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Research Article

Weakly Compact Uniform Attractor for the Nonautonomous Long-Short Wave Equations

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Solutions and weakly compact uniform attractor for the nonautonomous long-short wave equations with translation compact forces were studied in a bounded domain. We first established the existence and the uniqueness of the solution to the system by using Galerkin method and then obtained the uniform absorbing set and the weakly compact uniform attractor of the problem by applying techniques of constructing skew product flow in the extended phase space.

1. Introduction

The long wave-short wave (LS) resonance equations arise in the study of the interaction of the surface waves with both gravity and capillary modes presence and also in the analysis of internal waves, as well as Rossby wave [1]. In the plasma physics they describe the resonance of the high-frequency electron plasma oscillation and the associated low-frequency ion density perturbation [2]. Benney [3] presents a general theory for the interaction between the short wave and the long wave.

Due to their rich physical and mathematical properties the long wave-short wave resonance equations have drawn much attention of many physicists and mathematicians. For one-dimensional propagation of waves, there are many studies on this interaction. Guo [4, 5] obtains the existence of global solution for long-short wave equations and generalized long-short wave equations, respectively. The existence of global attractor was studied in [6–8]. The orbital stability of solitary waves for this system has been studied in [9]. In [10], Guo investigated the asymptotic behavior of solutions for the long-short wave equations with zero order dissipation in $H^2_{\text{per}} \times H^1_{\text{per}}$. The approximation inertial manifolds for LS type equations have been studied in [11]. The well posedness of the Cauchy problem for the long wave-short wave resonance equations was studied in [8, 12–17].

In this paper, nonautonomous LS equations with translation compact forces were studied. The essential difference between nonautonomous systems and autonomous ones is that the former get much influenced by the time-depended external forces, which breaks semigroup property of the flow or semiflow created by autonomous systems. Also, attractors of nonautonomous systems are no longer invariable; they change with the changing of the initial time. This makes it impossible for us to consider nonautonomous systems completely in the same way of autonomous ones. Fortunately, Chepyzhov and Vishik [18, 19] developed techniques by which skills in the study of autonomous systems can be used in dealing with nonautonomous problems. Their central idea is that constructing skew product flow in extended phase space is obtained by

$$S(t)(u,\sigma) = \left(U_\sigma(t,0)\,u, T(t)\,\sigma\right), \quad t \ge 0,\ (u,\sigma) \in E \times \sum, \tag{1}$$

where $\{U_{\sigma\in\sum}(t,\tau)\}$ is a family of processes, $\{T(t)\}$ is a translation semigroup, and the flow $\{S(t)\}$ can be proved to be a semigroup under some preconditions, such as the translation identity and $(E \times \sum, E)$-continuity of $\{U_\sigma(t,\tau)\}$, and more importantly, the compactness of the symbol space $\sum$. By this means, we can get the uniform attractor by projecting the global attractor of $\{S(t)\}$ to the phase space if the latter exists. We consider the following nonautonomous dissipative generalized long-short wave equations:

$$iu_t + u_{xx} - nu + i\alpha u + g\left(|u|^2\right)u = a(x,t), \tag{2}$$

$$n_t + \beta n + |u|_x^2 + f\left(|u|^2\right) = b(x,t), \tag{3}$$

with the initial conditions

$$u(x,\tau) = u_\tau(x), \qquad n(x,\tau) = n_\tau(x), \tag{4}$$

and the boundary value conditions

$$u(x,t)|_{\partial\Omega} = 0, \qquad n(x,t)|_{\partial\Omega} = 0, \tag{5}$$

where $x \in \Omega = (-D, D) \subset \mathbb{R}$. $t \geq \tau \in \mathbb{R}_+$. Nonautonomous terms $a(x,t)$ and $b(x,t)$ are time-depended external forces, which are supposed to be translation compact (cf. [18] or Assumption 1). Nonlinear terms $f(s)$ and $g(s)$ are given smooth and real, satisfying

$$\left|f(s)\right| \leqslant c_1\left(s^{p/2} + 1\right), \qquad \left|g(s)\right| \leqslant c_2\left(s^{1/2} + 1\right), \tag{6}$$

$$\left|f^{(k)}(s)\right| \leq c_3, \qquad \left|g^{(k)}(s)\right| \leq c_4, \tag{7}$$

where $0 \leq s < \infty$, $p < 2$, $k = 1, 2$, and constants α, β, and c_j are given in $\mathbb{R}_+$ for $j = 1, 2, 3, 4$.

Our aim here is, firstly, to get the unique existence of solutions for problem (2)~(5) and then to derive the existence of weakly compact uniform attractor for it with the above-mentioned method. Here, and throughout this paper, uniform means uniform about symbols (σ) in symbol space $(\sum)$ unless there is special explanation. In fact, it is the same if we say uniform about the initial time, since the translation identity and the $(E \times \sum, E)$-continuity of $\{U_\sigma(t,\tau)\}$ hold in our case (cf. [20]).

Throughout this paper, we denote by $\|\cdot\|$ the norm of $H = L^2(\Omega)$ with usual inner product $(\cdot,\cdot)$, denote by $\|\cdot\|$ the norm of $L^p(\Omega)$ for all $1 \leq p \leq \infty$ ($\|\cdot\|_2 = \|\cdot\|$), and denote by $\|\cdot\|_{H^k}$ the norm of a usual Sobolev space $H^k(\Omega)$ for all $1 \leq k \leq \infty$. And we denote different constants by a same letter C, and $C(\cdot,\cdot)$ represents that the constant relies only on the parameters appearing in the brackets.

This paper is organized as follows. In Section 2, we recall some facts about the nonautonomous system. In Section 3, we provide the uniform a priori estimates in time. In Section 4, we obtain the unique existence of the solutions for problem (2)~(5) by Galerkin method. Section 5 contains the weakly compact uniform attractor for the nonautonomous system (2)~(5), and in the proof of Theorem 13, the $(E_0 \times \sum, E_0)$-continuity of $\{U_{\sigma\in\sum}(t,\tau)\}$ is proved.

2. Preliminary Results

Let $\mathfrak{I}$ be a topological space, and let $\varphi(s) \in \mathfrak{I}$ be a function. The set

$$\mathscr{H}(\varphi) = \overline{\{\varphi(h+s) \mid h \in \mathbb{R}\}} \tag{8}$$

is called the *hull* of φ in $\mathfrak{I}$, denoted by $\mathscr{H}(\varphi)$. φ is *translation compact* if $\mathscr{H}(\varphi)$ is compact in $\mathfrak{I}$.

We denote all the translation compact functions in $L^2_{\mathrm{loc}}(\mathbb{R}; X)$ by $L^2_c(\mathbb{R}; X)$, where X is a Banach space. Apparently, $\varphi \in L^2_c(\mathbb{R}; X)$ implies that φ is translation bounded; that is,

$$\|\varphi\|^2_{L^2_b(\mathbb{R};X)} = \sup_{t\in\mathbb{R}} \int_t^{t+1} \|\varphi\|^2_X ds < \infty. \tag{9}$$

Let E be a Banach space, and let a family of two-parameter mappings $\{U(t,\tau)\} = \{U(t,\tau) \mid t \geq \tau, \tau \in \mathbb{R}\}$ act in E. We also need the following definitions and lemma (cf. [19, 20]).

Definition 1. Let $\sum$ be a parameter set. $\{U_\sigma(t,\tau), t \geq \tau, \tau \in \mathbb{R}\}$, $\sigma \in \sum$ is said to be a family of processes in Banach space E, if for each $\sigma \in \sum$, $\{U_\sigma(t,\tau)\}$ from E to E satisfies

$$\begin{gathered} U_\sigma(t,s) \circ U_\sigma(s,\tau) = U_\sigma(t,\tau), \quad \forall t \geq s \geq \tau,\ \tau \in \mathbb{R}, \\ U_\sigma(\tau,\tau) = I, \quad I \text{ is the identity operator},\ \tau \in \mathbb{R}. \end{gathered} \tag{10}$$

Definition 2. $\{U_{\sigma\in\sum}(t,\tau)\}$, a family of processes in Banach space E, is called $(E \times \sum, E)$-continuous, if for all fixed T and τ, $T \geq \tau$, projection $(u_\tau, \sigma) \to U_\sigma(T,\tau)u_\tau$ is continuous from $E \times \sum$ to E.

A set $B_0 \subset E$ is said to be uniformly absorbing set for the family of processes $\{U_{\sigma\in\sum}(t,\tau)\}$, if for any $\tau \in \mathbb{R}$ and $B \in \mathscr{B}(E)$ which denotes the set of all bounded subsets of E, there exists $t_0 = t_0(\tau, B) \geq \tau$, such that $\bigcup_{\sigma\in\sum} U_\sigma(t,\tau)B \subseteq B_0$ for all $t \geq t_0$. A set $Y \subset E$ is said to be uniformly attracting for the family of process $\{U_\sigma(t,\tau)\}$, $\sigma \in \sum$ if for any fixed $\tau \in \mathbb{R}$ and every $B \in \mathscr{B}(E)$,

$$\lim_{t\to+\infty}\left(\sup_{\sigma\in\sum} \operatorname{dist}_E\left(U_\sigma(t,\tau)B, Y\right)\right) = 0. \tag{11}$$

Definition 3. A closed set $\mathscr{A}_\sum \subset E$ is called the uniform attractor of the family of process $\{U_\sigma(t,\tau)\}$, $\sigma \in \sum$ if it is uniformly attracting (attracting property), and it is contained in any closed uniformly attracting set $\mathscr{A}'$ of the family of process $\{u_\sigma(t,\tau)\}$, $\sigma \in \sum : \mathscr{A}_E \subseteq \mathscr{A}'$ (minimality property).

