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Research Article

Invariant Operators of Five-Dimensional Nonconjugate Subalgebras of the Lie Algebra of the Poincaré Group P(1,4)

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We have classified all five-dimensional nonconjugate subalgebras of the Lie algebra of the Poincaré group P(1,4) into classes of isomorphic subalgebras. Using this classification, we have constructed invariant operators (generalized Casimir operators) for all five-dimensional nonconjugate subalgebras of the Lie algebra of the Poincaré group P(1,4) and presented them in the explicit form.

1. Introduction

At present, there are many papers devoted to the methods for construction and various applications of invariant operators (generalized Casimir operators) of the Lie algebras to the theory of representations of the Lie groups (and their Lie algebras), theory of special functions, theoretical and mathematical physics, and the theory of differential equations. The details can be found in [1–27] and references therein.

The generalized Poincaré group P(1,4) is a group of rotations and translations of the five-dimensional Minkowski space M(1,4). This group is applied to solve various problems of theoretical and mathematical physics (see, e.g., [28–30]). Invariant operators of the Lie algebra of the Poincaré group P(1,4) have been constructed by Fushchich and Krivskiy [4, 5, 28, 31]. Those operators are used for the classification of the irreducible representations of the Lie algebra of the Poincaré group P(1,4) and for the construction of P(1,4)-invariant differential equations.

The subgroup structure of the group P(1,4) has been studied in [32–36]. One of the nontrivial consequences of the description of the nonconjugate subalgebras of the Lie algebra of the group P(1,4) is that the Lie algebra of the group P(1,4) contains, as subalgebras, the Lie algebra of the Poincaré group P(1,3) and the Lie algebra of the extended Galilei group G(1,3) [37], that is, it naturally unites the Lie algebras of the symmetry groups of relativistic and nonrelativistic physics.

In [38, 39], invariant operators for some nonconjugate subalgebras of the Lie algebra of the group P(1,4) have been constructed. The description of invariant operators of eight-dimensional nonconjugate subalgebras of the Lie algebra of the group P(1,4) can be found in [40].

Invariant operators for all nonconjugate subalgebras of dimension ≤4 of the Lie algebra of the group P(1,4) have been constructed in [41, 42].

The aim of the paper is to construct the invariant operators of all five-dimensional nonconjugate subalgebras of the Lie algebra of the group P(1,4).

The outline of this paper is as follows. In Section 2, we present the brief information about the methods for calculating invariant operators. In Section 3, we define the Lie algebra of the Poincaré group P(1,4). In Section 4, we present the results of the classification of all five-dimensional decomposable nonconjugate subalgebras of the Lie algebra of the Poincaré group P(1,4) into isomorphism classes as well as their invariant operators. Section 5 is devoted to the presentation of the results of the classification of all five-dimensional indecomposable nonconjugate subalgebras of the Lie algebra of the Poincaré group P(1,4) into isomorphism classes as well as their invariant operators. The conclusions and results are discussed in Section 6.

2. About Methods for Calculating Invariant Operators

A method for calculating invariant operators of Lie algebras goes back to the original work of Lie; it has been discussed in detail in a paper of Patera et al. [7]. The method consists in reducing this problem to that of solving a set of linear first-order partial differential equations.

In the same work, the method has been applied for calculating invariant operators of all real Lie algebras of dimension less or equal five as well as real nilpotent algebras of dimension six. A short version of this method as well as the application for construction of invariant operators of the Lie algebra of the Poincaré group P(1,3) can be found in the paper by Patera et al. [8]. According to Patera et al. [7, 8] we also distinguish between Casimir operators (polynomials in the basic operators of the Lie algebra), rational invariants (rational functions of the basic operators of the Lie algebra), and general invariants (irrational and transcendental functions of the basic operators of the Lie algebra).

Recently, Boyko et al. [20] have proposed a new purely algebraic algorithm for computation of invariant operators (generalized Casimir operators) of Lie algebras. It uses the Cartan method of moving frames and the knowledge of the group of inner automorphisms of each Lie algebra. In particular, the algorithm has been applied to the computation of invariant operators for real low-dimensional Lie algebras, finite-dimensional solvable Lie algebras restricted only by a required structure of the nilradical, the class of triangular algebras, the class of solvable triangular Lie algebras with one nilindependent diagonal element, solvable Lie algebras with triangular nilradicals, and diagonal nilindependent elements, and so forth. The details can be found in Boyko et al. [20–24]. The discussion of a purely algebraic algorithm for the computation of invariant operators of Lie algebras by means of moving frames as well as the extension of the exploitation of Cartan's method in the Fels-Olver version can be found in the paper of Boyko et al. [25].

In order to construct invariant operators for five-dimensional nonconjugate subalgebras of the Lie algebra of the group P(1,4) we have done the following steps.

(i) Based on the complete classification of real structures of Lie algebras of dimension ≤ 5 obtained by Mubarakzyanov in [43, 44], we classify all five-dimensional nonconjugate subalgebras of the Lie algebra of the group P(1,4) into classes of isomorphic subalgebras.

In order to select nonconjugate subalgebras of the Lie algebra of the Poincaré group P(1,4) from the classification of five-dimensional Lie algebras provided by Mubarakzyanov we first choose any nonconjugate subalgebra of the Lie algebra of the group P(1,4) and investigate for which subalgebra from the Mubarakzyanov classification (or subalgebra from some Mubarakzyanov's class) this subalgebra is isomorphic. In order to realize it we directly use the following theorem.

Theorem 1 (see [45]). *If the structural constants of the Lie algebra L_r are equal to the structural constants of the Lie algebra L'_r correspondingly, then these Lie algebras are isomorphic. Inversely, if the Lie algebras L_r and L'_r are isomorphic, then in these algebras there exist such bases in which their structural constants will be equal, correspondingly.*

Next, we choose any other subalgebra from the remaining nonconjugate subalgebras of the Lie algebra of the group P(1,4) and do with it the same, and so on. We do the same with all nonconjugate subalgebras of the Lie algebra of the group P(1,4). In the result we obtain all classes of isomorphic five-dimensional subalgebras of the Lie algebra of the group P(1,4).

(ii) We use invariant operators for all real Lie algebras of dimension ≤ 5 constructed by Patera et al. [7] for the construction of invariant operators for all five-dimensional nonconjugate subalgebras of the Lie algebra of the group P(1,4).

