Research Article

Structure of Pareto Solutions of Generalized Polyhedral-Valued Vector Optimization Problems in Banach Spaces

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In general Banach spaces, we consider a vector optimization problem (SVOP) in which the objective is a set-valued mapping whose graph is the union of finitely many polyhedra or the union of finitely many generalized polyhedra. Dropping the compactness assumption, we establish some results on structure of the weak Pareto solution set, Pareto solution set, weak Pareto optimal value set, and Pareto optimal value set of (SVOP) and on connectedness of Pareto solution set and Pareto optimal value set of (SVOP). In particular, we improved and generalize, Arrow, Barankin, and Blackwell's classical results in Euclidean spaces and Zheng and Yang's results in general Banach spaces.

1. Introduction

Let X and Y be Banach spaces, Γ a closed convex subset of X, C a closed convex cone of Y, $b \in Y$, and $T : X \to Y$ a linear function. Consider the following optimization problem:

$$C - \min \{T(x) + b\} \quad \text{subject to } x \in \Gamma. \quad \text{(LMOP)}$$

Linear multiobjective optimization problem (LMOP) has been extensively studied and applied to various decision-making problems in economics, management science, and engineering (see [1–9] and references therein). One of the important topics in vector optimization is the study of the structure of Pareto solution sets. In 1953, Arrow et al. [2] studied the structure of the Pareto solution set, weak Pareto solution set, Pareto optimal value set, and weak Pareto optimal value set for a linear vector optimization problem in Euclidean spaces. In particular, let $X = \mathbb{R}^m$, $Y = \mathbb{R}^n$, and $C = \mathbb{R}_+^n$. If Γ is a polyhedron of R^m and the Pareto solution set S of (LMOP) is nonempty, then (i) S and the weak Pareto solution set S_w of (LMOP) are the unions of finitely many polyhedra of R^m; (ii) the Pareto optimal value set V and the weak Pareto optimal value set V_w are the unions of finitely many polyhedra of R^n. This theorem is well known as ABB theorem. Since the family of all piecewise linear functions

is much larger than that of all linear functions and there exists a wide class of functions that can be approximated by piecewise linear functions, it is of value to study piecewise linear multiobjective optimization (cf. [10–12]). Recently, In Banach spaces setting, Zheng and Yang [11] generalized ABB theorem to the case when the objective is a piecewise linear single-valued mapping. Note that the graph of a piecewise linear function is the union of finitely many polyhedra. Zheng [13] considered the following vector optimization problem:

$$C - \min F(x) \quad \text{subject to } x \in \Gamma, \quad \text{(SVOP)}$$

where $F : X \rightrightarrows Y$ is a multifunction whose graph is the union of finitely many convex polyhedra and Γ is a polyhedron of X. The following results of the structure of (weak) Pareto solution sets were obtained.

Theorem Z 1 (see [13]). *Let X and Y be Banach spaces, Γ the union of finitely many polyhedra of X, and $F : \Gamma \rightrightarrows Y$ a multifunction whose graph $\mathrm{Gr}(F)$ is the union of finitely many convex polyhedra of $X \times Y$. Suppose that the ordering cone $C \subset Y$ is closed, convex, pointed, and has a weakly compact base. If $F(\Gamma) + C$ is convex, then the Pareto solution set S and the Pareto optimal value set V of (SVOP) are the unions of finitely many polyhedra of X and Y, respectively.*

Theorem Z 2 (see [13]). *Let X and Y be Banach spaces, Γ the union of finitely many polyhedra of X, and $F : \Gamma \rightrightarrows Y$ a multifunction whose graph $\mathrm{Gr}(F)$ is the union of finitely many convex polyhedra of $X \times Y$. Suppose that the ordering cone $C \subset Y$ is closed, convex, and pointed. If the interior $\mathrm{int}(C)$ of C is nonempty, then the weak Pareto solution set S_w and the Pareto optimal value set V_w of (SVOP) are the unions of finitely many polyhedra of X and Y, respectively.*

A polyhedron is defined [12–15] as the intersection of finitely many closed half-spaces. Polyhedra exist in many contexts such as linear and quadratic programs, game theory, statistical decision theory, and mathematical biology as well. For more details on the theory of polyhedra, we refer the reader to [12, 14] and the references therein. A generalized polyhedron (G-polyhedron or semiclosed polyhedron) in [13, 15] is defined as the intersection of finitely many closed and/or open half-spaces. A G-polyhedron can be regarded as an extension of a polyhedron. It has nice properties analogous to a polyhedron; see [13, 15] and the references therein. It is necessarily noted that the graph $\mathrm{Gr}(F)$ of a multifunction F is closed when it is the union of finitely many polyhedra, while the graph $\mathrm{Gr}(F)$ is not necessarily closed when it is the union of finitely many G-polyhedra. In the case that F is single valued, the former implies that F is a piecewise continuous linear function but the latter does not imply that F is necessarily a piecewise continuous linear function. G-polyhedra are important in piecewise linear programs as well as polyhedra. For example, Fang et al. [15] proved that (weak) Pareto solution set of a piecewise linear multicriteria program with possible discontinuity is a union of finitely many G-polyhedra.

One of our main aims in this work is to investigate the structure of the (weak) Pareto solution set and the (weak) Pareto optimal set of (SVOP) whose graph $\mathrm{Gr}(F)$ is the union of finitely many G-polyhedra.

Another topic in linear optimization problems is to study the connectedness of (weak) Pareto solution sets. Many authors researched this issue; see [2, 11, 13, 16–20] and the references therein. Arrow et al. [2] proved that the (weak) Pareto solution set (S_w) S and the (weak) Pareto optimal value set (V_w) V of (LMOP) are pathwise connected, respectively, when $X = R^m$, $Y = R^n$, and $C = R^n_+$. Recently, Zheng and Yang [11] proved that the weak Pareto set S_w and the weak Pareto optimal value set V_w of (LMOP) are pathwise connected, respectively, when the ordering cone C has a nonempty interior and the mapping $T(\cdot) + b$ is C-convex. Zheng [13] proved that the Pareto set S and the Pareto optimal value set V of (SVOP) are pathwise connected, respectively, when the ordering cone C is a pointed, closed, convex cone with a weakly compact base and F is a C-convex multifunction whose graph is the union of finitely many convex polyhedra.

The other of our main aims is to study the connectedness of the Pareto set S and the Pareto optimal value set V of (SVOP) without the assumption of the ordering cone C having a weakly compact base but with that of the cone C being polyhedral.