Lemma 4. *Let $\sum$ be a compact metric space, and suppose $\{T(h) \mid h \geq 0\}$ is a family of operators acting on $\sum$, satisfying the following:*

(i)

$$T(h)\sum = \sum, \quad \forall h \in \mathbb{R}_+; \tag{12}$$

(ii) translation identity:

$$\begin{gathered} U_\sigma(t+h, \tau+h) = U_{T(h)\sigma}(t,\tau), \\ \forall \sigma \in \sum,\ t \geq \tau,\ \tau \in \mathbb{R},\ h \geq 0, \end{gathered} \tag{13}$$

where $U_\sigma(T,\tau)$ is arbitrarily a process in compact metric space E. Moreover, if the family of processes $\{U_{\sigma\in\sum}(T,\tau)\}$ is $(E\times\sum, E)$ continuous, and it has a uniform compact attracting set, then the skew product flow corresponding to it has a global attractor $\mathscr{A}$ on $E \times \sum$, and the projection of $\mathscr{A}$ on $\sum$, $\mathscr{A}_\sum$ is the compact uniform attractor of $\{U_{\sigma\in\sum}(T,\tau)\}$.

Remark 5. Assumption (13) holds if the system has a unique solution.

For brevity, we rewrite system (2)~(5) in the vector form by introducing $W(x,t) = (u(x,t), n(x,t))$ and $Y(x,t) = (a(x,t), b(x,t))$. We denote by $E_0 = H^2 \bigcap H_0^1(\Omega) \times H_0^1(\Omega)$ the space of vector functions $W(x,t) = (u(x,t), n(x,t))$ with norm

$$\|W\|_{E_0} = \left\{\|u\|_{H^2}^2 + \|n\|_{H^1}^2\right\}^{1/2}. \tag{14}$$

Similarly, we denote by $\sum_0$ the space of $Y(x,t)$ with norm

$$\|Y\|_{\sum_0} = \left\{\|a\|_{H^1}^2 + \|b\|_{H^1}^2\right\}^{1/2}. \tag{15}$$

Then system (2)~(5) can be considered as

$$\begin{gathered} \partial_t W = AW + \sigma(t), \\ W|_{t=\tau} = (u_\tau, n_\tau) = W_\tau, \\ W|_{\partial\Omega} = 0, \end{gathered} \tag{16}$$

where $\sigma(t) = Y(x,t)$ is the symbol of (16).

Assumption 1. Assume that the symbol $\sigma(t)$ comes from the symbol space $\sum$ defined by

$$\sum = \overline{\{Y_0(x, s+r) \mid r \in \mathbb{R}_+\}}, \tag{17}$$

where $Y_0 = (a_0(x,t), b_0(x,t)) \in L_c^2(\mathbb{R}; E_0)$ and the closure is taken in the sense of local quadratic mean convergence topology in the topological space $L_{\mathrm{loc}}^2(\mathbb{R}; \sum_0)$. Moreover, we suppose that $a_{0t}(x,t) \in L_b^2(\mathbb{R}; H^1)$.

Remark 6. By the conception of translation compact/boundedness we remark that

(i) $\forall Y_1 \in \sum$, $\|Y_1\|_{L_b^2(\mathbb{R};\sum_0)}^2 \le \|Y_0\|_{L_b^2(\mathbb{R};\sum_0)}^2$;

(ii) $T(t)\sum = \sum$, $\forall t \in \mathbb{R}$, where $T(t)\varphi(s) = \varphi(s+t)$ is an translation operator.

3. Uniform a Priori Estimates in Time

In this section, we derive uniform a priori estimates in time which enable us to show the existence of solutions and the uniform attractor. First we recall the following interpolation inequality (cf. [21]).

Lemma 7. *Let $j, m \in \mathbb{N} \cup \{0\}$, $q, r \in \mathbb{R}^+$, such that $0 \le j < m$, $1 \le q, r \le \infty$. Then one has*

$$\left\|D^j u\right\|_p \le C\left\|D^m u\right\|_r^a \|u\|_q^{1-a}, \tag{18}$$

for $u \in W^{m,r}(\Omega) \cap L^q(\Omega)$, where $\Omega \subset \mathbb{R}^1$, $j/m \le a \le 1$, and $1/p = j + a(1/r - m) + 1 - a/q$.

Lemma 8. *If $u_\tau(x) \in L^2(\Omega)$ and $Y(x,t) \in \Sigma$, then for the solutions of problem (2)~(5), one has*

$$\|u(t)\| \le C_1, \quad \forall t \ge t_1, \tag{19}$$

where $C_1 = C(\alpha, a_0)$, $t_1 = C(\alpha, a_0, \|u_\tau\|)$.

Proof. Taking the inner product of (2) with u in H we get that

$$\left(iu_t + u_{xx} - nu + i\alpha u + g\left(|u|^2\right)u, u\right) = (a(x,t), u). \tag{20}$$

Taking the imaginary part of (20), we obtain that

$$\frac{1}{2}\frac{d}{dt}\|u\|^2 + \alpha\|u\|^2 = \operatorname{Im}(a(x,t), u). \tag{21}$$

By Young inequality and Remark 6 we have

$$\frac{d}{dt}\|u\|^2 + \alpha\|u\|^2 \le \frac{1}{\alpha}\|a(x,t)\|_{L_b^2(\mathbb{R};H^1)}^2 \le \frac{1}{\alpha}\left\|a_0(x,t)\right\|_{L_b^2(\mathbb{R};H^1)}^2. \tag{22}$$

And then by Gronwall lemma we can complete the proof. □

In the following, we denote that $\int \cdot\, dx = \int_\Omega \cdot\, dx$, which will not cause confusions.

Lemma 9. *Under assumptions of (6), (7) and Assumption 1, if $W(\tau) \in H^1 \times H$, solutions of problem (2)~(5) satisfy*

$$\|W(t)\|_{H^1 \times H}^2 \le C_2, \quad \forall t \ge t_2, \tag{23}$$

where $C_2 = C(\alpha, \beta, f, g, Y_0, a_{0t})$ and $t_2 = C(\alpha, \beta, f, g, Y_0, a_{0t}, \|W_\tau\|_{H^1 \times H})$.

Proof. Taking the inner product of (2) with u_t in H and taking the real part, we get that

$$\begin{aligned} &-\frac{1}{2}\frac{d}{dt}\left\|u_x\right\|^2 - \frac{1}{2}\int n\frac{d}{dt}|u|^2 dx + \operatorname{Re}\left(i\alpha u, u_t\right) \\ &\quad + \frac{1}{2}\int g\left(|u|^2\right)\frac{d}{dt}|u|^2 dx = \operatorname{Re}\left(a(x,t), u_t\right). \end{aligned} \tag{24}$$

By (3) we know that

$$\begin{aligned} &\frac{d}{dt}\int n\frac{d}{dt}|u|^2 dx \\ &\quad = \frac{d}{dt}\int n|u|^2 dx - \int |u|^2 n_t dx \\ &\quad = \frac{d}{dt}\int n|u|^2 dx + \int |u|^2|u|_x^2 dx + \beta\int n|u|^2 dx \\ &\qquad + \int f\left(|u|^2\right)|u|^2 dx - \int b(x,t)|u|^2 dx \\ &\quad = \frac{d}{dt}\int n|u|^2 dx + \beta\int n|u|^2 dx \\ &\qquad + \int f\left(|u|^2\right)|u|^2 dx - \int b(x,t)|u|^2 dx, \end{aligned} \tag{25}$$

which shows that

$$-\frac{1}{2}\frac{d}{dt}\|u_x\|^2-\frac{1}{2}\left(\frac{d}{dt}\int n|u|^2dx+\beta\int n|u|^2dx+\int f\left(|u|^2\right)|u|^2dx-\int b(x,t)|u|^2dx\right)+\frac{1}{2}\frac{d}{dt}\int G\left(|u|^2\right)dx+\operatorname{Re}(i\alpha u,u_t)-\frac{d}{dt}\operatorname{Re}(a(x,t),u)+\operatorname{Re}\int a_t(x,t)\bar{u}\,dx=0, \tag{26}$$

where $G(s)$ is introduced by

$$G(s)=\int_0^s g(\xi)\,d\xi. \tag{27}$$

Taking the inner product of (2) with u in H and taking the real part, we get that

$$\operatorname{Re}(iu_t,u)-\|u_x\|^2-\int n|u|^2dx+\int g\left(|u|^2\right)|u|^2dx-\operatorname{Re}(a(x,t),u)=0. \tag{28}$$

Multiply (28) by α, and add the resulting identity to (26) to get

$$-\frac{1}{2}\frac{d}{dt}\|u_x\|^2-\frac{1}{2}\frac{d}{dt}\int n|u|^2dx-\frac{1}{2}(\beta+2\alpha)\int n|u|^2dx-\frac{1}{2}\int f\left(|u|^2\right)|u|^2dx+\frac{1}{2}\int b(x,t)|u|^2dx+\frac{1}{2}\frac{d}{dt}\int G\left(|u|^2\right)dx-\alpha\|u_x\|^2+\alpha\int g\left(|u|^2\right)|u|^2dx-\alpha\operatorname{Re}(a(x,t),u)-\frac{d}{dt}\operatorname{Re}(a(x,t),u)+\operatorname{Re}\int a_t(x,t)\bar{u}\,dx=0. \tag{29}$$