In order to present the results obtained, we consider the Lie algebra of the group P(1,4).

3. The Lie Algebra of the Poincaré Group P(1,4)

The Lie algebra of the group P(1,4) is given by 15 basic elements $M_{\mu\nu} = -M_{\nu\mu}$, $\mu, \nu = 0,1,2,3,4$ and P'_μ, $\mu = 0,1,2,3,4$ that satisfy the commutation relations

$$\begin{gathered} \left[P'_\mu, P'_\nu\right] = 0, \\ \left[M'_{\mu\nu}, P'_\sigma\right] = g_{\mu\sigma}P'_\nu - g_{\nu\sigma}P'_\mu, \\ \left[M'_{\mu\nu}, M'_{\rho\sigma}\right] = g_{\mu\rho}M'_{\nu\sigma} + g_{\nu\sigma}M'_{\mu\rho} - g_{\nu\rho}M'_{\mu\sigma} - g_{\mu\sigma}M'_{\nu\rho}, \end{gathered} \tag{1}$$

where $g_{\mu\nu}$, $\mu, \nu = 0,1,2,3,4$ is the metric tensor with components $g_{00} = -g_{11} = -g_{22} = -g_{33} = -g_{44} = 1$ and $g_{\mu\nu} = 0$ if $\mu \neq \nu$. Here and below, $M'_{\mu\nu} = iM_{\mu\nu}$.

We pass from $M'_{\mu\nu}$ and P'_μ to the following linear combinations:

$$\begin{gathered} G = M'_{40}, \qquad L_1 = M'_{32}, \qquad L_2 = -M'_{31}, \qquad L_3 = M'_{21}, \\ P_a = M'_{4a} - M'_{a0}, \qquad C_a = M'_{4a} + M'_{a0}, \quad a = 1,2,3, \\ X_0 = \frac{P'_0 - P'_4}{2}, \qquad X_k = P'_k, \quad k = 1,2,3, \\ X_4 = \frac{P'_0 + P'_4}{2}. \end{gathered} \tag{2}$$

Definition 2. We say that two subalgebras of the Lie algebra L which are map to each other by the group of inner automorphisms of the Lie algebra L are conjugate.

TABLE 1: Invariant operators of subalgebras of the type $A_{3,1} \oplus A_1 \oplus A_1$.

Basis elements of subalgebras	Invariant operators
$\langle -2X_4, X_1, P_1\rangle \oplus \langle P_3\rangle \oplus \langle P_2\rangle$	X_4, P_3, P_2
$\langle -2X_4, X_1, P_1\rangle \oplus \langle P_2\rangle \oplus \langle X_3\rangle$	X_4, P_2, X_3
$\langle 2X_4, P_3, X_3\rangle \oplus \langle X_1\rangle \oplus \langle X_2\rangle$	X_4, X_1, X_2
$\langle 2X_4, P_3 + X_0, X_3\rangle \oplus \langle X_1\rangle \oplus \langle X_2\rangle$	X_4, X_1, X_2
$\langle -2X_4, X_1, P_1\rangle \oplus \langle P_2\rangle \oplus \langle P_3 + X_3\rangle$	$X_4, P_2, P_3 + X_3$
$\langle 2X_4, P_1 + X_2, X_1\rangle \oplus \langle X_3\rangle \oplus \langle P_2 + X_1\rangle$	$X_4, X_3, P_2 + X_1$
$\langle 2\gamma X_4, P_1 + \gamma X_2 + \delta X_3, -P_2, \gamma > 0\rangle \oplus \langle P_3 + X_3 + \delta X_1\rangle \oplus \langle P_2 + \gamma X_1, \gamma > 0\rangle$	$X_4, P_3 + X_3 + \delta X_1, P_2 + \gamma X_1, \gamma > 0$
$\langle 2X_4, P_1 + \delta X_3, -P_2 + X_1, \delta > 0\rangle \oplus \langle P_3 + X_3 + \delta X_1, \delta > 0\rangle \oplus \langle P_2\rangle$	$X_4, P_2, P_3 + X_3 + \delta X_1, \delta > 0$
$\langle 2X_4, -P_1 - X_2, P_2\rangle \oplus \langle P_3\rangle \oplus \langle P_2 + X_1\rangle$	$X_4, P_3, P_2 + X_1$

In order to describe nonconjugate subalgebras of the Lie algebra of the group P(1,4), we have used a method proposed by Patera et al. [46].

In the paper, we use the complete list of nonconjugate (up to P(1,4)-conjugation) subalgebras of the Lie algebra of the group P(1,4) given in [47].

4. Invariant Operators of Five-Dimensional Decomposable Nonconjugate Subalgebras of the Lie Algebra of the Poincaré Group P(1,4)

In the paper, the symbol $A^a_{r,j}$ denotes the jth Lie algebra of dimension r and a is a continuous parameter for the algebra. It should be indicated that the notation $A^a_{r,j}$ corresponds to those used in the paper by Patera et al. [7]. In what follows, for the given specific Lie algebra, we write only nonzero commutation relations [7, 44].

Definition 3. We say that Lie algebra is decomposable if it is the direct sum of algebras of lower dimension.

Let us consider two Lie algebras L and L'.

Definition 4. We say that the Lie algebra $L \oplus L'$ is direct sum of Lie algebras L and L' if it consists of the vector space $L \oplus L'$ of the pairs (X, X'), $X \in L$, $X' \in L'$, satisfying the commutation relation

$$\left[\left(X, X'\right), \left(Y, Y'\right)\right] = \left([X, Y], \left[X', Y'\right]\right), \quad X, Y \in L,\ X', Y' \in L'. \tag{3}$$

We present results for five-dimensional decomposable nonconjugate subalgebras of the Lie algebra of the group P(1,4).

4.1. Lie Algebras of the Type $5A_1$. Consider $\langle X_0 + X_4\rangle \oplus \langle X_1\rangle \oplus \langle X_2\rangle \oplus \langle X_3\rangle \oplus \langle X_0 - X_4\rangle$.

Since the Lie algebras of the type $5A_1$ are Abelian, the invariant operators of these algebras are their basis elements.

