $3/5$, $J = [0, 1]$.

Let $X = U = L^2[0, \pi]$, define; the operator A by $Ax = x_{yy}$ with the domain

$$\begin{aligned} D (A) = \{&x \in X : x, x' \text{ are absolutely continuous}, \\ &x_{yy} \in X, x (t, 0) = x (t, \pi) = 0\}. \end{aligned} \tag{98}$$

Then,

$$Ax = -\sum_{n=1}^{\infty} n^2 \langle x, e_n \rangle e_n, \quad x \in D(A), \tag{99}$$

where $e_n(y) = \sqrt{2/\pi} \sin ny$, $0 \le y \le \pi$, $n = 1, 2, \ldots$, is the orthogonal set of eigenvectors of A. It is easily shown that operator A generates a strongly continuous semigroup $\{T(t) : t \ge 0\}$ on X, which are compact, analytic, and self-adjoint in X.

It can be easily seen that the linear system corresponding to (97) is approximately controllable on $[0, 1]$ [42].

Define $x(t)(y) = x(t, y)$; let $B : U \to X$ be defined by $Bu(t)(y) = Bu(t, y)$. Then system (97) can be written in the abstract form given by (4). We assume that $F : J \times D \to \mathscr{P}(X)$ satisfy the following conditions.

(F_1): $F : J \times D \to \mathscr{P}_{cp,cv}(X)$ is measurable to t for each fixed $x \in D$, u.s.c. to x for a.e. $t \in J$, and for each $x \in PC_{1-\alpha}(J, X)$ the set

$$S_{F,x} = \left\{ f \in L^1(J, X) : f(t) \in F(t, x) \right\} \tag{100}$$

is nonempty.

(F_2): There exists a positive constant L such that $\|F(t, x)\| \le L$ for all $(t, x) \in J \times D$, since, for any $x, y \in AC(J, X)$, we have

$$\left\| \Delta I_t^{1-\alpha} x(t_k, z) - \Delta I_t^{1-\alpha} y(t_k, z) \right\| \le \frac{\|x - y\|}{2}. \tag{101}$$

Obviously, the hypotheses of Theorem 15 are fulfilled. Thus, the system (97) is approximately controllable on $[0, 1]$.

5. Conclusions

In this paper, we study the approximate controllability for impulsive fractional differential inclusions with Riemann-Liouville fractional derivatives in Banach spaces. We mainly use fixed point theorem and semigroup theory; we obtain some new mild solutions for this kind of problems (4). An illustrative example is also discussed to show the effectiveness of the results in this paper. In the near future, we will consider Riemann-Liouville fractional partial differential problems, which will be more complicated.

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