That is,

$$\begin{aligned}&\frac{d}{dt}\left(\|u_x\|^2+\int n|u|^2dx-\int G\left(|u|^2\right)dx+2\operatorname{Re}\int a(x,t)\bar{u}\,dx\right)\\&\quad+\alpha\left(\|u_x\|^2+\int n|u|^2dx-\int G\left(|u|^2\right)dx+2\operatorname{Re}\int a(x,t)\bar{u}\,dx\right)+\alpha\|u_x\|^2\\&=-\int f\left(|u|^2\right)|u|^2dx+\int b(x,t)|u|^2dx-\alpha\int G\left(|u|^2\right)dx-(\alpha+\beta)\int n|u|^2dx+2\operatorname{Re}\int a_t(x,t)\bar{u}\,dx.\end{aligned} \tag{30}$$

In the following, we denote by C any constants depending only on the data (α,β,f,g), and $C(\cdot,\cdot)$ means it depends not only on (α,β,f,g) but also on parameters in the brackets. $\forall\rho>0$, when t is sufficiently large, by (6) and Lemmas 7 and 8, we have

$$\begin{aligned}&\left|-\int f\left(|u|^2\right)|u|^2dx\right|\\&\quad\le C\int|u|^{p+2}dx+C\int|u|^2dx\\&\quad\le C\int\left(|u|^{2p}+|u|^4\right)dx+C\|u\|^2\\&\quad\le C\|u\|_4^4+C\le C\|u_x\|\,\|u\|^3+C\\&\quad\le\rho\|u_x\|^2+C(\rho),\\&\left|\int b(x,t)|u|^2dx\right|\\&\quad\le\|b(x,t)\|^2_{L^2_b(\mathbb{R};\Sigma_0)}+\|u\|_4^4\\&\quad\le\|Y_0\|^2_{L^2_b(\mathbb{R};\Sigma_0)}+C\|u_x\|\\&\quad\le\rho\|u_x\|^2+C_2(\rho).\end{aligned} \tag{31}$$

By (6) we deduce that

$$|G(s)|\le\frac{2}{3}c_2s^{3/2}+c_2s,\quad\forall s\ge0. \tag{32}$$

And then

$$\begin{aligned}&\left|-\alpha\int_\Omega G\left(|u|^2\right)dx\right|\\&\quad\le C\int\left(|u|^3+|u|^2\right)dx\le C\|u\|_3^3+C\|u\|^2\\&\quad\le C\|u_x\|^{1/2}\|u\|^{5/2}+C\le\rho\|u_x\|^2+C_3(\rho),\end{aligned} \tag{33}$$

$$\begin{aligned}&\left|-(\alpha+\beta)\int_\Omega n|u|^2dx\right|\\&\quad\le\rho\|n\|^2+C(\rho)\|u\|_4^4\\&\quad\le\rho\|n\|^2+\rho\|u_x\|^2+C_4(\rho),\end{aligned} \tag{34}$$

$$\begin{aligned}&\left|2\operatorname{Re}\int_\Omega a_t(x,t)\bar{u}\,dx\right|\\&\quad\le\|a_t(x,t)\|^2_{L^2_c(\mathbb{R};\Sigma_0)}+\|u\|^2\\&\quad\le C\left(\|a_{0t}\|^2_{L^2_c(\mathbb{R};\Sigma_0)},\|u\|^2\right).\end{aligned} \tag{35}$$

By (30)~(35) we get that

$$\frac{d}{dt}\left(\|u_x\|^2+\int n|u|^2dx-\int G\left(|u|^2\right)dx+2\operatorname{Re}\int a(x,t)\bar{u}\,dx\right)$$
$$+\alpha\left(\|u_x\|^2+\int n|u|^2dx-\int G\left(|u|^2\right)dx\right.$$
$$\left.+2\operatorname{Re}\int a(x,t)\bar{u}\,dx\right)+\alpha\|u_x\|^2$$
$$\leq\rho\|n\|^2+4\rho\|u_x\|^2+C(\rho)+C\left(\|a_{0t}\|^2_{L^2_b(\mathbb{R};\Sigma_0)},\|u\|^2\right). \tag{36}$$

Similarly we can also deduce that

$$\frac{d}{dt}\left(\|u_x\|^2+\int n|u|^2dx-\int G\left(|u|^2\right)dx+2\operatorname{Re}\int a(x,t)\bar{u}\,dx\right)$$
$$+\beta\left(\|u_x\|^2+\int n|u|^2dx-\int G\left(|u|^2\right)dx\right.$$
$$\left.+2\operatorname{Re}\int a(x,t)\bar{u}\,dx\right)+(2\alpha-\beta)\|u_x\|^2$$
$$\leq\rho\|n\|^2+4\rho\|u_x\|^2+C(\rho)+C\left(\|a_{0t}\|^2_{L^2_b(\mathbb{R};\Sigma_0)},\|u\|^2\right). \tag{37}$$

Taking the inner product of (3) with n in H, we see that

$$\frac{1}{2}\frac{d}{dt}\|n\|^2+\int n|u|^2_x dx+\beta\|n\|^2$$
$$+\int f\left(|u|^2\right)n\,dx-\int b(x,t)\,n\,dx=0. \tag{38}$$

By (2) we get that

$$\int n|u|^2_x dx=\int nu_x\bar{u}\,dx+\int nu\bar{u}_x dx$$
$$=i\int\left(u_t\bar{u}_x-\bar{u}_tu_x\right)dx+2\operatorname{Re}\int i\alpha u\bar{u}_x dx \tag{39}$$
$$-2\operatorname{Re}\int b(x,t)\bar{u}_x dx,$$

$$\frac{d}{dt}\int\left(iu\bar{u}_x-iu_x\bar{u}\right)dx$$
$$=i\int\left(u_t\bar{u}_x+u\bar{u}_{xt}-u_{xt}\bar{u}-u_x\bar{u}_t\right)dx$$
$$=i\int\left(u_t\bar{u}_x-u_x\bar{u}_t+u_t\bar{u}_x-u_x\bar{u}_t\right)dx \tag{40}$$
$$=2i\int\left(u_t\bar{u}_x-\bar{u}_tu_x\right)dx.$$

It comes from (38)~(40) that

$$\frac{d}{dt}\|n\|^2+\frac{d}{dt}\int i\left(u\bar{u}_x-u_x\bar{u}\right)dx$$
$$+2\beta\|n\|^2+i\alpha\int\left(u\bar{u}_x-u_x\bar{u}\right)dx$$
$$\leq i\alpha\int\left(u\bar{u}_x-u_x\bar{u}\right)dx-4\operatorname{Re}\int i\alpha u\bar{u}_x dx \tag{41}$$
$$+4\operatorname{Re}\int a(x,t)\bar{u}_x dx$$
$$-2\int f\left(|u|^2\right)n\,dx+2\int b(x,t)\,n\,dx.$$

Deal with the right hand side of inequality (41), by Lemmas 7 and 8,

$$\left|i\alpha\int\left(u\bar{u}_x-u_x\bar{u}\right)dx\right|\leq 2\alpha\|u\|\|u_x\|\leq\rho\|u_x\|+C_1(\rho), \tag{42}$$

$$\left|-4\operatorname{Re}\int i\alpha u\bar{u}_x dx\right|\leq 4\alpha\|u\|\|u_x\|\leq\rho\|u_x\|+C_2(\rho), \tag{43}$$

$$\left|-2\int f\left(|u|^2\right)n\,dx\right|$$
$$\leq C\int|u|^p|n|\,dx+C\int|n|\,dx$$
$$\leq\frac{1}{2}\rho\|n\|^2+C(\rho)\int|u|^{2p}dx+\frac{1}{2}\rho\|n\|^2+C(\rho)$$
$$\leq\rho\|n\|^2+C(\rho)\|u_x\|^{p-1}\|u\|^{p+1}+C(\rho)$$
$$\leq\rho\|n\|^2+\rho\|u_x\|^2+C_3(\rho), \tag{44}$$

$$\left|4\operatorname{Re}\int a(x,t)\bar{u}_x dx\right|$$
$$\leq 4\|a(x,t)\|_{L^2_b(\mathbb{R};\Sigma_0)}\|u_x\|\leq\rho\|u_x\| \tag{45}$$
$$+C_4\left(\rho,\|a_0(x,t)\|_{L^2_b(\mathbb{R};\Sigma_0)}\right),$$

$$\left|2\int b(x,t)\,n\,dx\right|$$
$$\leq 2\|b(x,t)\|_{L^2_b(\mathbb{R};\Sigma_0)}\|n\|\leq\rho\|n\|^2 \tag{46}$$
$$+C_5\left(\rho,\|b_0(x,t)\|_{L^2_b(\mathbb{R};\Sigma_0)}\right).$$

So

$$\frac{d}{dt}\left(\|n\|^2+i\int\left(u\bar{u}_x-u_x\bar{u}\right)dx\right)$$
$$+2\beta\|n\|^2+i\alpha\int\left(u\bar{u}_x-u_x\bar{u}\right)dx \tag{47}$$
$$\leq 2\rho\|n\|^2+4\rho\|u_x\|^2+C\left(\rho,\|Y_0(x,t)\|_{L^2_b(\mathbb{R};\Sigma_0)}\right).$$