2. Preliminaries

Let X and Y be Banach spaces and let $C \subset Y$ be a convex cone. We say that C is pointed if $C \cap -C = \{0\}$. In this case, one can define a partial order $\leq_C$ in Y as follows: for $y_1, y_2 \in Y$, $y_1 \leq_C y_2$ if and only if $y_2 - y_1 \in C$. Let $\mathrm{int}(C)$ denote the interior of C. When $\mathrm{int}(C) \neq \emptyset$, by $y_1 <_C y_2$, we mean that $y_2 - y_1 \in \mathrm{int}(C)$. Let Y^* denote the dual space of Y and C^+ the dual cone of C, defined by

$$C^+ := \{c^* \in Y^* : \langle c^*, c \rangle \geq 0, \forall c \in C\}. \tag{1}$$

We denote by C^{+1} the set of all strictly positive continuous linear functionals; that is,

$$C^{+1} := \{c^* \in Y^* : \langle c^*, c \rangle > 0, \forall c \in C \setminus \{0\}\}. \tag{2}$$

We say a convex subset Θ of C is a base of C if it satisfies that

$$C = \{t\theta : t \in \mathbb{R}_+, \theta \in \Theta\}, \quad 0 \notin \mathrm{cl}(\Theta), \tag{3}$$

where $\mathrm{cl}(\cdot)$ denotes the closure. C is said to have a bounded (resp., weakly compact) base, if it has a base that is bounded (resp., weakly compact). It is known that $C^{+1} \neq \emptyset$ if and only if C has a base.

Let A be a subset of A and a in A. As usual, we denote by $\mathrm{E}(A, C)$, $\mathrm{WE}(A, C)$, and $\mathrm{Pos}(A, C)$ the set of all Pareto points of A, the set of all weak Pareto points of A, and the set of all positively proper Pareto points of A, respectively; that is,

$$a \in \mathrm{E}(A, C) \iff (a - C) \cap A = \{a\},$$

$$a \in \mathrm{WE}(A, C) \iff (a - \mathrm{int}(C)) \cap A = \emptyset,$$

$$a \in \mathrm{Pos}(A, C)$$

$$\iff \exists\, c^* \in C^{+1} \text{ satisfying } \langle c^*, y - a \rangle \geq 0 \quad \forall y \in A. \tag{4}$$

It is clear that

$$\mathrm{Pos}(A, C) \subset \mathrm{E}(A, C) = \mathrm{E}(A + C, C) \subset \mathrm{WE}(A, C). \tag{5}$$

For a multifunction $F : X \rightrightarrows Y$, we denote by $\mathrm{Gr}(F)$ and $\mathrm{epi}_C(F)$ the graph and C-epigraph of F, respectively; that is,

$$\mathrm{Gr}(F) := \{(x, y) : x \in X, y \in F(x)\},$$
$$\mathrm{epi}_C(F) := \{(x, y) : x \in X, y \in F(x) + C\}. \tag{6}$$

We say that F is C-convex, if $\mathrm{epi}_C(F)$ is convex. Obviously, F is C-convex if and only if

$$tF(x_1) + (1 - t)F(x_2) \subset F(tx_1 + (1 - t)x_2) + C$$
$$\forall x_1, x_2 \in X, \forall t \in [0, 1]. \tag{7}$$

Recall [14] that a subset P of a Banach space Z is a (convex) polyhedron, if there exist $u_1^*, \ldots, u_m^* \in Z^*$ and $s_1, \ldots, s_m \in \mathbb{R}$ such that

$$P = \{z \in Z : \langle u_i^*, z \rangle \leq s_i, i = 1, \ldots, m\}. \tag{8}$$

A subset Q of Z is said to be a G-polyhedron, if Q is a polyhedron, or there exist a polyhedron P of Z and $v_1^*, \ldots, v_m^* \in Z^*$, $r_1, \ldots, r_m \in \mathbb{R}$ such that

$$Q = P \cap \{z \in Z : \langle v_i^*, z \rangle < r_i, \ i = 1, \ldots, m\}; \quad (9)$$

see [11, 13, 15]. Clearly, each polyhedron is closed but a G-polyhedron is not necessarily closed. From the definitions, it is easy to verify that if Q_1 and Q_2 are G-polyhedra of Banach space Z, $Q_1 \cap Q_2$ is a G-polyhedron of Z, and $Q_1 \setminus Q_2$ is the union of finitely many G-polyhedra of Z.

In this paper, we will consider set-valued vector optimization problem (SVOP). In the remainder of the paper, we always assume that the graph $\mathrm{Gr}(F)$ of the objective mapping F is the union of finitely many G-polyhedra or polyhedra of $X \times Y$.

We say that $\overline{x} \in \Gamma$ is a Pareto (resp., weak Pareto or positively proper Pareto) solution of (SVOP), if there exists $\overline{y} \in F(\overline{x})$ such that $\overline{y} \in \mathrm{E}(F(\Gamma), C)$ (resp., $\overline{y} \in \mathrm{WE}(F(\Gamma), C)$ or $\overline{y} \in \mathrm{Pos}(F(\Gamma), C)$); in this case, we say that $\overline{y}$ is a Pareto (resp., weak Pareto or positively proper Pareto) optimal value of (SVOP). Let S, S_w, and S_p, respectively, denote the sets of all Pareto, weak Pareto and positively proper Pareto solutions of (SVOP). Let V, V_w, and V_p, respectively, denote the sets of all Pareto, weak Pareto, and positively proper Pareto optimal values of (SVOP). Obviously,

$$S_p \subset S \subset S_w, \qquad V_p \subset V \subset V_w,$$
$$S_p = P \cap F^{-1}(V_p), \qquad S = \Gamma \cap F^{-1}(V), \quad (10)$$
$$S_w = \Gamma \cap F^{-1}(V_w).$$

Next, we provide some properties on the union of finitely many G-polyhedra in general Banach spaces, which generalize the corresponding results in [13].

Lemma 1. *A subset P of $X \times Y$ is the union of finitely many G-polyhedra (resp., polyhedra) if and only if there exist closed subspaces X_1 and X_2 of X, closed subspaces Y_1 and Y_2 of Y, and the union of finitely many G-polyhedra (resp., polyhedron) P_2 of $X_2 \times Y_2$ such that*

$$X \times Y = X_1 \times Y_1 + X_2 \times Y_2,$$
$$(X_1 \times Y_1) \cap (X_2 \times Y_2) = \{(0,0)\},$$
$$\max \{\dim (X_2), \dim (Y_2)\} < +\infty, \quad (11)$$
$$P = X_1 \times Y_1 + P_2.$$

The proof of Lemma 1 is similar to that for polyhedron or G-polyhedron in [13] and is omitted.

If $Y = \{0\}$ in Lemma 1, we have the following corollary.

Corollary 2. *A subset P of X is the union of finitely many G-polyhedra (resp., polyhedra) if and only if there exist closed*

subspaces X_1 and X_2 of X and the union of finitely many G-polyhedra (resp., polyhedra) P_2 of X_2 such that

$$X = X_1 + X_2, \qquad X_1 \cap X_2 = \{0\}, \qquad \dim (X_2) < +\infty,$$
$$P = X_1 + P_2. \quad (12)$$

For a subset A of $X \times Y$, let

$$A_X := \{x \in X : \exists y \in Y \text{ s.t. } (x, y) \in A\},$$
$$A_Y := \{y \in Y : \exists x \in X \text{ s.t. } (x, y) \in A\}. \quad (13)$$

Lemma 3 (see [13]). *Let P be a G-polyhedron (resp., polyhedron) of $X \times Y$. Then P_X is the union of finitely many G-polyhedra (resp., polyhedra) of X.*

Corollary 4. *Let $G : X \rightrightarrows Y$ be a multifunction whose graph is a convex G-polyhedron (resp., polyhedra) of $X \times Y$. Let P and Q be convex G-polyhedra (resp., polyhedra) of X and Y, respectively. Then $G(P)$ and $G^{-1}(Q)$ are the union of finitely many G-polyhedra (resp., polyhedra) of Y and X, respectively.*

Noting that

$$G(P) = (\mathrm{Gr}(G) \cap (P \times Y))_Y,$$
$$G^{-1}(Q) = (\mathrm{Gr}(G) \cap (X \times Q))_X, \quad (14)$$

Corollary 4 is a consequence of Lemma 3.