4.2. Lie Algebras of the Type $A_2 \oplus A_1 \oplus A_1 \oplus A_1$. The nonzero commutation relation for algebra A_2 has the following form:

$$[e_1, e_2] = e_2. \tag{4}$$

The nonconjugate subalgebra of the type $A_2 \oplus A_1 \oplus A_1 \oplus A_1$ of the Lie algebra of the group P(1,4) can be written as

$$\langle -G, X_4\rangle \oplus \langle X_1\rangle \oplus \langle X_2\rangle \oplus \langle X_3\rangle. \tag{5}$$

It is known that the invariant operators for Lie algebras of the type $A_2 \oplus A_1$ are invariant operators of the subalgebras A_2 and A_1 (see, e.g., Patera et al. [7]). The Lie algebras of the type A_2 do not have invariant operators according to Patera et al. [7, 8]. Each Lie algebra of the type A_1 has one invariant operator, which is its basis element. Therefore, the invariant operators for Lie algebra of the type $A_2 \oplus A_1 \oplus A_1 \oplus A_1$ are basis elements of subalgebras A_1, A_1, and A_1.

4.3. Lie Algebras of the Type $A_{3,1} \oplus A_1 \oplus A_1$. The nonzero commutation relation for algebra $A_{3,1}$ has the following form:

$$[e_2, e_3] = e_1. \tag{6}$$

There exist nine five-dimensional nonconjugate subalgebras of the Lie algebra of the group P(1,4) which are isomorphic to algebra of the type $A_{3,1} \oplus A_1 \oplus A_1$. Two of them depend on parameters and hence constitute continua of subalgebras.

For all nonconjugate subalgebras invariant operators are Casimir operators.

The nonconjugate subalgebras of the type $A_{3,1} \oplus A_1 \oplus A_1$ of the Lie algebra of the group P(1,4) and their invariant operators are given in Table 1.

4.4. Lie Algebras of the Type $A_{3,2} \oplus A_1 \oplus A_1$. The nonzero commutation relations for algebra $A_{3,2}$ have the following form:

$$[e_1, e_3] = e_1, \qquad [e_2, e_3] = e_1 + e_2. \tag{7}$$

There exists only one class of five-dimensional nonconjugate subalgebras of the Lie algebra of the group P(1,4) which are isomorphic to subalgebra of the type $A_{3,2} \oplus A_1 \oplus A_1$.

Among invariant operators of nonconjugate subalgebras there are Casimir operators, and general invariants.

The nonconjugate subalgebras of the type $A_{3,2} \oplus A_1 \oplus A_1$ of the Lie algebra of the group P(1,4) and their invariant operators are given in Table 2.

Table 2: Invariant operators of subalgebras of the type $A_{3,2} \oplus A_1 \oplus A_1$.

Basis elements of subalgebras	Invariant operators
$\langle 2aX_4, P_3, G + aX_3, a < 0\rangle \oplus \langle X_1\rangle \oplus \langle X_2\rangle$	$X_4 \exp\left[\frac{-P_3}{2aX_4}\right], X_1, X_2$

Table 3: Invariant operators of subalgebra of the type $A_{3,3} \oplus A_1 \oplus A_1$.

Basis elements of subalgebras	Invariant operators
$\langle P_3, X_4, G\rangle \oplus \langle X_1\rangle \oplus \langle X_2\rangle$	$\frac{X_4}{P_3}, X_1, X_2$

Table 4: Invariant operators of subalgebras of the type $A_{3,4} \oplus A_1 \oplus A_1$.

Basis elements of subalgebras	Invariant operators
$\langle X_0, X_4, -G\rangle \oplus \langle L_3\rangle \oplus \langle X_3\rangle$	X_0X_4, L_3, X_3
$\langle X_0, X_4, -G\rangle \oplus \langle X_1\rangle \oplus \langle X_2\rangle$	X_0X_4, X_1, X_2
$\langle X_0, X_4, -G - a_3X_3, a_3 < 0\rangle \oplus \langle X_1\rangle \oplus \langle X_2\rangle$	X_0X_4, X_1, X_2

4.5. Lie Algebras of the Type $A_{3,3} \oplus A_1 \oplus A_1$. The nonzero commutation relations for algebra $A_{3,3}$ have the following form:

$$[e_1, e_3] = e_1, \qquad [e_2, e_3] = e_2. \tag{8}$$

There exists only one five-dimensional nonconjugate subalgebra of the Lie algebra of the group P(1,4) which is isomorphic to subalgebra of the type $A_{3,3} \oplus A_1 \oplus A_1$.

Among invariant operators of nonconjugate subalgebra there are Casimir operators and rational invariant.

The nonconjugate subalgebra of the type $A_{3,3} \oplus A_1 \oplus A_1$ of the Lie algebra of the group P(1,4) and its invariant operators are given in Table 3.

4.6. Lie Algebras of the Type $A_{3,4} \oplus A_1 \oplus A_1$. The nonzero commutation relations for algebra $A_{3,4}$ have the following form:

$$[e_1, e_3] = e_1, \qquad [e_2, e_3] = -e_2. \tag{9}$$

There exist three five-dimensional nonconjugate subalgebras of the Lie algebra of the group P(1,4) which are isomorphic to subalgebra of the type $A_{3,4} \oplus A_1 \oplus A_1$. One of them depends on parameters and hence constitute continua of subalgebras.

For all nonconjugate subalgebras invariant operators are Casimir operators.

The nonconjugate subalgebras of the type $A_{3,4} \oplus A_1 \oplus A_1$ of the Lie algebra of the group P(1,4) and their invariant operators are given in Table 4.

4.7. Lie Algebras of the Type $A_{3,6} \oplus A_1 \oplus A_1$. The nonzero commutation relations for algebra $A_{3,6}$ have the following form:

$$[e_1, e_3] = -e_2, \qquad [e_2, e_3] = e_1. \tag{10}$$

There exist thirteen five-dimensional nonconjugate subalgebras of the Lie algebra of the group P(1,4) which are isomorphic to subalgebra of the type $A_{3,6} \oplus A_1 \oplus A_1$. Three of them depend on parameters and hence constitute continua of subalgebras.

For all nonconjugate subalgebras invariant operators are Casimir operators.

The nonconjugate subalgebras of the type $A_{3,6} \oplus A_1 \oplus A_1$ of the Lie algebra of the group P(1,4) and their invariant operators are given in Table 5.