Analogously, we can also deduce that

$$\begin{aligned}&\frac{d}{dt}\left(\|n\|^2 + i\int\left(u\bar{u}_x - u_x\bar{u}\right)dx\right)\\&\quad + 2\beta\|n\|^2 + i\beta\int\left(u\bar{u}_x - u_x\bar{u}\right)dx\\&\quad \le 2\rho\|n\|^2 + 4\rho\left\|u_x\right\|^2 + C\left(\rho, \left\|Y_0(x,t)\right\|_{L_b^2(\mathbb{R};\Sigma_0)}\right).\end{aligned} \tag{48}$$

Set $\gamma = \min\{\alpha, \beta\}$, and

$$\begin{aligned}E &= \left\|u_x\right\|^2 + \|n\|^2 + \int n|u|^2 dx\\&\quad - \int G\left(|u|^2\right)dx + 2\operatorname{Re}\int a\bar{u}\,dx\\&\quad + i\int\left(u\bar{u}_x - u_x\bar{u}\right)dx.\end{aligned} \tag{49}$$

Then by (36), (47) and (37), (48) we can, respectively, get

$$\begin{aligned}&\frac{d}{dt}E + \alpha E + \alpha\left\|u_x\right\|^2 + \beta\|n\|^2\\&\quad \le 8\rho\left\|u_x\right\|^2 + 3\rho\|n\|^2 + C\left(\rho, \left\|Y_0(x,t)\right\|_{L_b^2(\mathbb{R};\Sigma_0)}\right),\\&\frac{d}{dt}E + \beta E + \alpha\left\|u_x\right\|^2 + \beta\|n\|^2\\&\quad \le 8\rho\left\|u_x\right\|^2 + 3\rho\|n\|^2 + C\left(\rho, \left\|Y_0(x,t)\right\|_{L_b^2(\mathbb{R};\Sigma_0)}\right),\end{aligned} \tag{50}$$

which shows that if we set $\rho \le \min\{\alpha/8, \beta/3\}$, we can deduce that

$$\frac{d}{dt}E + \gamma E \le C_0, \quad \forall t \ge t_0, \tag{51}$$

where $C_0 = C(\rho, \|Y_0(x,t)\|_{L_b^2(\mathbb{R};\Sigma_0)}, \|a_{0t}(x,t)\|_{L_b^2(\mathbb{R};\Sigma_0)})$. By Gronwall lemma we see that

$$E(t) \le E(t_0)e^{-\gamma(t-t_0)} + \frac{C_0}{\gamma}, \quad \forall t \ge t_0. \tag{52}$$

Similar to (33), (34), (45), and (42), for $t \ge t_0$ we have

$$\begin{aligned}&\left|\int n|u|^2 dx - \int G\left(|u|^2\right)dx\right.\\&\quad \left. +2\operatorname{Re}\int a\bar{u}\,dx + i\int\left(u\bar{u}_x - u_x\bar{u}\right)dx\right|\\&\quad \le \rho\|n\|^2 + \rho\left\|u_x\right\|^2 + C\left(\rho, \left\|a_0(x,t)\right\|_{L_b^2(\mathbb{R};\Sigma_0)}\right).\end{aligned} \tag{53}$$

And then

$$\begin{aligned}\left|E(t_0)\right| &\le \left\|u_x(t_0)\right\|^2 + \left\|n(t_0)\right\|^2\\&\quad + \left|\int n(t_0)\left|u(t_0)\right|^2 dx - \int G\left(\left|u(t_0)\right|^2\right)dx\right.\\&\quad + 2\operatorname{Re}\int a(t_0)\bar{u}(t_0)\,dx\\&\quad \left. +i\int\left(u(t_0)\bar{u}_x(t_0) - u_x(t_0)\bar{u}(t_0)\right)dx\right|\\&\le C(R),\end{aligned} \tag{54}$$

where $C(R) = C(\rho, \|Y_0(x,t)\|_{L_b^2(\mathbb{R};\Sigma_0)}, \|a_{0t}(x,t)\|_{L_b^2(\mathbb{R};\Sigma_0)}, R)$ when $\|W_\tau\|_{H^1\times H} \le R$. Then by (52) we infer that

$$\begin{aligned}E(t) &\le C(R)e^{-\gamma(t-t_0)} + \frac{C_0}{\gamma}, \quad \forall t \ge t_0,\\&\le \frac{2C_0}{\gamma}, \quad \forall t \ge t_*,\end{aligned} \tag{55}$$

where $t_* = \inf\{t \mid t \ge t_0 \text{ and } C(R)e^{-\gamma(t_*-t_0)} \le C_0/\gamma\}$. By (49), (53), and (55) we infer that

$$\left\|u_x(t)\right\|^2 + \|n(t)\|^2 \le \rho\|n\|^2 + \rho\left\|u_x\right\|^2 + C_0. \tag{56}$$

Choose $\rho = \min\{\alpha/8, \beta/3, 1/2\}$; then we have

$$\begin{aligned}&\left\|u_x\right\|^2 + \|n(t)\|^2\\&\quad \le C\left(\left\|Y_0(x,t)\right\|_{L_b^2(\mathbb{R};\Sigma_0)}, \left\|a_{0t}(x,t)\right\|_{L_b^2(\mathbb{R};\Sigma_0)}\right), \quad \forall t \ge t_*,\end{aligned} \tag{57}$$

which concludes the proof by using Lemma 8. □

Lemma 10. *Under assumptions of Lemma 9, if $W(\tau) \in E_0 = H^2 \times H^1$, solutions of problem (2)~(5) satisfy*

$$\|W(t)\|_{E_0}^2 \le C_2, \quad \forall t \ge t_3, \tag{58}$$

where $C_2 = C(\alpha, \beta, f, g, Y_0, a_{0t})$ and $t_3 = C(\alpha, \beta, f, g, Y_0, a_{0t}, \|W_\tau\|_{E_0})$.

Proof. Taking the real part of the inner product of (2) with u_{xxt} in H, we have

$$\begin{aligned}&\frac{d}{dt}\left\|u_{xx}\right\|^2 - \operatorname{Re}\int n u\bar{u}_{xxt}dx + \operatorname{Re}\left(i\alpha u, u_{xxt}\right)\\&\quad + \operatorname{Re}\int g\left(|u|^2\right)u\bar{u}_{xxt}dx\\&\quad - \operatorname{Re}\int a(x,t)\bar{u}_{xxt}dx = 0.\end{aligned} \tag{59}$$

By (2) and (3), we have

$$\begin{aligned}&-\operatorname{Re}\int n u\bar{u}_{xxt}dx\\&\quad = -\frac{d}{dt}\int\operatorname{Re}\left(n u\bar{u}_{xx}\right)dx + \operatorname{Re}\int n_t u\bar{u}_x x dx\\&\qquad + \operatorname{Re}\int n u_t\bar{u}_{xx}dx\\&\quad = -\frac{d}{dt}\int\operatorname{Re}\left(n u\bar{u}_{xx}\right)dx\\&\qquad - \operatorname{Re}\int u\bar{u}_{xx}\left(|u|_x^2 + \beta n + f\left(|u|^2\right) - b\right)dx\\&\qquad + \operatorname{Re}\int n\bar{u}_{xx}\left(-inu - \alpha u + ig\left(|u|^2\right)u - ia\right)dx.\end{aligned} \tag{60}$$

Since

$$\operatorname{Re}\left(i\alpha u, u_{xxt}\right) = \operatorname{Re}\int i\alpha u\overline{u}_{txx}dx = -\operatorname{Re}\int i\alpha u_t\overline{u}_{xx}dx, \quad (61)$$

we see that

$$\begin{aligned}\operatorname{Re}\left(i\alpha u, u_{xxt}\right) &= \alpha\|u_{xx}\|^2 - \alpha\operatorname{Re}\int nu\overline{u}_{xx}dx \\ &\quad + \alpha\operatorname{Re}\int g\left(|u|^2\right)u\overline{u}_{xx}dx \\ &\quad - \alpha\operatorname{Re}\int a\overline{u}_{xx}dx.\end{aligned} \quad (62)$$

Multiplying (2) by $\overline{u}$ and taking the real part, we find that

$$|u|_t^2 = 2\operatorname{Re}\left(iu_{xx}\overline{u}\right) - 2\alpha|u|^2 - 2\operatorname{Re}\left(ia\overline{u}\right), \quad (63)$$

therefore,

$$\begin{aligned}&\operatorname{Re}\int g\left(|u|^2\right)u\overline{u}_{xxt}dx \\ &= -\int g'\left(|u|^2\right)|u|_x^2\operatorname{Re}\left(u\overline{u}_{xt}\right)dx \\ &\quad - \int g\left(|u|^2\right)\frac{1}{2}\frac{d}{dt}|u_x|^2dx \\ &= -\int g'\left(|u|^2\right)|u|_x^2\operatorname{Re}\left(u\overline{u}_{xt}\right)dx \\ &\quad - \frac{1}{2}\frac{d}{dt}\int g\left(|u|^2\right)|u_x|^2dx \\ &\quad + \int g'\left(|u|^2\right)|u_x|^2\left(\operatorname{Re}\left(iu_{xx}\overline{u}\right) - \alpha|u|^2\right. \\ &\qquad \left.- \operatorname{Re}\left(ia\overline{u}\right)\right)dx.\end{aligned} \quad (64)$$