Lemma 5. *Let A be a subset of Y with a closed convex cone C. Then*

$$\mathrm{E}(A, C) = A \setminus (A + C \setminus \{0\}). \quad (15)$$

Proof. Let $a \in A$. Then, one can easily verify that

$$(a - C \setminus \{0\}) \cap A = \emptyset \Longleftrightarrow a \in A \setminus (A + C \setminus \{0\}). \quad (16)$$

Noting that

$$a \in \mathrm{E}(A, C) \Longleftrightarrow (a - C \setminus \{0\}) \cap A = \emptyset, \quad (17)$$

it follows that (15) holds. The proof is completed. $\square$

Lemma 6. *Let A be a subset of Banach space Y with the ordering cone C. Then,*

$$\mathrm{WE}(A, C) = A \cap \mathrm{WE}(\mathrm{cl}\, A, C). \quad (18)$$

Proof. It is obvious that $A \cap \mathrm{WE}(\mathrm{cl}\, A, C) \subseteq \mathrm{WE}(A, C)$. We only need to show that $\mathrm{WE}(A, C) \subseteq A \cap \mathrm{WE}(\mathrm{cl}\, A, C)$. Let $x \in \mathrm{WE}(A, C)$. Then $(x - \mathrm{int}(C)) \cap A = \emptyset$; that is

$$A \subset Y \setminus (x - \mathrm{int}(C)). \quad (19)$$

Since $x - \mathrm{int}(C)$ is open, one has $\mathrm{cl}(A) \subset Y \setminus (x - \mathrm{int}(C))$ and hence,

$$(x - \mathrm{int}(C)) \cap \mathrm{cl}(A) = \emptyset. \quad (20)$$

Thus $x \in \mathrm{WE}(\mathrm{cl}\, A, C)$. The proof is completed. $\square$

We will also use the following lemma [13, Proposition 2.1] in the sequel.

Lemma 7. *Let Y_1 and Y_2 be closed subspaces of a Banach space Y such that*

$$Y = Y_1 + Y_2, \qquad Y_1 \cap Y_2 = \{0\}. \qquad (21)$$

Let $A := Y_1 + A_2$ for some subset A_2 of Y_2. Let C_2 be the projection of the ordering cone C on Y_2; that is,

$$C_2 = \{c_2 \in Y_2 : y_1 + c_2 \in C \text{ for some } y_1 \in Y_1\}. \qquad (22)$$

Then, the following assertions hold.

(i) $Y_1 + Pos(A_2, C_2) \subset Pos(A, C)$ *if* $Y_1 \cap C = \{0\}$.

(ii) $\mathrm{E}(A, C) = \emptyset$ *if* $Y_1 \cap C \neq \{0\}$.

(iii) $\mathrm{E}(A, C) = Y_1 + \mathrm{E}(A_2, C_2)$ *if* $Y_1 \cap C = \{0\}$.

(iv) $WE(A, C) = \emptyset$ *if* $Y_1 \cap \mathrm{int}(C) \neq \emptyset$.

(v) $WE(A, C) = Y_1 + WE(A_2, C_2)$ *if* $\mathrm{int}(C) \neq \emptyset$ *and* $Y_1 \cap \mathrm{int}(C) = \emptyset$.

Proof. For the proofs of (ii)–(v), see [13, Proposition 2.1]. We only need to show (i).

For (i), we assume that $Y_1 \cap C = \{0\}$. Let $y_1 \in Y_1$ and $y_2 \in Pos(A_2, C_2)$. Then there exists $\widetilde{c}^* \in C_2^{+1}$ such that

$$\langle \widetilde{c}^*, a_2 - y_2 \rangle \geq 0 \quad \forall a_2 \in A_2. \qquad (23)$$

Define c^* by

$$\langle c^*, x_1 + x_2 \rangle := \langle \widetilde{c}^*, x_2 \rangle, \quad \forall (x_1, x_2) \in Y_1 \times Y_2. \qquad (24)$$

Then c^* is well defined and it is easy to verify that $c^* \in C^{+1}$. On the other hand, (23) and (24) imply that

$$\langle c^*, x_1 + a_2 - (y_1 + y_2) \rangle \geq 0 \quad \forall x_1 \in Y_1, \ \forall a_2 \in A_2. \quad (25)$$

Hence $y_1 + y_2 \in Pos(A, C)$. □

Remark 8. In view of Lemma 7, it is practical to investigate some topics on (SVOP) in a finite dimensional framework. This is our main ideal to consider (SVOP) in general Banach spaces.

3. The Structure of Solution Sets and Optimal Value Sets

In this section, our aim is to investigate the structure of the (weak) Pareto solution set and the (weak) Pareto optimal set of (SVOP) whose graph $\mathrm{Gr}(F)$ is the union of finitely many G-polyhedra. Throughout the remainder of this paper, we assume that S_w, S, and S_p are nonempty.

Let Y be a Banach space. For $y^* \in Y^*$ and $A \subset Y$, let

$$\lambda_{y^*}(A) := \inf_{a \in A} \langle y^*, a \rangle,$$

$$L_{y^*}(A) := \left\{ a \in A : \langle y^*, a \rangle = \inf_{y \in A} \langle y^*, y \rangle \right\}. \qquad (26)$$

It is easy to verify that

$$\mathrm{Pos}(A, C) = \bigcup_{y^* \in C^{+1}} L_{y^*}(A). \qquad (27)$$

In addition, if $A + C$ is convex and C has a nonempty interior, by [21, Corollary 5.29], one has

$$\mathrm{WE}(A, C) = \bigcup_{y^* \in C^+ \backslash \{0\}} L_{y^*}(A). \qquad (28)$$

The following lemma is also useful in our later analysis and can be found in [13].

Lemma 9. *Let Y be a Banach space with the ordering cone C which has a weakly compact base. Let A be the union of finitely many polyhedra of Y. Suppose that $A + C$ is convex and that $\mathrm{E}(A, C)$ is nonempty. Then $\mathrm{E}(A, C)$ is the union of finitely many convex polyhedra. More precisely, there exist $y_1^*, \ldots, y_p^* \in C^{+1}$ such that*

$$\mathrm{E}(A, C) = Pos(A, C) = \bigcup_{i=1}^{p} L_{y_i^*}(A). \qquad (29)$$

We have the following proposition.

Proposition 10. *Let Y be a Banach space with the ordering cone C and let $A := \bigcup_{i=1}^{k} A_i$, $A_i \subset Y$. Set*

$$E_i$$

$$:= \mathrm{E}(A_i, C) \setminus \bigcup_{i \leq j \leq k, j \neq i} \left[(A_j + C) \setminus \left(\mathrm{E}(A_i, C) \cap \mathrm{E}(A_j, C) \right) \right]. \qquad (30)$$

Then, $\mathrm{E}(A, C) = \bigcup_{i=1}^{k} E_i$.