4.8. Lie Algebras of the Type $A_{3,6} \oplus A_2$. The nonzero commutation relations for algebra $A_{3,6}$ have the following form:

$$[e_1, e_3] = -e_2, \qquad [e_2, e_3] = e_1. \tag{11}$$

The nonzero commutation relations for algebra A_2 have the following form:

$$[e_1, e_2] = e_2. \tag{12}$$

There exist five five-dimensional nonconjugate subalgebras of the Lie algebra of the group P(1,4) which are isomorphic to subalgebra of the type $A_{3,6} \oplus A_2$. Three of them depend on parameters and hence constitute continua of subalgebras.

For all nonconjugate subalgebras invariant operators are Casimir operators. In this case all nonconjugate subalgebras have the same invariant operator.

The nonconjugate subalgebras of the type $A_{3,6} \oplus A_2$ of the Lie algebra of the group P(1,4) and their invariant operators are given in Table 6.

4.9. Lie Algebras of the Type $A_{3,8} \oplus A_1 \oplus A_1$. The nonzero commutation relations for algebra $A_{3,8}$ have the following form:

$$[e_1, e_3] = -2e_2, \qquad [e_1, e_2] = e_1, \qquad [e_2, e_3] = e_3. \tag{13}$$

There exists only one five-dimensional nonconjugate subalgebra of the Lie algebra of the group P(1,4) which is isomorphic to subalgebra of the type $A_{3,8} \oplus A_1 \oplus A_1$.

For this nonconjugate subalgebra invariant operators are Casimir operators.

Table 5: Invariant operators of subalgebras of the type $A_{3,6} \oplus A_1 \oplus A_1$.

Basis elements of subalgebras	Invariant operators
$\langle P_1, P_2, L_3\rangle \oplus \langle P_3\rangle \oplus \langle X_4\rangle$	$P_1^2 + P_2^2, P_3, X_4$
$\langle P_1, P_2, L_3\rangle \oplus \langle X_3\rangle \oplus \langle X_4\rangle$	$P_1^2 + P_2^2, X_3, X_4$
$\langle X_1, -X_2, -L_3\rangle \oplus \langle P_3\rangle \oplus \langle X_4\rangle$	$X_1^2 + X_2^2, P_3, X_4$
$\langle X_1, -X_2, -L_3\rangle \oplus \langle G\rangle \oplus \langle X_3\rangle$	$X_1^2 + X_2^2, G, X_3$
$\langle X_1, X_2, L_3\rangle \oplus \langle X_0 + X_4\rangle \oplus \langle P_3 + C_3\rangle$	$X_1^2 + X_2^2, X_0 + X_4, P_3 + C_3$
$\langle X_1, X_2, L_3\rangle \oplus \langle X_4\rangle \oplus \langle X_0\rangle$	$X_1^2 + X_2^2, X_4, X_0$
$\langle X_1, X_2, L_3\rangle \oplus \langle X_4\rangle \oplus \langle X_3\rangle$	$X_1^2 + X_2^2, X_4, X_3$
$\langle X_1, X_2, L_3\rangle \oplus \langle X_0 - X_4\rangle \oplus \langle X_3\rangle$	$X_1^2 + X_2^2, X_0 - X_4, X_3$
$\langle P_1, P_2, L_3\rangle \oplus \langle X_4\rangle \oplus \langle P_3 + X_3\rangle$	$P_1^2 + P_2^2, X_4, P_3 + X_3$
$\langle X_1, X_2, L_3\rangle \oplus \langle X_4\rangle \oplus \langle P_3 + X_0\rangle$	$X_1^2 + X_2^2, X_4, P_3 + X_0$
$\langle X_1, X_2, L_3 + d_3X_3, d_3 < 0\rangle \oplus \langle X_4\rangle \oplus \langle X_0\rangle$	$X_1^2 + X_2^2, X_4, X_0$
$\langle X_1, X_2, L_3 + \tilde{d}_0X_0, \tilde{d}_0 < 0\rangle \oplus \langle X_0 - X_4\rangle \oplus \langle X_3\rangle$	$X_1^2 + X_2^2, X_0 - X_4, X_3$
$\langle X_1, X_2, L_3 + \alpha(X_0 + X_4), \alpha < 0\rangle \oplus \langle X_4\rangle \oplus \langle X_3\rangle$	$X_1^2 + X_2^2, X_4, X_3$

Table 6: Invariant operators of subalgebras of the type $A_{3,6} \oplus A_2$.

Basis elements of subalgebras	Invariant operators
$\langle X_1, X_2, L_3\rangle \oplus \langle -G, P_3\rangle$	$X_1^2 + X_2^2$
$\langle X_1, X_2, L_3\rangle \oplus \langle -G, X_4\rangle$	$X_1^2 + X_2^2$
$\langle X_1, X_2, L_3 + dX_3, d < 0\rangle \oplus \langle -G - aX_3, X_4, a < 0\rangle$	$X_1^2 + X_2^2$
$\langle X_1, -X_2, -L_3\rangle \oplus \langle -G - aX_3, X_4, a < 0\rangle$	$X_1^2 + X_2^2$
$\langle X_1, X_2, L_3 + dX_3, d < 0\rangle \oplus \langle -G, X_4\rangle$	$X_1^2 + X_2^2$

Table 7: Invariant operators of subalgebra of the type $A_{3,8} \oplus A_1 \oplus A_1$.

Basis elements of subalgebras	Invariant operators
$\langle -P_3, G, C_3\rangle \oplus \langle X_1\rangle \oplus \langle X_2\rangle$	$2G^2 - P_3C_3 - C_3P_3, X_1, X_2$

The nonconjugate subalgebra of the type $A_{3,8} \oplus A_1 \oplus A_1$ of the Lie algebra of the group P(1,4) and its invariant operators are given in Table 7.

4.10. Lie Algebras of the Type $A_{3,9} \oplus A_1 \oplus A_1$. The nonzero commutation relations for algebra $A_{3,9}$ have the following form:

$$[e_1, e_2] = e_3, \qquad [e_2, e_3] = e_1, \qquad [e_3, e_1] = e_2. \tag{14}$$

There exist only two five-dimensional nonconjugate subalgebras of the Lie algebra of the group P(1,4) which are isomorphic to subalgebra of the type $A_{3,9} \oplus A_1 \oplus A_1$.

For all nonconjugate subalgebras invariant operators are Casimir operators.