Now we deal with (64) to get (70). Due to equalities

$$\begin{aligned}|u|_x^2 &= 2\operatorname{Re}\left(u\overline{u}_x\right), \\ \frac{d}{dt}\operatorname{Re}\left(u\overline{u}_x\right) &= \operatorname{Re}\left(u_t\overline{u}_x\right) + \operatorname{Re}\left(u\overline{u}_{xt}\right),\end{aligned} \quad (65)$$

we deduce that

$$\begin{aligned}&\int g'\left(|u|^2\right)|u|_x^2\operatorname{Re}\left(u\overline{u}_{xt}\right)dx \\ &= \frac{d}{dt}\int g'\left(|u|^2\right)2\operatorname{Re}\left(u\overline{u}_x\right)\operatorname{Re}\left(u\overline{u}_x\right)dx \\ &\quad - \int g''\left(|u|^2\right)|u|_t^2 2\operatorname{Re}\left(u\overline{u}_x\right)\operatorname{Re}\left(u\overline{u}_x\right)dx \\ &\quad - \int g'\left(|u|^2\right)2\operatorname{Re}\left(u\overline{u}_x\right)_t\operatorname{Re}\left(u\overline{u}_x\right)dx \\ &\quad - \int g'\left(|u|^2\right)2\operatorname{Re}\left(u\overline{u}_x\right)\operatorname{Re}\left(u_t\overline{u}_x\right)dx.\end{aligned} \quad (66)$$

We take care of terms in (66) as follows:

$$\begin{aligned}&\int g''\left(|u|^2\right)|u|_t^2 2\operatorname{Re}\left(u\overline{u}_x\right)\operatorname{Re}\left(u\overline{u}_x\right)dx \\ &\quad = 4\int g''\left(|u|^2\right)\left(\operatorname{Re}\left(u\overline{u}_x\right)\right)^2 \\ &\qquad \times\left(\operatorname{Re}\left(iu_{xx}\overline{u}\right) - \alpha|u|^2 - \operatorname{Re}\left(ia\overline{u}\right)\right)dx, \\ &\int g'\left(|u|^2\right)2\operatorname{Re}\left(u\overline{u}_x\right)_t\operatorname{Re}\left(u\overline{u}_x\right)dx \\ &\quad = \int g'\left(|u|^2\right)2\operatorname{Re}\left(u_t\overline{u}_x\right)\operatorname{Re}\left(u\overline{u}_x\right)dx \\ &\qquad + \int g'\left(|u|^2\right)2\operatorname{Re}\left(u\overline{u}_{xt}\right)\operatorname{Re}\left(u\overline{u}_x\right)dx \\ &\quad = 2\int g'\left(|u|^2\right)\operatorname{Re}\left(u\overline{u}_x\right) \\ &\qquad \times\operatorname{Re}\left(i\overline{u}_x\left(u_{xx} - nu + i\alpha u + g\left(|u|^2\right)u - a\right)\right)dx \\ &\qquad + \int g'\left(|u|^2\right)|u|_x^2\operatorname{Re}\left(u\overline{u}_{xt}\right)dx, \\ &\int g'\left(|u|^2\right)2\operatorname{Re}\left(u\overline{u}_x\right)\operatorname{Re}\left(u_t\overline{u}_x\right)dx \\ &\quad = 2\int g'\left(|u|^2\right)\operatorname{Re}\left(u\overline{u}_x\right) \\ &\qquad \times\operatorname{Re}\left(i\overline{u}_x\left(u_{xx} - nu + i\alpha u + g\left(|u|^2\right)u - a\right)\right)dx.\end{aligned} \quad (67)$$

It follows from (66)~(67) that

$$\begin{aligned}&-\int g'\left(|u|^2\right)|u|_x^2\operatorname{Re}\left(u\overline{u}_{xt}\right)dx \\ &\quad = -\frac{d}{dt}\int g'\left(|u|^2\right)2\operatorname{Re}\left(u\overline{u}_x\right)\operatorname{Re}\left(u\overline{u}_x\right)dx \\ &\qquad + 4\int g''\left(|u|^2\right)\left(\operatorname{Re}\left(u\overline{u}_x\right)\right)^2 \\ &\qquad\quad \times\left(\operatorname{Re}\left(iu_{xx}\overline{u}\right) - \alpha|u|^2 - \operatorname{Re}\left(ia\overline{u}\right)\right)dx \\ &\qquad + 4\int g'\left(|u|^2\right)\operatorname{Re}\left(u\overline{u}_x\right) \\ &\qquad\quad \times\operatorname{Re}\left(i\overline{u}_x\left(u_{xx} - nu + i\alpha u + g\left(|u|^2\right)u - a\right)\right)dx \\ &\qquad + \int g'\left(|u|^2\right)|u|_x^2\operatorname{Re}\left(u\overline{u}_{xt}\right)dx.\end{aligned} \quad (68)$$

And then

$$\begin{aligned}&-\int g'\left(|u|^2\right)|u|_x^2\operatorname{Re}\left(u\overline{u}_{xt}\right)dx \\ &\quad = -\frac{d}{dt}\int g'\left(|u|^2\right)\operatorname{Re}\left(u\overline{u}_x\right)\operatorname{Re}\left(u\overline{u}_x\right)dx\end{aligned}$$

$$+2\int g''\left(|u|^2\right)\left(\operatorname{Re}\left(u\bar{u}_x\right)\right)^2 \times\left(\operatorname{Re}\left(iu_{xx}\bar{u}\right)-\alpha|u|^2-\operatorname{Re}\left(ia\bar{u}\right)\right)dx + 2\int g'\left(|u|^2\right)\operatorname{Re}\left(u\bar{u}_x\right) \times\operatorname{Re}\left(i\bar{u}_x\left(u_{xx}-nu+i\alpha u+g\left(|u|^2\right)u-a\right)\right)dx. \tag{69}$$

From (64) and (69) we have

$$\begin{aligned}
&\operatorname{Re}\int g\left(|u|^2\right)u\bar{u}_{xxt}dx\\
&=-\frac{d}{dt}\int g'\left(|u|^2\right)\operatorname{Re}\left(u\bar{u}_x\right)\operatorname{Re}\left(u\bar{u}_x\right)dx
-\frac{1}{2}\frac{d}{dt}\int g\left(|u|^2\right)\left|u_x\right|^2dx\\
&\quad+2\int g''\left(|u|^2\right)\left(\operatorname{Re}\left(u\bar{u}_x\right)\right)^2
\times\left(\operatorname{Re}\left(iu_{xx}\bar{u}\right)-\alpha|u|^2-\operatorname{Re}\left(ia\bar{u}\right)\right)dx\\
&\quad+2\int g'\left(|u|^2\right)\operatorname{Re}\left(u\bar{u}_x\right)
\times\operatorname{Re}\left(i\bar{u}_x\left(u_{xx}-nu+i\alpha u+g\left(|u|^2\right)u-a\right)\right)dx\\
&\quad+\int g'\left(|u|^2\right)\left|u_x\right|^2
\times\left(\operatorname{Re}\left(iu_{xx}\bar{u}\right)-\alpha|u|^2-\operatorname{Re}\left(ia\bar{u}\right)\right)dx.
\end{aligned}\tag{70}$$

By (59), (60), (62), and (70) we conclude that

$$\begin{aligned}
&\frac{d}{dt}\left(\left\|u_{xx}\right\|^2-2\operatorname{Re}\int n\bar{u}_{xx}-2\int g'\left(|u|^2\right)\left(\operatorname{Re}\left(u\bar{u}_x\right)\right)^2
-\int g\left(|u|^2\right)|u|_x^2-2\operatorname{Re}\int a\bar{u}_{xx}\right)\\
&\quad+2\alpha\left(\left\|u_{xx}\right\|^2-2\operatorname{Re}\int n\bar{u}_{xx}
-2\int g'\left(|u|^2\right)\left(\operatorname{Re}\left(u\bar{u}_x\right)\right)^2
-\int g\left(|u|^2\right)\left|u_x\right|^2-2\operatorname{Re}\int a\bar{u}_{xx}\right)\\
&\quad+2\alpha\int n\bar{u}_{xx}dx+4\alpha\int g'\left(|u|^2\right)\left(\operatorname{Re}\left(u\bar{u}_x\right)\right)^2dx
+2\alpha\int g\left(|u|^2\right)\left|u_x\right|^2dx+2\alpha\operatorname{Re}\int a\bar{u}_{xx}dx\\
&\quad-2\operatorname{Re}\int u\bar{u}_{xx}\left(|u|_x^2+\beta n+f\left(|u|^2\right)-b\right)dx
+2\operatorname{Re}\int n\bar{u}_{xx}\left(-inu-\alpha u+ig\left(|u|^2\right)-ia\right)dx\\
&\quad+2\alpha\operatorname{Re}\int g\left(|u|^2\right)\bar{u}u_{xx}dx
+4\int g''\left(|u|^2\right)\left(\operatorname{Re}\left(u\bar{u}_x\right)\right)^2
\times\left(\operatorname{Re}\left(iu_{xx}\bar{u}\right)-\alpha|u|^2-\operatorname{Re}\left(ia\bar{u}\right)\right)dx\\
&\quad+4\int g'\left(|u|^2\right)\operatorname{Re}\left(u\bar{u}_x\right)
\times\operatorname{Re}\left(i\bar{u}_x\left(u_{xx}-nu+i\alpha u+g\left(|u|^2\right)u-a\right)\right)dx\\
&\quad+2\int g'\left(|u|^2\right)\left|u_x\right|^2\left(\operatorname{Re}\left(iu_{xx}\bar{u}\right)-\alpha|u|^2-\operatorname{Re}\left(ia\bar{u}\right)\right)dx
+2\operatorname{Re}\int a_t\bar{u}_{xx}dx=0,
\end{aligned}\tag{71}$$

where $\int\cdot=\int\cdot\,dx$.