Proof. Let $x \in \mathrm{E}(A, C)$. Then, there exists an integer $i_0 \in [1 - k]$ such that $x \in \mathrm{E}(A_{i_0}, C)$. For each j with $x \in A_j + C$, noting that

$$\left(A_j + C \right) \cap \mathrm{E}(A, C)$$

$$= \left(A_j + C \right) \cap \mathrm{E}(A + C, C) \subset \mathrm{E}\left(A_j + C, C \right) \qquad (31)$$

$$= \mathrm{E}\left(A_j, C \right),$$

one has $x \in \mathrm{E}(A_j, C)$ and $x \in \mathrm{E}(A_{i_0}, C) \cap \mathrm{E}(A_j, C)$. It follows that

$$x \notin \left(A_j + C \right) \left(\mathrm{E}\left(A_{i_0}, C \right) \cap \mathrm{E}\left(A_j, C \right) \right) \qquad (32)$$

$$\text{for each integer } j \in [1 - k].$$

Therefore, one has $x \in E_{i_0}$ which implies that $\mathrm{E}(A, C) \subset \bigcup_{i=1}^{k} E_i$.

Conversely, let $x \in \bigcup_{i=1}^{k} E_i$. Then, $x \in E_i$ for some integer i in $[1 - k]$. Let $z \in A \cap (x - C)$. We only need to show that $z = x$. By the definition of A, there exists an integer j in $[1-k]$ such that $z \in A_j$. Noting that $E_i \subset \mathrm{E}(A_i, C)$, one has $z = x$

if $j = i$. Now suppose that $j \neq i$. Then, by the definition of E_i, it follows that $x \notin (A_j + C) \setminus (\mathrm{E}(A_i, C) \cap \mathrm{E}(A_j, C))$. Since $x \in z + C \subset A_j + C$, one has $x \in \mathrm{E}(A_j, C)$. It follows that $x = z$. The proof is completed. $\qquad\square$

Lemma 11. *Let Y be a Banach space, let A be the union of finitely many G-polyhedra of Y, and let Λ be a subset of Y^*. Then, $\bigcup_{y^* \in \Lambda} L_{y^*}(A)$ is the union of finitely many G-polyhedra of Y and, more precisely, there exists a finite subset Λ_0 of Λ such that*

$$\bigcup_{y^* \in \Lambda} L_{y^*}(A) = \bigcup_{y^* \in \Lambda_0} L_{y^*}(A). \tag{33}$$

Proof. By the assumptions, there exist finitely many G-polyhedra $P_1, \ldots, P_n$ such that $A = \bigcup_{i=1}^n P_i$. Then for each $i \in [1 - n]$, let

$$\Lambda_i := \left\{ y^* \in \Lambda : P_i \cap L_{y^*}(A) \neq \emptyset \right\}. \tag{34}$$

Then

$$\bigcup_{y^* \in \Lambda} L_{y^*}(A) = \bigcup_{i=1}^n \left(\bigcup_{y^* \in \Lambda_i} P_i \cap L_{y^*}(A) \right). \tag{35}$$

Let $y^* \in \Lambda_i$. Since $\lambda_{y^*}(A) = \lambda_{y^*}(\mathrm{cl}(A))$ and $\mathrm{cl}(P_i)$ is a polyhedron, (26) implies that $\mathrm{cl}(P_i) \cap L_{y^*}(A)$ is a face of $\mathrm{cl}(P_i)$. Noting that every polyhedron has finitely many faces. Hence, there exists a finite subset $\widetilde{\Lambda}_i$ of Λ_i, such that

$$\begin{aligned} &\left\{ \mathrm{cl}(P_i) \cap L_{y^*}(A) : y^* \in \Lambda_i \right\} \\ &= \left\{ \mathrm{cl}(P_i) \cap L_{y^*}(A) : y^* \in \widetilde{\Lambda}_i \right\}. \end{aligned} \tag{36}$$

Now, we show that for any $y_1^*, y_2^* \in \Lambda_i$,

$$\begin{aligned} \mathrm{cl}(P_i) \cap L_{y_1^*}(A) &= \mathrm{cl}(P_i) \cap L_{y_2^*}(A) \\ \text{implies } P_i \cap L_{y_1^*}(A) &= P_i \cap L_{y_2^*}(A). \end{aligned} \tag{37}$$

If (37) holds, by (36), we have

$$\bigcup_{y^* \in \Lambda_i} P_i \cap L_{y^*}(A) = \bigcup_{y^* \in \widetilde{\Lambda}_i} P_i \cap L_{y^*}(A). \tag{38}$$

From (35), it follows that

$$\bigcup_{y^* \in \Lambda} L_{y^*}(A) = \bigcup_{i=1}^n \left(\bigcup_{y^* \in \widetilde{\Lambda}_i} P_i \cap L_{y^*}(A) \right). \tag{39}$$

Let $\Lambda_0 := \bigcup_{i=1}^n \widetilde{\Lambda}_i$. Noting that $P_i \cap L_{y^*}(A) \subset L_{y^*}(A)$, we have that (33) holds. Hence, we only need to show (37). Suppose there would exist $y_1^*, y_2^* \in \Lambda_i$ such that $\mathrm{cl}(P_i) \cap L_{y_1^*}(A) = \mathrm{cl}(P_i) \cap L_{y_2^*}(A)$, but $P_i \cap L_{y_1^*}(A) \neq P_i \cap L_{y_2^*}(A)$. Let $y \in P_i \cap L_{y_1^*}(A)$ with $y \notin P_i \cap L_{y_2^*}(A)$. By (26) and the definition of Λ_i, we have $\langle y_2^*, y \rangle > \lambda_{y^*}(A)$, which implies that $y \notin \mathrm{cl}(P_i) \cap \lambda_{y_2^*}(A) = \mathrm{cl}(P_i) \cap \lambda_{y_1^*}(A)$. This is a contradiction. The proof is completed. $\qquad\square$

Theorem 12. *Let X and Y be Banach spaces, Γ a G-polyhedron, and $\mathrm{Gr}(F)$ the union of finitely many G-polyhedra of $X \times Y$ and let the ordering cone C have nonempty interior. Suppose that $F(\Gamma) + C$ is convex. Then, S_w and V_w are the union of finitely many G-polyhedra of X and Y, respectively. More precisely, there exist $y_i^* \in C^+$ with $\|y_i^*\| = 1$ ($i \in [i - q]$ for some integer q) such that*

$$V_w = \bigcup_{y^* \in C^+ \setminus \{0\}} L_{y^*}(F(\Gamma)) = \bigcup_{i=1}^q L_{y_i^*}(F(\Gamma)). \tag{40}$$

Proof. Since $F(\Gamma) + C$ is convex, by (28), we have

$$\mathrm{WE}(F(\Gamma) + C, C) = \bigcup_{y^* \in C^+ \setminus \{0\}} L_{y^*}(F(\Gamma) + C). \tag{41}$$

Noting that $\mathrm{WE}(F(\Gamma), C) = F(\Gamma) \cap \mathrm{WE}(F(\Gamma) + C, C)$, we have

$$\begin{aligned} &\mathrm{WE}(F(\Gamma), C) \\ &= \bigcup_{y^* \in C^+ \setminus \{0\}} F(\Gamma) \cap L_{y^*}(F(\Gamma) + C) \\ &= \bigcup_{y^* \in C^+ \setminus \{0\}} L_{y^*}(F(\Gamma)). \end{aligned} \tag{42}$$