The nonconjugate subalgebras of the type $A_{3,9} \oplus A_1 \oplus A_1$ of the Lie algebra of the group P(1,4) and their invariant operators are given in Table 8.

4.11. Lie Algebras of the Type $A_{3,9} \oplus A_2$. The nonzero commutation relations for algebra $A_{3,9}$ have the following form:

$$[e_1, e_2] = e_3, \qquad [e_2, e_3] = e_1, \qquad [e_3, e_1] = e_2. \tag{15}$$

The nonzero commutation relations for algebra A_2 have the following form:

$$[e_1, e_2] = e_2. \tag{16}$$

There exist only one five-dimensional nonconjugate subalgebra of the Lie algebra of the group P(1,4) which is isomorphic to subalgebra of the type $A_{3,9} \oplus A_2$.

For this nonconjugate subalgebra invariant operator is Casimir operator.

The nonconjugate subalgebra of the type $A_{3,9} \oplus A_2$ of the Lie algebra of the group P(1,4) and its invariant operator are given in Table 9.

4.12. Lie Algebras of the Type $A_{4,1} \oplus A_1$. The nonzero commutation relations for algebra $A_{4,1}$ have the following form:

$$[e_2, e_4] = e_1, \qquad [e_3, e_4] = e_2. \tag{17}$$

There exist five five-dimensional nonconjugate subalgebras of the Lie algebra of the group P(1,4) which are isomorphic to subalgebra of the type $A_{4,1} \oplus A_1$. One of them depends on parameters and hence constitute continua of subalgebras.

For all nonconjugate subalgebras invariant operators are Casimir operators.

Table 8: Invariant operators of subalgebras of the type $A_{3,9} \oplus A_1 \oplus A_1$.

Basis elements of subalgebras	Invariant operators
$\langle L_3, L_1, L_2 \rangle \oplus \langle X_4 \rangle \oplus \langle X_0 \rangle$	$L_1^2 + L_2^2 + L_3^2, X_4, X_0$
$\left\langle \frac{1}{2}L_3 + \frac{1}{4}(P_3 + C_3), \frac{1}{2}L_2 + \frac{1}{4}(P_2 + C_2), -\frac{1}{2}L_1 - \frac{1}{4}(P_1 + C_1) \right\rangle \oplus \langle X_0 + X_4 \rangle \oplus \left\langle L_3 - \frac{1}{2}(P_3 + C_3) \right\rangle$	$(2L_1 + P_1 + C_1)^2 + (2L_2 + P_2 + C_2)^2 + (2L_3 + P_3 + C_3)^2, X_0 + X_4, 2L_3 - (P_3 + C_3)$

Table 9: Invariant operator of subalgebra of the type $A_{3,9} \oplus A_2$.

Basis elements of subalgebras	Invariant operators
$\langle -L_3, -L_1, L_2\rangle \oplus \langle -G, X_4\rangle$	$L_1^2 + L_2^2 + L_3^2$

Table 10: Invariant operators of subalgebras of the type $A_{4,1} \oplus A_1$.

Basis elements of subalgebras	Invariant operators
$\langle 2X_4, -X_3, X_0, P_3\rangle \oplus \langle L_3\rangle$	$X_4, X_3^2 - 4X_4X_0, L_3$
$\langle 2X_4, -X_3, X_0, P_3\rangle \oplus \langle X_1\rangle$	$X_4, X_3^2 - 4X_4X_0, X_1$
$\langle 2X_4, -X_3, X_0, P_3 + X_2\rangle \oplus \langle X_1\rangle$	$X_4, X_3^2 - 4X_4X_0, X_1$
$\langle 2X_4, -X_1, P_2 + X_0, P_1\rangle \oplus \langle X_3\rangle$	$X_4, X_1^2 - 4X_4(P_2 + X_0), X_3$
$\langle 2\gamma X_4, -\gamma X_1, P_2 + \gamma X_0 + X_1, P_1 + X_2, \gamma > 0\rangle \oplus \langle X_3\rangle$	$X_4, \gamma^2 X_1^2 - 4\gamma X_4(P_2 + \gamma X_0 + X_1), X_3$

Table 11: Invariant operators of subalgebras of the type $A_{4,2}^a \oplus A_1$.

Basis elements of subalgebras	Invariant operators
$a = 1 : \langle P_2, 2a_1X_4, P_1, G + a_1X_1, a_1 < 0\rangle \oplus \langle X_3\rangle$	$X_4 \exp\left(\frac{-P_1}{2a_1X_4}\right), \frac{X_4}{P_2}, X_3$

Table 12: Invariant operators of subalgebra of the type $A_{4,5}^{ab} \oplus A_1$.

Basis elements of subalgebras	Invariant operators
$a = 1, b = 1 : \langle X_4, P_1, P_2, G\rangle \oplus \langle X_3\rangle$	$\frac{X_4}{P_1}, \frac{X_4}{P_2}, X_3$

The nonconjugate subalgebras of the type $A_{4,1} \oplus A_1$ of the Lie algebra of the group P(1,4) and their invariant operators are given in Table 10.

4.13. Lie Algebras of the Type $A_{4,2}^a \oplus A_1$. The nonzero commutation relations for algebra $A_{4,2}^a$ have the following form:

$$[e_1, e_4] = ae_1, \qquad [e_2, e_4] = e_2, \qquad [e_3, e_4] = e_2 + e_3, \quad (a \neq 0). \tag{18}$$

There exists only one class of five-dimensional nonconjugate subalgebras of the Lie algebra of the group P(1,4) which are isomorphic to subalgebras of the type $A_{4,2}^a \oplus A_1$.

Among invariant operators of nonconjugate subalgebras there are Casimir operator, rational invariant and general invariants.

The nonconjugate subalgebras of the type $A_{4,2}^a \oplus A_1$ of the Lie algebra of the group P(1,4) and their invariant operators are given in Table 11.

4.14. Lie Algebras of the Type $A_{4,5}^{ab} \oplus A_1$. The nonzero commutation relations for algebra $A_{4,5}^{ab}$ have the following form:

$$[e_1, e_4] = e_1, \qquad [e_2, e_4] = ae_2, \qquad [e_3, e_4] = be_3, \quad (ab \neq 0, -1 \leq a \leq b \leq 1). \tag{19}$$

There exists only one five-dimensional nonconjugate subalgebra of the Lie algebra of the group P(1,4) which is isomorphic to subalgebra of the type $A_{4,5}^{ab} \oplus A_1$.