For later purpose, we let

$$\begin{aligned}
F(u,n)&=-2\operatorname{Re}\int n\bar{u}_{xx}dx
-2\int g'\left(|u|^2\right)\operatorname{Re}\left(u\bar{u}_x\right)^2dx-\int g\left(|u|^2\right)\left|u_x\right|^2dx
-2\operatorname{Re}\int a\bar{u}_{xx}dx,
\end{aligned}\tag{72}$$

$$\begin{aligned}
-G(u,n)
&=2\alpha\int n\bar{u}_{xx}dx+4\alpha\int g'\left(|u|^2\right)\left(\operatorname{Re}\left(u\bar{u}_x\right)\right)^2dx
+2\alpha\int g\left(|u|^2\right)\left|u_x\right|^2dx+2\alpha\operatorname{Re}\int a\bar{u}_{xx}dx\\
&\quad-2\operatorname{Re}\int u\bar{u}_{xx}\left(|u|_x^2+\beta n+f\left(|u|^2\right)-b\right)dx
+2\operatorname{Re}\int n\bar{u}_{xx}\left(-inu-\alpha u+ig\left(|u|^2\right)-ia\right)dx\\
&\quad+2\alpha\operatorname{Re}\int g\left(|u|^2\right)\bar{u}u_{xx}dx
+4\int g''\left(|u|^2\right)\left(\operatorname{Re}\left(u\bar{u}_x\right)\right)^2
\times\left(\operatorname{Re}\left(iu_{xx}\bar{u}\right)-\alpha|u|^2-\operatorname{Re}\left(ia\bar{u}\right)\right)dx\\
&\quad+4\int g'\left(|u|^2\right)\operatorname{Re}\left(u\bar{u}_x\right)
\times\operatorname{Re}\left(i\bar{u}_x\left(u_{xx}-nu+i\alpha u
+g\left(|u|^2\right)u-a\right)\right)dx
\end{aligned}$$

$$+2\int g'\left(|u|^2\right)|u_x|^2 \times\left(\operatorname{Re}\left(iu_{xx}\bar{u}\right)-\alpha|u|^2-\operatorname{Re}\left(ia\bar{u}\right)\right)dx + 2\operatorname{Re}\int a_t\bar{u}_{xx}dx. \tag{73}$$

Then from (71) we have

$$\frac{d}{dt}\left(\|u_{xx}\|^2+F\right)+2\alpha\left(\|u_{xx}\|^2+F\right)=G, \tag{74}$$

or

$$\frac{d}{dt}\left(\|u_{xx}\|^2+F\right)+\alpha\left(\|u_{xx}\|^2+F\right)+\alpha\|u_{xx}\|^2=G-\alpha F. \tag{75}$$

By Lemma 9 and Agmon inequality we have

$$\|u(t)\|_{H^1}^2+\|u(t)\|_{\infty}^2+\|n(t)\|_H^2\leq 2C_2,\quad \forall t\geq t_2. \tag{76}$$

In the following, we denote by $C=C(\alpha,\beta,f,g,Y_0,a_{0t})$. By Lemma 7 and (76) we estimate the size of $|G-\alpha F|$ to get

$$\begin{aligned}
&\frac{d}{dt}\left(\|u_{xx}\|^2+F\right)+\alpha\left(\|u_{xx}\|^2+F\right)+\alpha\|u_{xx}\|^2\\
&\leq C\int|n|^2|u_{xx}|\,dx+C\int|u_x|^2|u_{xx}|\,dx\\
&\quad +C\int|u_x|^2|nu|\,dx+C\\
&\leq C\|u_{xx}\|\,\|n\|_4^2+C\|u_{xx}\|\,\|u_x\|_4^2\\
&\quad +C\|u\|_{L^\infty}\|n\|\,\|u_x\|_4^2+C\\
&\leq C\|u_{xx}\|\,\|n_x\|^{1/2}\|n\|^{3/2}+C\|u_{xx}\|^{3/2}\|u_x\|^{3/2}\\
&\quad +C\|u\|_{L^\infty}\|n\|\,\|u_{xx}\|^{7/4}\|u\|^{1/4}+C\\
&\leq C\|u_{xx}\|\,\|n_x\|^{1/2}+C\|u_{xx}\|^{7/4}+C\\
&\leq\frac{\alpha}{2}\|u_{xx}\|^2+\frac{\beta}{2}\|n_x\|^2+C.
\end{aligned} \tag{77}$$

Taking the inner product of (3) with n_{xx} in H, we see that

$$-\frac{1}{2}\frac{d}{dt}\|n_x\|^2+\int|u|_x^2n_{xx}dx-\beta\|n_x\|^2 +\int f\left(|u|^2\right)n_{xx}dx-\int bn_{xx}dx=0. \tag{78}$$

Since

$$\int|u|_x^2n_{xx}dx=2\int\operatorname{Re}\left(u\bar{u}_xn_{xx}\right)dx =-2\int\operatorname{Re}\left(u\bar{u}_{xx}n_x+|u_x|^2n_x\right)dx, \tag{79}$$

by (78) we can deduce that

$$\begin{aligned}
&\frac{d}{dt}\|n_x\|^2+4\int\operatorname{Re}\left(u\bar{u}_{xx}n_x\right)dx\\
&\quad+4\int|u_x|^2n_xdx+2\beta\|n_x\|^2\\
&\quad+2\int f'\left(|u|^2\right)\left(u_x\bar{u}+u\bar{u}_x\right)n_xdx\\
&\quad-2\int b_xn_xdx=0.
\end{aligned} \tag{80}$$

From (2) we know that

$$\begin{aligned}
&iu_{tx}+u_{xxx}-n_xu-nu_x+i\alpha u_x\\
&\quad+g'\left(|u|^2\right)|u|_x^2u+g\left(|u|^2\right)u_x-a_x(x,t)=0.
\end{aligned} \tag{81}$$

Taking the real part of the inner product to (81) with u_{xx} in H, we have

$$\begin{aligned}
&\operatorname{Re}\int iu_{tx}\bar{u}_{xx}-\operatorname{Re}\int n_xu\bar{u}_{xx}dx\\
&\quad-\operatorname{Re}\int nu_x\bar{u}_{xx}dx+\operatorname{Re}\int i\alpha u_x\bar{u}_{xx}dx\\
&\quad+\operatorname{Re}\int g'\left(|u|^2\right)|u|_x^2u\bar{u}_{xx}dx\\
&\quad+\operatorname{Re}\int g\left(|u|^2\right)u_x\bar{u}_{xx}dx-\operatorname{Re}\int a_x\bar{u}_{xx}dx=0.
\end{aligned} \tag{82}$$

Because of

$$\frac{d}{dt}\operatorname{Re}\int iu_x\bar{u}_{xx}dx=2\operatorname{Re}\int iu_{tx}\bar{u}_{xx}dx, \tag{83}$$

it holds that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\operatorname{Re}\int iu_x\bar{u}_{xx}dx-\operatorname{Re}\int n_xu\bar{u}_{xx}dx\\
&\quad-\operatorname{Re}\int nu_x\bar{u}_{xx}dx+\operatorname{Re}\int i\alpha u_x\bar{u}_{xx}dx\\
&\quad+\operatorname{Re}\int g'\left(|u|^2\right)|u|_x^2u\bar{u}_{xx}dx\\
&\quad+\operatorname{Re}\int g\left(|u|^2\right)u_x\bar{u}_{xx}dx\\
&\quad-\operatorname{Re}\int a_x\bar{u}_{xx}dx=0.
\end{aligned} \tag{84}$$

By (84) and (80), we find that

$$\begin{aligned}&\frac{d}{dt}\|n_x\|^2+2\frac{d}{dt}\operatorname{Re}\int iu_x\overline{u}_{xx}dx+4\int|u_x|^2n_xdx+2\beta\|n_x\|^2\\&\quad+2\int f'\left(|u|^2\right)|u|_x^2n_xdx\\&\quad-2\int b_xn_xdx-4\operatorname{Re}\int nu_x\overline{u}_{xx}dx\\&\quad+4\alpha\operatorname{Re}\int iu_x\overline{u}_{xx}dx+4\operatorname{Re}\int g'\left(|u|^2\right)|u|_x^2u\overline{u}_{xx}dx\\&\quad+4\operatorname{Re}\int g\left(|u|^2\right)u_x\overline{u}_{xx}dx-4\operatorname{Re}\int a_x\overline{u}_{xx}dx=0.\end{aligned}\tag{85}$$