Since the feasible set Γ is a G-polyhedron of X and $\mathrm{Gr}(F)$ is the union of finitely many G-polyhedra of $X \times Y$, Corollary 4 implies that $F(\Gamma)$ is the union of finitely many G-polyhedra of Y. Since $V_w = \mathrm{WE}(F(\Gamma), C)$, it follows from (26) and Lemma 11 that V_w is the union of finitely many G-polyhedra of Y and there exist $y_i^* \in C^+$ with $\|y_i^*\| = 1$ ($i \in [i - q]$ for some integer q) such that (40) holds. On the other hand, from Corollary 4, one can easily see that $S_w = \Gamma \cap F^{-1}(V_w)$ is the union of finitely many G-polyhedra of X. The proof is completed. $\qquad\square$

By Lemma 3, dropping convexity of $F(\Gamma) + C$ but requiring that the ordering cone C is polyhedral, we generalize Theorem 3.3 in [13] to the union of finitely many G-polyhedra setting.

Theorem 13. *Let X and Y be Banach spaces. Let the ordering cone C be polyhedral and have nonempty interior. Suppose that Γ and $\mathrm{Gr}(F)$ are the unions of finitely many G-polyhedra of X and $X \times Y$, respectively. Then, S_w and V_w are the unions of finitely many G-polyhedra of X and Y, respectively.*

Proof. Suppose that $\mathrm{Gr}(F)$ is the union of finitely many G-polyhedra of $X \times Y$. Since $F(\Gamma) = (\mathrm{Gr}(F) \cap (\Gamma \times Y))_Y$, by Lemma 3, one has that $\mathrm{cl}(F(\Gamma))$ is the union of finitely many polyhedra of Y. It is similar to the proof of Theorem 3.3 in [13]; we can show that $\mathrm{WE}(\mathrm{cl}(F(\Gamma)))$ is the union of finitely many polyhedra of Y. By Lemma 6, one has

$$V_w = \mathrm{WE}(F(\Gamma), C) = F(\Gamma) \cap \mathrm{WE}(\mathrm{cl}(F(\Gamma)), C) \tag{43}$$

which implies that V_w is the union of finitely many G-polyhedra of Y. Noting that $S_w = (\mathrm{Gr}(F) \cap (\Gamma) \times V_w)_X$, Lemma 3 implies that S_w is the union of finitely many G-polyhedra of X. The proof is completed. $\qquad\square$

Dropping the assumption of the ordering cone C having a weakly compact base but inquiring C being a polyhedral cone, we have the following result analogous to that of Lemma 11.

Lemma 14. *Let Y be a Banach space, let the ordering cone C be pointed and polyhedral, and let A be the union of finitely many polyhedra of Y. Suppose that $A + C$ is convex and that $\mathrm{E}(A, C)$ is nonempty. Then $\mathrm{E}(A, C)$ is the union of finitely many polyhedra of Y. More precisely, there exist $c_1^*, \ldots, c_p^* \in C^{+1}$ such that*

$$\mathrm{E}(A, C) = \mathrm{Pos}(A, C) = \bigcup_{i=1}^{p} L_{c_i^*}(A), \tag{44}$$

$$L_{c_i^*}(\mathrm{cl} \ \mathrm{co} \ A) = L_{c_i^*}(A). $$

Proof. By Corollary 2 there exist a closed subspace Y_1, a finite dimensional subspace Y_2 of Y, and the union A_2 of finitely many polyhedra of Y_2 such that

$$Y = Y_1 + Y_2, \qquad Y_1 \cap Y_2 = \{0\}, \tag{45}$$

$$A = Y_1 + A_2. \tag{46}$$

Let C_2 be the projection of the ordering cone C on Y_2; that is,

$$C_2 = \{c_2 \in Y_2 : c_1 + c_2 \in C \text{ for some } c_1 \in Y_1\}. \tag{47}$$

Since C is polyhedral, together with Corollary 4, we have that the cone C_2 is the union of finitely many polyhedra of Y_2. Hence it is also closed. From the assumption of $\mathrm{E}(A,C) \neq \emptyset$, by Lemma 7(ii), we have $Y_1 \cap C = \{0\}$. From this, with Lemma 7(iii), we have $\mathrm{E}(A_2, C_2) \neq \emptyset$. By Lemma 7(i), we have

$$\mathrm{Pos}(A, C) \supset Y_1 + \mathrm{Pos}(A_2, C_2). \tag{48}$$

We claim that $\mathrm{Pos}(A_2, C_2)$ is the union of finitely many polyhedra of Y_2. For any $(x, c) \in A \times C$, there exist $(x_1, x_2) \in Y_1 \times A_2$ and $(c_1, c_2) \in Y_1 \times C_2$ such that

$$x = x_1 + x_2, \qquad c = c_1 + c_2. \tag{49}$$

Since Y_1 is a subspace of Y, we have

$$x + c = (x_1 + c_1) + x_2 + c_2 \in Y_1 + A_2 + C_2. \tag{50}$$

Hence we have

$$A + C \subset Y_1 + A_2 + C_2. \tag{51}$$

Conversely, for any $(x_1, a_2, c_2) \in Y_1 \times A_2 \times C_2$, there exists $c_1 \in Y_1$ such that $c_1 + c_2 \in C$. Hence we have

$$y_1 + a_2 + c_2$$
$$= (y_1 - c_1) + a_2 + (c_1 + c_2) \in Y_1 + A_2 + C \tag{52}$$
$$= A + C.$$

Thus we have shown that

$$A + C = Y_1 + A_2 + C_2. \tag{53}$$

Let $x_2, y_2 \in A_2 + C_2$. Then there exist $x_1, y_1 \in Y_1$ such that

$$x_1 + x_2, y_1 + y_2 \in A + C. \tag{54}$$

Since $A + C$ is convex, by (53), one has

$$\lambda(x_1 + x_2) + (1 - \lambda)(y_1 + y_2) \in Y_1 + (A_2 + C_2)$$
$$\forall \lambda \in [0, 1]. \tag{55}$$

Noting that

$$\lambda x_1 + (1 - \lambda) y_1 \in Y_1,$$
$$\lambda x_2 + (1 - \lambda) y_2 \in Y_2 \quad \forall \lambda \in [0, 1], \tag{56}$$

from (45) and (55), one has

$$\lambda x_2 + (1 - \lambda) y_2 \in A_2 + C_2 \quad \forall \lambda \in [0, 1]. \tag{57}$$

Hence $A_2 + C_2$ is convex.

Let $x_2 \in C_2 \cap -C_2$. Then there exist $x_1', x_1'' \in Y_1$ such that $x_1' + x_2, x_1'' - x_2 \in C$. Since C is a convex cone, it follows that

$$x_1' + x_1'' = (x_1' + x_2) + (x_1'' - x_2) \in C. \tag{58}$$

Hence

$$x_1' + x_1'' \in Y_1 \cap C = \{0\}. \tag{59}$$

We have

$$x_1'' = -x_1', \qquad x_1' + x_2 \in C \cap -C = \{0\}. \tag{60}$$

Together with (46), we have $x_2 = 0$ and $C_2 \cap -C_2 = \{0\}$.