Among invariant operators of nonconjugate subalgebra there are Casimir operator and rational invariants.

The nonconjugate subalgebra of the type $A_{4,5}^{ab} \oplus A_1$ of the Lie algebra of the group P(1,4) and its invariant operators are given in Table 12.

4.15. Lie Algebras of the Type $A_{4,6}^{ab} \oplus A_1$. The nonzero commutation relations for algebra $A_{4,6}^{ab}$ have the following form:

$$[e_1, e_4] = ae_1, \qquad [e_2, e_4] = be_2 - e_3, \qquad [e_3, e_4] = e_2 + be_3, \quad (a \neq 0, b \geq 0). \tag{20}$$

There exist only two classes of five-dimensional nonconjugate subalgebras of the Lie algebra of the group P(1,4) which are isomorphic to subalgebras of the type $A_{4,6}^{ab} \oplus A_1$.

Among invariant operators of nonconjugate subalgebras there are Casimir operators, rational invariant and general invariants.

The nonconjugate subalgebras of the type $A_{4,6}^{ab} \oplus A_1$ of the Lie algebra of the group P(1,4) and their invariant operators are given in Table 13.

4.16. Lie Algebras of the Type $A_{4,9}^b \oplus A_1$. The nonzero commutation relations for algebra $A_{4,9}^b$ have the following form:

$$[e_2, e_3] = e_1, \qquad [e_1, e_4] = (1 + b)e_1, \qquad [e_2, e_4] = e_2, \qquad [e_3, e_4] = be_3, \quad (-1 < b \leq 1). \tag{21}$$

Table 13: Invariant operators of subalgebras of the type $A_{4,6}^{ab} \oplus A_1$.

Basis elements of subalgebras	Invariant operators
$a = e, b = e : \langle X_4, P_1, P_2, L_3 + eG, e > 0\rangle \oplus \langle X_3\rangle$	$\frac{X_4^2}{P_1^2 + P_2^2}, (P_1^2 + P_2^2)\left(\frac{P_1 + iP_2}{P_1 - iP_2}\right)^{ie}, X_3, e > 0$
$a = e, b = 0 : \langle X_4, X_1, X_2, L_3 + eG, e > 0\rangle \oplus \langle X_3\rangle$	$X_1^2 + X_2^2, \ln X_4 + e \arcsin \frac{X_1}{\sqrt{X_1^2 + X_2^2}}, X_3, e > 0$

Table 14: Invariant operators of subalgebras of the type $A_{4,9}^{b} \oplus A_1$ $(b = 0)$.

Basis elements of subalgebras	Invariant operators
$\langle 2X_4, P_3, X_3, G\rangle \oplus \langle L_3\rangle$	L_3
$\langle 2X_4, P_3, X_3, G\rangle \oplus \langle X_1\rangle$	X_1
$\langle 2eX_4, P_3, X_1 + eX_3, G, e > 0\rangle \oplus \langle X_2\rangle$	X_2
$\langle 2X_4, P_3, X_3, G + aX_2, a < 0\rangle \oplus \langle X_1\rangle$	X_1
$\langle 2\mu X_4, P_3, X_1 + \mu X_3, G + \alpha X_1, \alpha < 0, \mu > 0\rangle \oplus \langle X_2\rangle$	X_2

Table 15: Invariant operators of subalgebras of the type $A_{4,10} \oplus A_1$.

Basis elements of subalgebras	Invariant operators
$\langle -4X_4, P_1 + X_2, P_2 - X_1, L_3\rangle \oplus \langle P_3 + \tilde{h}_3 X_3\rangle$	$X_4, -8X_4L_3 + (P_1 + X_2)^2 + (P_2 - X_1)^2, P_3 + \tilde{h}_3 X_3$
$\langle -4X_4, P_1 + X_2, P_2 - X_1, L_3\rangle \oplus \langle X_3\rangle$	$X_4, -8X_4L_3 + (P_1 + X_2)^2 + (P_2 - X_1)^2, X_3$

Table 16: Invariant operator of subalgebra of the type $A_{4,12} \oplus A_1$.

Basis elements of subalgebras	Invariant operators
$\langle -P_1, P_2, G, -L_3\rangle \oplus \langle X_3\rangle$	X_3

There exist five five-dimensional nonconjugate subalgebras of the Lie algebra of the group P(1,4) which are isomorphic to algebra of the type $A_{4,9}^{b} \oplus A_1$. Three of them depend on parameters and hence constitute continua of subalgebras.

For all nonconjugate subalgebras invariant operators are Casimir operators.

The nonconjugate subalgebras of the type $A_{4,9}^{b} \oplus A_1$ ($b = 0$) of the Lie algebra of the group P(1,4) and their invariant operators are given in Table 14.

4.17. Lie Algebras of the Type $A_{4,10} \oplus A_1$. The nonzero commutation relations for algebra $A_{4,10}$ have the following form:

$$[e_2, e_3] = e_1, \qquad [e_2, e_4] = -e_3, \qquad [e_3, e_4] = e_2. \tag{22}$$

There exist two five-dimensional nonconjugate subalgebras of the Lie algebra of the group P(1,4) which are isomorphic to subalgebra of the type $A_{4,10} \oplus A_1$. One of them depends on parameters and hence constitute continua of subalgebras.

For all nonconjugate subalgebras invariant operators are Casimir operators.

The nonconjugate subalgebras of the type $A_{4,10} \oplus A_1$ of the Lie algebra of the group P(1,4) and their invariant operators are given in Table 15.

4.18. Lie Algebras of the Type $A_{4,12} \oplus A_1$. The nonzero commutation relations for algebra $A_{4,12}$ have the following form:

$$\begin{aligned} [e_1, e_3] &= e_1, & [e_2, e_3] &= e_2, \\ [e_1, e_4] &= -e_2, & [e_2, e_4] &= e_1. \end{aligned} \tag{23}$$

There exists only one five-dimensional nonconjugate subalgebra of the Lie algebra of the group P(1,4) which is isomorphic to algebra of the type $A_{4,12} \oplus A_1$.

For this nonconjugate subalgebra invariant operator is Casimir operator.

The nonconjugate subalgebra of the type $A_{4,12} \oplus A_1$ of the Lie algebra of the group P(1,4) and its invariant operator are given in Table 16.