That is,

$$\begin{aligned}&\frac{d}{dt}\left(\|n_x\|^2+2\operatorname{Re}\int iu_x\overline{u}_{xx}dx\right)\\&\quad+2\beta\left(\|n_x\|^2+2\operatorname{Re}\int iu_x\overline{u}_{xx}dx\right)\\&=4\beta\operatorname{Re}\int iu_x\overline{u}_{xx}dx-4\int|u_x|^2n_xdx\\&\quad-2\int f'\left(|u|^2\right)|u|_x^2n_xdx+2\int b_xn_xdx\\&\quad+4\operatorname{Re}\int nu_x\overline{u}_{xx}dx-4\alpha\operatorname{Re}\int iu_x\overline{u}_{xx}dx\\&\quad-4\operatorname{Re}\int g'\left(|u|^2\right)|u|_x^2u\overline{u}_{xx}dx\\&\quad-4\operatorname{Re}\int g\left(|u|^2\right)u_x\overline{u}_{xx}dx+4\operatorname{Re}\int a_x\overline{u}_{xx}dx.\end{aligned}\tag{86}$$

Similar to (77), we estimate each term in (86), and then we get

$$\begin{aligned}&\frac{d}{dt}\left(\|n_x\|^2+2\operatorname{Re}\int iu_x\overline{u}_{xx}dx\right)\\&\quad+\beta\left(\|n_x\|^2+2\operatorname{Re}\int iu_x\overline{u}_{xx}dx\right)+\beta\|n_x\|^2\\&\le2\beta\operatorname{Re}\int iu_x\overline{u}_{xx}dx+C\|u_{xx}\|+C\|u_x\|_4^2\|n_x\|\\&\quad+C\|n_x\|+C\|u_x\|_\infty\|n\|\|u_{xx}\|\\&\le C\|u_{xx}\|+C\|u_x\|^{3/2}\|u_{xx}\|^{1/2}\|n_x\|\\&\quad+C\|n_x\|+C\|u\|^{1/4}\|u_{xx}\|^{3/4}\|n\|\|u_{xx}\|\\&\le\frac{\alpha}{2}\|u_{xx}\|^2+\frac{\beta}{2}\|n_x\|^2+C.\end{aligned}\tag{87}$$

Let $\gamma=\min\{\alpha,\beta\}$, and

$$E=\|u_{xx}\|^2+\|n_x\|^2+F+2\operatorname{Re}\int iu_x\overline{u}_{xx}dx.\tag{88}$$

By (77) and (87) we deduce that

$$\frac{d}{dt}E+\gamma E\le C,\quad\forall t\ge t_2,\tag{89}$$

which has the same form with (51) in the proof of Lemma 9. Similar to the study of (51), we can derive that

$$E\left(t_2\right)\le C\left(R_2\right),\qquad E\left(t\right)\le\frac{2C}{\gamma},\quad\forall t\ge t_{2*},\tag{90}$$

where $t_{2*}=\inf\{t\mid t\ge t_{2*},C(R_2)e^{-\gamma(t_{2*}-t_0)}\le C_0/\gamma\}$ and $C(R_2)=C(\alpha,\beta,f,g,Y_0,a_{0t},R_2)$ when $\|W_\tau\|_{H^2\times H^1}\le R_2$. By (72) we deduce that

$$\begin{aligned}&\left|F+2\operatorname{Re}\int iu_x\overline{u}_{xx}dx\right|\\&\le2\|u\|_\infty\|n\|\|u_{xx}\|+C\|u\|_\infty^2\|u_x\|^2\\&\quad+\|u\|_\infty\|u_x\|^2+\|a\|_{L_b^2(\mathbb{R};\Sigma_0)}\|u_{xx}\|\\&\quad+2\|u_x\|\|u_{xx}\|+C\\&\le C\|u_{xx}\|+C\le\frac{1}{2}\|u_{xx}\|^2+C,\end{aligned}\tag{91}$$

and then by (88), (90), and (91) we deduce that

$$\|u_{xx}\|^2+2\|n_x\|^2\le C,\quad\forall t\ge t_{2*},\tag{92}$$

which concludes the proof by Lemma 9. □

4. Solutions for (2)~(5)

Theorem 11. *Under assumptions of Lemma 10, for each $W_\tau\in E_0$, system (2)~(5) has a unique global solution $W(x,t)\in L^\infty(\tau,T;E_0)$, $\forall T>\tau$.*

Proof. We prove this theorem briefly by two steps.

Step 1. The existence of the solution.

By Galërkin's method, we apply the following approximate solution:

$$W^l\left(x,t\right)=\sum_{j=1}^{l}w_j^l\left(t\right)\eta_j\left(x\right),\tag{93}$$

to approach the solution of the problem (2)~(5), where $\{\eta_j\}_{j=1}^\infty$ is a orthogonal basis of $H(\Omega)$ satisfying $-\Delta\eta_j=\lambda_j\eta_j$ $(j=1,2,\ldots)$. And $W^l(x,t)$ satisfies

$$\begin{aligned}&\left(iu_t^l+u_{xx}^l-n^lu^l+i\alpha u^l+g\left(\left|u^l\right|^2\right)u^l-a,\eta_j\right)=0,\\&\left(n_t^l+\beta n^l+\left|u^l\right|_x^2+f\left(\left|u^l\right|^2\right)-b,\eta_j\right)=0,\\&\left(W^l\left(x,\tau\right),\eta_j\right)=\left(W_\tau,\eta_j\right),\qquad W^l|_{\partial\Omega}=0,\end{aligned}\tag{94}$$

where $j=1,2,\ldots,l$. Then (94) becomes an initial boundary value problem of ordinary differential equations. According to the standard existence theory for the ordinary differential equations, there exists a unique solution of (94). Similar to [4, 22], by the a priori estimates in Section 3 we know that $\{W^l\}_{l=1}^{\infty}$ converges (weakly star) to a $W(x,t)$ which solves (2)~(5).

Step 2. The uniqueness of the solution.

Suppose W_1, W_2 are two solutions of the problem (2)~(5). Let $W = W_1 - W_2$, then $W(x,t) = (u(x,t), n(x,t))$ satisfies

$$\begin{aligned} & iu_t + u_{xx} - n_1 u_1 + n_2 u_2 + i\alpha u \\ & \quad + g\left(|u_1|^2\right)u_1 - g\left(|u_2|^2\right)u_2 = 0, \\ & n_t + \beta n + |u_1|_x^2 - |u_2|_x^2 \\ & \quad + f\left(|u_1|^2\right) - f\left(\left|u_2^2\right|\right) = 0, \\ & W|_{t=\tau} = 0, \qquad W|_{\partial\Omega} = 0. \end{aligned} \tag{95}$$

Similar to [4, 5, 22], we can deduce that $\|W\| = 0$. □

5. Uniform Absorbing Set and Uniform Attractor

From Theorem 11 we know that $\{U_{\sigma\in\sum}(t,\tau)\}$, the family of processes corresponding to (2)~(5), is well defined. And assumption (13) is satisfied.

Theorem 12. *Under assumptions of Theorem 11, $\{U_{\sigma\in\sum}(t,\tau)\}$ possesses a bounded uniformly absorbing set B_0 in E_0.*

Proof. Let $B_0 = \{W \in E_0 \mid \|W\|_{E_0}^2 \le C(\|W_\tau\|_{E_0}, \|Y_0\|_{L_c^2(\mathbb{R};\sum_0)})\}$. From Theorem 11 we know that B_0 is a bounded absorbing set of the process $U_{\sigma=Y_0}$.

On the other hand, from Assumption 1 we know that for each $Y \in \sum$, $\|Y\|_{L_b^2(\mathbb{R};\sum_0)}^2 \le \|Y_0\|_{L_b^2(\mathbb{R};\sum_0)}^2$ holds. Thus, the solution of our system satisfies

$$\begin{aligned} \|W\|_{E_0} &\le C\left(\|W_\tau\|_{E_0}, \|Y\|_{L_b^2(\mathbb{R};\sum_0)}\right) \\ &\le C\left(\|W_\tau\|_{E_0}, \|Y_0\|_{L_b^2(\mathbb{R};\sum_0)}\right). \end{aligned} \tag{96}$$

So the set $B_0 = \{W \in E_0 \mid \|W\|_{E_0}^2 \le \rho_0^2 \triangleq C(\|W_\tau\|_{E_0}, \|Y_0\|_{L_b^2(\mathbb{R};\sum_0)})\}$ is a bounded uniformly absorbing set of $\{U_{\sigma\in\sum}(t,\tau)\}$. □

Theorem 13. *Under assumptions of Theorem 12, $\{U_{\sigma\in\sum}(t,\tau)\}$ admits a weakly compact uniform attractor $\mathscr{A}_{\sum}$.*

Proof. To prove the existence of weakly compact uniform attractor in E_0, from Lemma 4 and Theorems 11 and 12, the only thing we should do is to verify that $\{U_{\sigma\in\sum}(t,\tau)\}$ is $(E\times\sum, E)$-continuous. Through the following proof, $\rightharpoonup$ means weak converges, and $\overset{*}{\rightharpoonup}$ means *weak converges.

For any fixed $t_1 \ge \tau \in \mathbb{R}$, let

$$(W_{\tau k}, \sigma_k) \rightharpoonup (W_\tau, \sigma) \quad \text{in } E_0 \times \sum. \tag{97}$$

We will complete the proof if we deduce that

$$W_{\sigma_k}(t_1) \rightharpoonup W_\sigma(t_1) \quad \text{in } E_0, \tag{98}$$

where $W_{\sigma_k}(t_1) = (u_k(t_1), n_k(t_1)) = U_{\sigma_k}(t_1,\tau)W_{\tau k}$, $W_\sigma(t_1) = (u(t_1)$, and $n(t_1)) = U_\sigma(t_1,\tau)W_\tau$.