We have shown that C_2 is a closed, convex, and pointed cone of Y_2. Noting that Y_2 is finite dimensional, it follows that C_2 has a compact base. Applying Lemma 9, it follows that there exist $\widetilde{c}_1^*, \ldots, \widetilde{c}_p^*$ in C_2^{+1} such that

$$\mathrm{Pos}(A_2, C_2) = \mathrm{E}(A_2, C_2) = \bigcup_{i=1}^{p} L_{\widetilde{c}_i^*}(A_2). \tag{61}$$

Thus $\mathrm{Pos}(A_2, C_2)$ is the union of finitely many polyhedra of Y_2. By Lemma 7((ii), (iii)), (48), and (61), one has that

$$\mathrm{Pos}(A, C) \supset Y_1 + \mathrm{Pos}(A_2, C_2)$$
$$= Y_1 + \mathrm{E}(A_2, C_2) = \mathrm{E}(A, C). \tag{62}$$

Hence,

$$\mathrm{E}(A, C) = \mathrm{Pos}(A, C) = Y_1 + \mathrm{Pos}(A_2, C_2). \tag{63}$$

From (45), (46), and (63), applying Corollary 2, it follows that $\mathrm{Pos}(A, C)$ is the union of finitely many polyhedra of Y.

For each $\widetilde{c}_i^*$, $i \in [1 - p]$, we define c^* by

$$\langle c_i^*, y_1 + y_2 \rangle := \langle \widetilde{c}_i^*, y_2 \rangle, \quad \forall (y_1, y_2) \in Y_1 \times Y_2. \tag{64}$$

Then, one can easily show that $c_i^* \in C^{+1}$ and

$$L_{c_i^*}(A) = Y_1 + L_{\widetilde{c}_i^*}(A_2). \tag{65}$$

In fact, for any $y \in C \setminus \{0\}$, there exists a pair $(y_1, y_2) \in Y_1 \times C_2$ such that $y = y_1 + y_2$. If $y_2 = 0$, then $y = y_1 \in Y_1$ and thus $y \in Y_1 \cap C = \{0\}$, a contradiction. Hence $y_2 \neq 0$. Then we have

$$\langle c_i^*, y \rangle = \langle c_i^*, y_1 + y_2 \rangle = \langle \widetilde{c_i}^*, y_2 \rangle > 0. \tag{66}$$

It follows that $c_i^* \in C^{+1}$. For (65), let $a \in L_{c_i^*}(A)$. By (26), (46), and (64), there exists $(a_1, a_2) \in Y_1 \times A_2$ such that $a = a_1 + a_2$ and

$$\begin{aligned}
\langle \widetilde{c_i}^*, a_2 \rangle &= \langle c_i^*, a_1 + a_2 \rangle \\
&= \inf_{y_1 \in Y_1, y_2 \in A_2} \langle c_i^*, y_1 + y_2 \rangle = \inf_{y_2 \in A_2} \langle \widetilde{c_i}^*, y_2 \rangle.
\end{aligned} \tag{67}$$

We have $a_2 \in L_{\widetilde{c_i}^*}(A_2)$ and $a \in Y_1 + L_{\widetilde{c_i}^*}(A_2)$. The conclusion, $Y_1 + L_{\widetilde{c_i}^*}(A_2) \subset A$ holds obviously. Thus (65) holds. From (48) and (63)–(65), we have

$$\begin{aligned}
\mathrm{E}(A, C) = \mathrm{Pos}(A, C) &= Y_1 + \bigcup_{i=1}^{p} L_{\widetilde{c_i}^*}(A_2) \\
&= \bigcup_{i=1}^{p} \left(Y_1 + L_{\widetilde{c_i}^*}(A_2) \right) = \bigcup_{i=1}^{p} L_{c_i^*}(A).
\end{aligned} \tag{68}$$

Finally, we show that $L_{c_i^*}(\mathrm{cl}\ \mathrm{co}\ A) = L_{c_i^*}(A)$. Indeed, for any $y \in Y$, (45) implies that there exists unique pair $(y_1, y_2) \in Y_1 \times Y_1$, such that $y = y_1 + y_2$. Since Y_2 is finitely dimensional, one can take a constant $M > 0$ which is independent on y satisfying

$$\|y_1\| + \|y_2\| \leq M \|y\|. \tag{69}$$

From (46), we have

$$\mathrm{co}\ A = Y_1 + \mathrm{co}\ A_2 \subset Y_1 + \mathrm{cl}\ \mathrm{co}\ A_2. \tag{70}$$

It follows that

$$y_1 + \mathrm{cl}\ \mathrm{co}\ A_2 \subset \mathrm{cl}\ \mathrm{co}\ A \quad \forall y_1 \in Y_1, \tag{71}$$

and thus

$$Y_1 + \mathrm{cl}\ \mathrm{co}\ A_2 \subset \mathrm{cl}\ \mathrm{co}\ A. \tag{72}$$

On the other hand, for each sequenc $\{(y_{1,n}, a_{2,n})\} \subset Y_1 \times \mathrm{cl}\ \mathrm{co}\ A_2$ with $y_n := y_{1,n} + a_{2,n}$ converging to some $y \in Y$, there exists a pair $(y_1, y_2) \in Y_1 \times Y_2$ such that $y = y_1 + y_2$. Together with (69), we have

$$\|y_{1,n} - y_1\| + \|y_{2,n} - y_2\| \leq M \|y_n - y\|, \tag{73}$$

and hence $\{y_{1,n}\}$ and $\{y_{2,n}\}$ converge to $y_1 \in Y_1$ (since Y_1 is closed) and $y_2 \in \mathrm{cl}\ \mathrm{co}\ A_2$, respectively. Thus we have $y \in Y_1 + \mathrm{cl}\ \mathrm{co}\ A_2$ and hence $Y_1 + \mathrm{cl}\ \mathrm{co}\ A_2$ is closed. Together with (70), we have

$$\mathrm{cl}\ \mathrm{co}\ A \subset Y_1 + \mathrm{cl}\ \mathrm{co}\ A_2. \tag{74}$$

From (72) and (74), we have

$$\mathrm{cl}\ \mathrm{co}\ A = Y_1 + \mathrm{cl}\ \mathrm{co}\ A_2, \tag{75}$$

and together with (65), one has

$$L_{c_i^*}(\mathrm{cl}\ \mathrm{co}\ A) = Y_1 + L_{\widetilde{c_i}^*}(\mathrm{cl}\ \mathrm{co}\ A_2). \tag{76}$$

By Theorem 3.2(i) in [13], it follows that

$$L_{\widetilde{c_i}^*}(\mathrm{cl}\ \mathrm{co}\ A_2) = L_{\widetilde{c_i}^*}(A_2). \tag{77}$$

This and (65) imply that

$$\begin{aligned}
L_{c_i^*}(\mathrm{cl}\ \mathrm{co}\ A) &= Y_1 + L_{\widetilde{c_i}^*}(\mathrm{cl}\ \mathrm{co}\ A_2) \\
&= Y_1 + L_{\widetilde{c_i}^*}(A_2) = L_{c_i^*}(A).
\end{aligned} \tag{78}$$

The proof is completed. $\qquad\square$

Theorem 15. *Let X and Y be Banach spaces and let the ordering cone C be pointed and polyhedral. Suppose that Γ and $\mathrm{Gr}(F)$ are the unions of finitely many polyhedra of X and $X \times Y$, respectively, and that $F(\Gamma) + C$ is convex. Then, S and V are the union of finitely many convex polyhedra of X and Y, respectively, and more precisely, there exist $c_i^* \in C^{+1}$ ($i \in [1-q]$ for some integer q) such that*

$$V = V_p = \bigcup_{i=1}^{q} L_{c_i^*}(F(\Gamma)). \tag{79}$$

Consequently, $S = S_p$ and V are the unions of finitely many polyhedra.