Thus, we have described the invariant operators for all decomposable five-dimensional subalgebras of the Lie algebra of the group P(1,4).

5. Invariant Operators of Five-Dimensional Indecomposable Nonconjugate Subalgebras of the Lie Algebra of the Poincaré Group P(1,4)

We present results for five-dimensional indecomposable nonconjugate subalgebras of the Lie algebra of the group P(1,4).

Table 17: Invariant operators of subalgebras of the type $A_{5,4}$.

Basis elements of subalgebras	Invariant operators
$\langle 2X_4, P_1, P_2, X_1, X_2\rangle$	X_4
$\langle 2X_4, P_1, P_2, X_1 + eX_3, X_2, e > 0\rangle$	X_4
$\langle 2X_4, P_1 + X_3, P_2, X_1, X_2\rangle$	X_4
$\langle 2X_4, P_1, P_2 + X_3, X_1 + \mu X_3, X_2\rangle$	X_4
$\left\langle 2X_4, P_1, P_2 + X_3, X_1, -\frac{1}{2}(P_3 - X_2)\right\rangle$	X_4
$\left\langle 2X_4, P_1, P_2 + X_3, X_1, -\frac{1}{2}(P_3 - X_2 + \mu X_3), \mu \neq 0\right\rangle$	X_4
$\langle 4X_4, P_1 + \beta X_2, P_2 + X_3 + \beta X_1, P_2 + X_3 + (\beta+2)X_1, P_1 - P_3 + (\beta+1)X_2, \beta > 0\rangle$	X_4
$\langle 4X_4, P_2 + \gamma X_1 + X_3, P_1 + \gamma X_2 + \delta X_3, P_1 - P_3 - \delta X_1 + (\gamma+1)X_2 + (\delta-\mu)X_3,\ P_2 + (\gamma+2)X_1 + X_3, \gamma > 0, \mu > 0\rangle$	X_4
$\langle 4X_4, P_1 + P_2 + (\delta+2)X_1 + (\delta+1)X_3, P_3 - P_1 - X_2 + (\mu-\delta)X_3, P_2 + 2X_1 + X_3, P_2 + X_3\rangle$	X_4

Table 18: Invariant operators of subalgebras of the type $A_{5,5}$.

Basis elements of subalgebras	Invariant operators
$\langle 2X_4, -X_2, P_1 + X_0, X_1, P_2 + \beta X_3, \beta > 0\rangle$	X_4
$\langle 2X_4, -X_2, P_1 + X_0, X_1, P_2\rangle$	X_4
$\langle 2X_4, -X_2, P_1 + X_0, X_1 + \mu X_3, P_2 + \beta X_3, \beta > 0, \mu > 0\rangle$	X_4
$\langle 2X_4, -X_2, P_1 + X_0, X_1 + \mu X_3, P_2, \mu > 0\rangle$	X_4

Table 19: Invariant operators of subalgebra of the type $A^{abc}_{5,7}$.

Basis elements of subalgebras	Invariant operators
$a = 1, b = 1, c = 1 : \langle P_1, P_2, P_3, X_4, G\rangle$	$\frac{P_1}{P_2}, \frac{P_1}{P_3}, \frac{P_1}{X_4}$

5.1. Lie Algebras of the Type $A_{5,4}$. The nonzero commutation relations for algebra $A_{5,4}$ have the following form:

$$[e_2, e_4] = e_1, \qquad [e_3, e_5] = e_1. \tag{24}$$

This Lie algebra is nilpotent.

There are nine nonconjugate subalgebras of the Lie algebra of the group P(1,4) which are isomorphic to Lie algebra of type $A_{5,4}$, six of which depend on parameters and hence constitute continua of subalgebras.

Invariant operators of all subalgebras are Casimir operators. Moreover, all subalgebras have the same invariant operator.

The nonconjugate subalgebras of the type $A_{5,4}$ of the Lie algebra of the group P(1,4) and their invariant operators are given in Table 17.

5.2. Lie Algebras of the Type $A_{5,5}$. The nonzero commutation relations for algebra $A_{5,5}$ have the following form:

$$[e_3, e_4] = e_1, \qquad [e_2, e_5] = e_1, \qquad [e_3, e_5] = e_2. \tag{25}$$

This Lie algebra is nilpotent.

There are four nonconjugate subalgebras of the Lie algebra of the group P(1,4) which are isomorphic to Lie algebra of the type $A_{5,5}$, three of which depend on parameters and hence constitute continua of subalgebras.

Invariant operators of all subalgebras are Casimir operators. Moreover, all subalgebras have the same invariant operator.

The nonconjugate subalgebras of the type $A_{5,5}$ of the Lie algebra of the group P(1,4) and their invariant operators are given in Table 18.

5.3. Lie Algebras of the Type $A^{abc}_{5,7}$. The nonzero commutation relations for algebra $A^{abc}_{5,7}$ have the following form:

$$\begin{gathered}[e_1, e_5] = e_1, \qquad [e_2, e_5] = ae_2, \qquad [e_3, e_5] = be_3, \\ [e_4, e_5] = ce_4, \quad (abc \neq 0, -1 \leq c \leq b \leq a \leq 1).\end{gathered} \tag{26}$$

This Lie algebra is solvable.

The isomorphism of five-dimensional nonconjugate subalgebras of the Lie algebra of the group P(1,4) and Lie algebras of the type $A^{abc}_{5,7}$ is possible only when $a = 1$, $b = 1$, and $c = 1$. Only one nonconjugate subalgebra is isomorphic to subalgebra of this type.

Invariant operators for this subalgebra are rational invariants.

The nonconjugate subalgebra of the type $A^{abc}_{5,7}$ of the Lie algebra of the group P(1,4) and its invariant operators are given in Table 19.

5.4. Lie Algebras of the Type $A^{bc}_{5,9}$. The nonzero commutation relations for algebra $A^{bc}_{5,9}$ have the following form:

$$\begin{gathered}[e_1, e_5] = e_1, \qquad [e_2, e_5] = e_1 + e_2, \\ [e_3, e_5] = be_3, \qquad [e_4, e_5] = ce_4, \quad (0 \neq c \leq b).\end{gathered} \tag{27}$$

This Lie algebra is solvable.

Table 20: Invariant operators of subalgebras of the type $A_{5,9}^{bc}$.