From (97) and Theorem 11 we know that

$$\|W_{\tau k}\|_{E_0} \le C, \tag{99}$$

$$\sup_{t\in[\tau,T]} \left\|W_{\sigma_k}(t)\right\|_{E_0} \le C. \tag{100}$$

By Agmon inequality,

$$\|v\|_\infty \le C\|v\|_{H^1}. \tag{101}$$

We see that

$$\left\|W_{\sigma_k}(t)\right\|_\infty \le C, \quad \forall 0 \le t \le T. \tag{102}$$

Note that

$$iu_{kt} = -u_{kxx} + n_k u_k - i\alpha u_k - g\left(|u_k|^2\right)u_k + a_k(x,t), \tag{103}$$

$$n_{kt} = -|u_k|_x^2 - \beta n_k - f\left(|u_k|^2\right) + b_k(x,t), \tag{104}$$

and $\sigma_k = (a_k(x,t), b_k(x,t)) \in \sum$. By (100) and (102), we find that $\partial_t W_{\sigma_k}(t) \in L^\infty(\tau,T;H)$ and

$$\left\|\partial_t W_{\sigma_k}(t)\right\|_{L^\infty(\tau,T;H)} \le C. \tag{105}$$

Due to Theorem 11 and (105), we know that there exist $\widetilde{W}(t) \triangleq (\tilde{u}(t), \tilde{n}(t)) \in L^\infty(\tau,T;E_0)$, and subsequences of $\{W_{\sigma_k}(t)\}$, which are still denoted by $\{W_{\sigma_k}(t)\}$, such that

$$W_{\sigma_k}(t) \overset{*}{\rightharpoonup} \widetilde{W}(t) \quad \text{in } L^\infty(\tau,T;E_0), \tag{106}$$

$$\partial_t W_{\sigma_k}(t) \overset{*}{\rightharpoonup} \partial_t\widetilde{W}(t) \quad \text{in } L^\infty(\tau,T;H). \tag{107}$$

Besides, for $\forall t_1 \in [\tau,T]$, by (100) we know that there exists $W^0 \triangleq (u^0, n^0) \in E_0$, such that

$$W_{\sigma k}(t_1) \rightharpoonup W^0 \quad \text{in } E_0. \tag{108}$$

By (106) and a compactness embedding theorem, we claim that

$$u_k(t) \longrightarrow \tilde{u}(t) \quad \text{strongly in } L^2(0,T;H). \tag{109}$$

In the following, we shall show that $\widetilde{W}(t)$ is a solution of the problem (2)~(5).

For $\forall v \in H$, $\forall \psi \in C_0^\infty(\tau,T)$, by (103) we find that

$$\begin{aligned} & \int_\tau^T (iu_{kt}, \psi(t)v)\,dt + \int_\tau^T (u_{kxx}, \psi(t)v)\,dt \\ & \quad - \int_\tau^T (n_k u_k, \psi(t)v)\,dt + \int_\tau^T (i\alpha u_k, \psi(t)v)\,dt \\ & \quad + \int_\tau^T \left(g\left(|u_k|^2\right)u_k, \psi(t)v\right)dt \\ & \quad - \int_\tau^T (a_k(x,t), \psi(t)v)\,dt = 0. \end{aligned} \tag{110}$$

Since

$$\int_\tau^T (n_k u_k, \psi(t) v)\, dt - \int_\tau^T (\tilde{n}\tilde{u}, \psi(t) v)\, dt = \int_\tau^T ((u_k - \tilde{u}) n_k, \psi(t) v)\, dt + \int_\tau^T (\tilde{u}(n_k - \tilde{n}), \psi(t) v)\, dt, \tag{111}$$

by (102), (109), and (106),

$$\begin{aligned}&\int_\tau^T ((u_k - \tilde{u}) n_k, \psi(t) v)\, dt \\ &\quad \leq \sup_{0\leq t\leq T} \|n_k(t)\|_\infty \|\psi(t)v\|_{L^2(0,T;H)} \|u_k - \tilde{u}\|_{L^2(0,T;H)} \longrightarrow 0, \\ &\int_\tau^T (\tilde{u}(n_k - \tilde{n}), \psi(t) v)\, dt \\ &\quad = \int_\tau^T \left((n_k - \tilde{n}), \psi(t) v\bar{\tilde{u}}\right) dt \longrightarrow 0.\end{aligned} \tag{112}$$

Then we have

$$\int_\tau^T (n_k u_k, \psi(t) v)\, dt \longrightarrow \int_\tau^T (\tilde{n}\tilde{u}, v)\, \psi(t)\, dt. \tag{113}$$

By using the similar methods to the other terms of (110), we have

$$\begin{aligned}&\int_\tau^T (i\tilde{u}_t, v)\, \psi(t)\, dt + \int_\tau^T (\tilde{u}_{xx}, v)\, \psi(t)\, dt \\ &\quad - \int_\tau^T (\tilde{n}\tilde{u}, v)\, \psi(t)\, dt + \int_\tau^T (i\alpha\tilde{u}, v)\, \psi(t)\, dt \\ &\quad + \int_\tau^T \left(g\left(|\tilde{u}|^2\right)\tilde{u}, v\right) \psi(t)\, dt \\ &\quad - \int_\tau^T (a(x,t), v)\, \psi(t)\, dt = 0.\end{aligned} \tag{114}$$

Therefore, we obtain

$$i\tilde{u}_t + \tilde{u}_{xx} - \tilde{u}\tilde{n} + i\alpha\tilde{u} + g\left(|\tilde{u}|^2\right)\tilde{u} = a(x,t), \tag{115}$$

which shows that $(\tilde{u}, \tilde{n}, a(t))$ satisfies (2).

For $\forall v \in H$, $\forall \psi \in C_0^\infty(\tau, T)$ with $\psi(T) = 0$, $\psi(\tau) = 1$, by (103) we find that

$$\begin{aligned}&- \int_\tau^T (iu_k, v)\, \psi'(t)\, dt + \int_\tau^T (u_{kxx}, v)\, \psi(t)\, dt \\ &\quad - \int_\tau^T (n_k u_k, v)\, \psi(t)\, dt + \int_\tau^T (i\alpha u_k, v)\, \psi(t)\, dt \\ &\quad + \int_\tau^T \left(g\left(|u_k|^2\right) u_k, v\right) \psi(t)\, dt - \int_\tau^T (a_k(x,t), v)\, \psi(t)\, dt \\ &= i(u_k(\tau), v).\end{aligned} \tag{116}$$

Assumption (97) implies that

$$u_k(\tau) = u_{\tau k} \rightharpoonup u_\tau \quad \text{in } H. \tag{117}$$

Then by (116) and (117), we have

$$\begin{aligned}&- \int_\tau^T (i\tilde{u}, v)\, \psi'(t)\, dt + \int_\tau^T (\tilde{u}_{xx}, v)\, \psi(t)\, dt \\ &\quad - \int_\tau^T (\tilde{n}\tilde{u}, v)\, \psi(t)\, dt + \int_\tau^T (i\alpha\tilde{u}, v)\, \psi(t)\, dt \\ &\quad + \int_\tau^T \left(g\left(|\tilde{u}|^2\right)\tilde{u}, v\right) \psi(t)\, dt - \int_\tau^T (a(x,t), v)\, \psi(t)\, dt \\ &= i(u_\tau, v).\end{aligned} \tag{118}$$

While from (115) we know that

$$\begin{aligned}&- \int_\tau^T (i\tilde{u}, v)\, \psi'(t)\, dt + \int_\tau^T (\tilde{u}_{xx}, v)\, \psi(t)\, dt \\ &\quad - \int_\tau^T (\tilde{n}\tilde{u}, v)\, \psi(t)\, dt + \int_\tau^T (i\alpha\tilde{u}, v)\, \psi(t)\, dt \\ &\quad + \int_\tau^T \left(g\left(|\tilde{u}|^2\right)\tilde{u}, v\right) \psi(t)\, dt - \int_\tau^T (a(x,t), v)\, \psi(t)\, dt \\ &= i(\tilde{u}(\tau), v).\end{aligned} \tag{119}$$

It come from (118) and (119) that

$$(u_\tau, v) = (\tilde{u}(\tau), v), \quad \forall v \in H. \tag{120}$$

And then

$$\tilde{u}(\tau) = u_\tau. \tag{121}$$

By (115) and (121), we have

$$\tilde{u}(t) = u(t). \tag{122}$$

For $\forall v \in H$, $\forall \psi \in C_0^\infty(\tau, t_1)$, with $\psi(\tau) = 0$, $\psi(t_1) = 1$, then repeating the procedure of proofs of (116)~(119), by (108) we deduce that

$$u^0 = \tilde{u}(t_1). \tag{123}$$

It comes from (108), (122), and (123) that

$$u_k(t_1) \rightharpoonup u(t_1) \quad \text{in } H^2(\Omega). \tag{124}$$

Similarly, we can also deduce that

$$n_k(t_1) \rightharpoonup n(t_1) \quad \text{in } H^1(\Omega). \tag{125}$$

By (124) and (125), we derive (98). We complete the proof of Theorem 13. □

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