Proof. Suppose that Γ and $\mathrm{Gr}(F)$ are the unions of finitely many polyhedra of X and $X \times Y$, respectively. Noting that

$$F(\Gamma) = (\mathrm{Gr}(F) \cap (\Gamma \times Y))_Y, \tag{80}$$

and by Lemma 3, one has that $F(\Gamma)$ is the union of finitely many polyhedra of Y. From this and Lemma 14, it follows that there exist $c_i^* \in C^{+1}$ ($i \in [1 - q]$ for some integer q) such that (79) holds. Hence, $V = V_p$ is the union of finitely many polyhedra of Y. This and Corollary 4 imply that $S = S_p = \Gamma \cap F^{-1}(V)$ is the union of finitely many polyhedra of X. The proof is completed. $\qquad\square$

Without the convexity of $A + C$ in Lemma 14, we have the following lemma which will also be applied to consider (SVOP).

Lemma 16. *Let Y be a Banach space, let the ordering cone C be pointed and polyhedral, and let A be the union of finitely many G-polyhedra of Y. Suppose that $\mathrm{E}(A, C)$ is nonempty. Then $\mathrm{E}(A, C)$ is the union of finitely many G-polyhedra of Y.*

Proof. By Corollary 2 there exist a closed subspace Y_1, a finite dimensional subspace Y_2 of Y, and the union A_2 of finitely many G-polyhedra of Y_2, such that (45) and (46) hold.

Let C be the projection of C on Y_2. We first show that $A_2 + C_2$ is the union of finitely many G-polyhedra of Y_2. Indeed, since C is a polyhedron on $Y_1 \times Y_2$, Lemma 3 implies that C_2 is the union of finitely many polyhedra of Y_2. We can assume

$$A_2 = \bigcup_{i=1}^{p} A_{2,i}, \qquad C_2 = \bigcup_{j=1}^{q} C_{2,j}, \tag{81}$$

where each $A_{2,i}$ is a G-polyhedron of Y_2 and each $C_{2,j}$ is a polyhedron of Y_2. It follows that

$$A_2 + C_2 = \bigcup_{i=1}^{p}\bigcup_{j=1}^{q} \left(A_{2,i} + C_{2,j}\right). \tag{82}$$

Noting that Y_2 is finitely dimensional, by Proposition 2.1 in [15], we have that each $A_{2,i} + C_{2,j}$ is a G-polyhedron of Y_2. This with (82) implies that $A_2 + C_2$ is the union of finitely many G-polyhedra of Y_2. Together with (45), (46), and (53), applying Corollary 2, it follows that $A + C$ is the union of finitely many G-polyhedra of Y. Let $A = \bigcup_{i=1}^{k} A_i$, where each A_i is a G-polyhedron of Y. Noting that $\{0\}$ is a face of C, by the above proof, applying Proposition 2.1 in [15] again, we have that each $A_i + C \setminus \{0\}$ as well as $A_i + C$ is the union of finitely many G-polyhedra of Y. So is the complimentary $(A_i + C \setminus \{0\})^c$. By Lemma 5, we have

$$\begin{aligned}(A_i, C) &= A_i \setminus (A_i + C \setminus \{0\}) \\ &= A_i \cap (A_i + C \setminus \{0\})^c,\end{aligned} \tag{83}$$

which implies that (A_i, C) is the union of finitely many G-polyhedra of Y. So is the set

$$\bigcup_{i \leq j \leq k, j \neq i} \left[\left(A_j + C\right) \setminus \left(E(A_i, C) \cap E(A_j, C)\right)\right]. \tag{84}$$

Let

E_i

$$:= E(A_i, C) \setminus \bigcup_{i \leq j \leq k, j \neq i} \left[\left(A_j + C\right) \setminus \left(E(A_i, C) \cap E(A_j, C)\right)\right]. \tag{85}$$

Then E_i is also the union of finitely many G-polyhedra of Y. Applying Proposition 10, we have

$$E(A, C) = \bigcup_{i=1}^{k} E_i. \tag{86}$$

Thus $E(A, C)$ is the union of finitely many G-polyhedra of Y. The proof is completed. $\square$

Theorem 17. *Let X and Y be Banach spaces and let the ordering cone C be pointed and polyhedral. Suppose that Γ and $\mathrm{Gr}(F)$ are the unions of finitely many G-polyhedra of X and $X \times Y$, respectively. Then, S and V are the union of finitely many convex G-polyhedra of X and Y, respectively.*

Proof. Suppose that Γ and $\mathrm{Gr}(F)$ are the unions of finitely many G-polyhedra of X and $X \times Y$, respectively. Noting that

$$F(\Gamma) = (\mathrm{Gr}(F) \cap (\Gamma \times Y))_Y, \tag{87}$$

and by Lemma 3, one has that $F(\Gamma)$ is the union of finitely many polyhedra of Y. From this and Lemma 16, we have that V is the union of finitely many G-polyhedra of Y. This and Corollary 4 imply that $S = \Gamma \cap F^{-1}(V)$ is the union of finitely many G-polyhedra of X too. The proof is completed. $\square$

4. Connectedness of Pareto Solution Sets and Pareto Optimal Value Sets

For various applications, it is of special interest to move continuously from an optimal solution to another along optimal alternatives. In order to do this movement, one needs that the optimal solution set is pathwise connected or at least connected. Many authors studied connectedness of optimal solution sets in multiobjective optimization under some restrictive assumptions (cf. [2, 16–20, 22, 23]).

In this section, dropping the assumption that the ordering cone C has a weakly compact base but requiring that C is pointed and polyhedral, under the C-convexity assumption on the set-valued objective mapping F, we will establish connectedness of Pareto solution set S and Pareto optimal value set V of (SVOP).

Let $F' : X \rightrightarrows Y$ be such that

$$\mathrm{Gr}\left(F'\right) = \mathrm{cl}\,\mathrm{co}\left(\mathrm{Gr}(F) \cap (\Gamma \times Y)\right). \tag{88}$$

Consider the following vector optimization problem:

$$C - \min F'(x) \quad \text{subject to } x \in \Gamma. \qquad \left(\mathrm{SVOP}'\right)$$

Let S' and V' denote the set of all Pareto solutions of $\left(\mathrm{SVOP}'\right)$ and the set of all Pareto optimal values of $\left(\mathrm{SVOP}'\right)$, respectively.

Lemma 18. *Let Γ be a polyhedron of X, let C be a closed, convex, pointed, and polyhedral cone of Y, and let $\mathrm{Gr}F(\Gamma)$ be the union of finitely many polyhedra of $X \times Y$. Suppose that F is C-convex. Then $S' = S$ and $V' = V$.*

Proof. By Corollary 4, $F(\Gamma)$ is the union of finitely many polyhedra of Y. From Corollary 2, there exist a closed subspace Y_1, a finite dimensional subspace of Y, and the union of finitely many polyhedra $F(\Gamma)_2$ of Y_2 such that

$$\begin{aligned} Y = Y_1 + Y_2, \qquad Y_1 \cap Y_2 = \{0\}, \\ F(\Gamma) = Y_1 + F(\Gamma)_2. \end{aligned} \tag{89}$$

By (75) in the proof of Lemma 14, we have

$$\mathrm{cl}\,\mathrm{co}\,F(\Gamma) = Y_1 + \mathrm{cl}\,\mathrm{co}\,F(\Gamma)_2. \tag{90}$$

Since Γ is a polyhedron of X, by Lemma 2.1 in [13] and Theorem 19.6 in [14], $\mathrm{Gr}(F')$ is a polyhedron of $X \times Y$. By Lemma 3, $F'(\Gamma)$ is the union of finitely many polyhedra of X. Hence it is closed. On the other hand, From the convexity of $\mathrm{Gr}(F')$, we have that $F'(\Gamma)$ is convex. Noting that $F(\Gamma) \subset F'(\Gamma) \subset \mathrm{cl}\,\mathrm{co}\,F(\Gamma)$, it follows that $F'(\Gamma) = \mathrm{cl}\,\mathrm{co}\,F(\Gamma)$. Thus by (90), one has

$$F'(\Gamma) = Y_1 + \mathrm{cl}\,\mathrm{co}\,F(\Gamma)_2. \tag{91}$$

Let C_2 be the projection of C on Y_2. Analogously to the proof of Lemma 14, one can show

$$V' = \mathrm{E}\left(F'(\Gamma), C\right) = \mathrm{Pos}\left(F'(\Gamma), C\right)$$
$$= Y_1 + \mathrm{Pos}\left(\mathrm{cl\ co}\, F(\Gamma)_2, C_2\right),$$
$$V = \mathrm{E}\left(F(\Gamma), C\right) = \mathrm{Pos}\left(F(\Gamma), C\right) \qquad (92)$$
$$= Y_1 + \mathrm{Pos}\left(F(\Gamma)_2, C_2\right).$$

In order to show $V' = V$, we only need to show

$$\mathrm{Pos}\left(\mathrm{cl\ co}\, F(\Gamma)_2, C_2\right) = \mathrm{Pos}\left(F(\Gamma)_2, C_2\right). \qquad (93)$$

Indeed, since F is C-convex, it follows that $F'(\Gamma) + C$ and $F(\Gamma) + C$ are convex. It is the same to prove that $A_2 + C_2$ is convex in the proof of Lemma 14; we have that both $F'(\Gamma)_2 + C_2$ and $F(\Gamma)_2 + C_2$ are also convex. Noting that Y_2 is finite dimensional, by Theorem 3.2(i) in [13], one has

$$L_{\widetilde{c}^*}\left(\mathrm{cl\ co}\, F(\Gamma)\right) = L_{\widetilde{c}^*}\left(F(\Gamma)\right) \quad \forall \widetilde{c}^* \in C_2^{+1}. \qquad (94)$$

On the other hand, (27) implies that

$$\mathrm{Pos}\left(\mathrm{cl\ co}\, F(\Gamma)_2, C_2\right) = \bigcup_{\widetilde{c}^* \in C_2^{+1}} L_{\widetilde{c}^*}\left(\mathrm{cl\ co}\, F(\Gamma)_2\right), \qquad (95)$$

$$\mathrm{Pos}\left(F(\Gamma)_2, C_2\right) = \bigcup_{\widetilde{c}^* \in C_2^{+1}} L_{\widetilde{c}^*}\left(F(\Gamma)_2\right). \qquad (96)$$

From (94)–(96), it follows that (93) holds. Hence, we have shown $V' = V$. It remains to show $S' = S$. Noting that $S' = \Gamma \cap F'^{-1}(V')$, $S = \Gamma \cap F^{-1}(V)$, the following assertion in [13, see the proof of Lemma 4.1]

$$\Gamma \cap F^{-1}\left(L_{c^*}\left(F(\Gamma)\right)\right) = \Gamma \cap F'^{-1}\left(L_{c^*}\left(F'(\Gamma)\right)\right) \quad \forall c^* \in C^+,$$
$$V' = \bigcup_{c^* \in C^{+1}} L_{c^*}\left(F'(\Gamma)\right),$$
$$V = \bigcup_{c^* \in C^{+1}} L_{c^*}\left(F(\Gamma)\right), \qquad (97)$$

it follows that $S' = S$. The proof is completed. $\qquad \square$

For our main result, we need the following lemma; see [13, Lemma 4.2].

Lemma 19. *Let Y be a Banach space and A a polyhedron of Y. Let Λ be a convex subset of Y^*. Then $\bigcup_{y^* \in \Lambda} L_{y^*}(A)$ is pathwise connected.*

Theorem 20. *Let X and Y be Banach spaces, Γ a polyhedron of X, C a closed, convex, pointed, and polyhedral cone of Y, and $\mathrm{Gr}F(\Gamma)$ the union of finitely many polyhedra of $X \times Y$. Suppose that the set-valued objective function F is C-convex. Then, the Pareto solution set S and the Pareto optimal value set V of (SVOP) are pathwise connected.*

Proof. Let $F' : X \rightrightarrows Y$ be defined by (88). Then, $\mathrm{Gr}(F')$ is a polyhedron of $X \times Y$. By (27), one has $V' = \bigcap_{y^* \in C^{+1}} L_{y^*}(F'(\Gamma))$. It follows that

$$\mathrm{Gr}\left(F'\right) \cap \left(\Gamma \times V'\right)$$
$$= \bigcup_{y^* \in C^{+1}} L_{(0, y^*)}\left(\mathrm{Gr}\left(F'\right) \cap \left(\Gamma \times Y\right)\right). \qquad (98)$$

This and Lemma 19 imply that $\mathrm{Gr}(F') \cap (\Gamma \times V')$ is pathwise connected. Hence $S' = (\mathrm{Gr}(F') \cap (\Gamma \times V'))_X$ and $V' = (\mathrm{Gr}(F') \cap (\Gamma \times V'))_Y$ are pathwise connected. By Lemma 18, one sees that S and V are pathwise connected. $\qquad \square$

Corollary 21. *Let Y be a Banach space with the pointed ordering C being closed and polyhedral and let A be the union of finitely many convex polyhedra of Y. Suppose that $A + C$ is convex. Then, $\mathrm{E}(A, C)$ is pathwise connected.*

Remark 22. In [13, Theorem 4.1 and Corollary 4.1], under the assumptions that the ordering cone C has a weakly compact base and the Banach space Y is finite dimensional, respectively, the corresponding results of connectedness of (SVOP) were established. In our results, Theorem 20 and Corollary 21, we drop these two assumptions, respectively.

Acknowledgments

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