Basis elements of subalgebras	Invariant operators
$b=1, c=1: \langle 2a_2X_4, P_1, P_2, P_3, G+a_2X_1, a_2<0\rangle$	$\frac{X_4}{P_2}, \frac{X_4}{P_3}, X_4\exp\left(\frac{-P_1}{2a_2X_4}\right)$

Table 21: Invariant operators of subalgebras of the type $A_{5,13}^{apq}$.

Basis elements of subalgebras	Invariant operators
$a=1, p=1, q=\frac{1}{e}: \left\langle P_3, X_4, P_1, P_2, G+\frac{1}{e}L_3, e<0\right\rangle$	$\frac{P_3}{X_4}, \frac{P_3^2}{P_1^2+P_2^2}, P_3^{2/e}\left(\frac{P_1+iP_2}{P_1-iP_2}\right)^i$
$a=1, p=0, q=\frac{1}{e}: \left\langle P_3, X_4, X_1, X_2, G+\frac{1}{e}L_3, e<0\right\rangle$	$\frac{P_3}{X_4}, \frac{1}{X_1^2+X_2^2}, P_3^{2/e}\left(\frac{X_1+iX_2}{X_1-iX_2}\right)^i$
$a=-1, p=0, q=\frac{1}{e}: \left\langle X_0, X_4, -X_1, X_2, -G-\frac{1}{e}L_3, e>0\right\rangle$	$\frac{1}{X_0X_4}, \frac{1}{X_1^2+X_2^2}, X_0^{2/e}\left(\frac{X_1-iX_2}{X_1+iX_2}\right)^i$
$a=-1, p=0, q=-\frac{1}{e}: \left\langle X_0, X_4, X_1, X_2, -G-\frac{1}{e}L_3-\frac{\kappa_3}{e}X_3, e>0, \kappa_3<0\right\rangle$	$\frac{1}{X_0X_4}, \frac{1}{X_1^2+X_2^2}, X_0^{-2/e}\left(\frac{X_1+iX_2}{X_1-iX_2}\right)^i$

Table 22: Invariant operators of subalgebras of the type $A_{5,14}^{p}$ $(p=0)$.

Basis elements of subalgebras	Invariant operators
$\langle -2X_4, X_3, -P_1, P_2, P_3-L_3\rangle$	$X_4, P_1^2+P_2^2, \arcsin\frac{P_1}{\sqrt{P_1^2+P_2^2}}+\frac{X_3}{2X_4}$
$\langle 2X_4, X_3, X_1, X_2, L_3-P_3\rangle$	$X_4, X_1^2+X_2^2, \arcsin\frac{X_1}{\sqrt{X_1^2+X_2^2}}+\frac{X_3}{2X_4}$
$\langle 2d_3X_4, P_3+X_3, P_1, P_2, L_3-d_3X_3, d_3<0\rangle$	$X_4, P_1^2+P_2^2, \arcsin\frac{P_1}{\sqrt{P_1^2+P_2^2}}+\frac{P_3+X_3}{2d_3X_4}$
$\langle 2d_3X_4, P_3, P_1, P_2, L_3+d_3X_3, d_3<0\rangle$	$X_4, P_1^2+P_2^2, \arcsin\frac{P_1}{\sqrt{P_1^2+P_2^2}}+\frac{P_3}{2d_3X_4}$
$\langle 2dX_4, P_3+X_0, X_1, X_2, L_3+dX_3, d<0\rangle$	$X_4, X_1^2+X_2^2, \arcsin\frac{X_1}{\sqrt{X_1^2+X_2^2}}+\frac{P_3+X_0}{2dX_4}$
$\langle -2dX_4, P_3, -X_1, X_2, -L_3-dX_3, d<0\rangle$	$X_4, X_1^2+X_2^2, \arcsin\frac{X_1}{\sqrt{X_1^2+X_2^2}}+\frac{P_3}{2dX_4}$
$\langle 2X_4, X_3, X_1, X_2, L_3-P_3+\alpha_0X_0, \alpha_0<0\rangle$	$X_4, X_1^2+X_2^2, \arcsin\frac{X_1}{\sqrt{X_1^2+X_2^2}}+\frac{X_3}{2X_4}$

The isomorphism of five-dimensional nonconjugate subalgebras of the Lie algebra of the group P(1,4) and Lie algebras of the type $A_{5,9}^{bc}$ is possible only when $b=1$ and $c=1$. Only one class of nonconjugate subalgebras is isomorphic to Lie algebra of this type.

Among the invariant operators of these subalgebras there are rational invariants as well as general invariants.

The nonconjugate subalgebras of the type $A_{5,9}^{bc}$ of the Lie algebra of the group P(1,4) and its invariant operators are given in Table 20.

5.5. Lie Algebras of the Type $A_{5,13}^{apq}$. The nonzero commutation relations for algebra $A_{5,13}^{apq}$ have the following form:

$$[e_1,e_5]=e_1,\qquad [e_2,e_5]=ae_2,\qquad [e_3,e_5]=pe_3-qe_4,$$
$$[e_4,e_5]=qe_3+pe_4,\quad (aq\neq 0,\ |a|\leq 1). \tag{28}$$

This Lie algebra is solvable.

There exist four classes of five-dimensional nonconjugate subalgebras of the Lie algebra of the group P(1,4) which are isomorphic to Lie algebras of the type $A_{5,13}^{apq}$. They correspond to four different values of parameters a, p, q.

Among the invariant operators of these subalgebras there are rational invariants as well as general invariants.

The nonconjugate subalgebras of the type $A_{5,13}^{apq}$ of the Lie algebra of the group P(1,4) and their invariant operators are given in Table 21.

5.6. Lie Algebras of the Type $A_{5,14}^{p}$. The nonzero commutation relations for algebra $A_{5,14}^{p}$ have the following form:

$$[e_2,e_5]=e_1,\qquad [e_3,e_5]=pe_3-e_4,\qquad [e_4,e_5]=e_3+pe_4. \tag{29}$$

This Lie algebra is solvable.

There exist seven five-dimensional nonconjugate subalgebras of the Lie algebra of the group P(1,4) which are

Table 23: Invariant operators of subalgebras of the type $A^{pq}_{5,16}$.

Basis elements of subalgebras	Invariant operators
$p=1, q=-\frac{1}{b}: \left\langle 2\frac{k}{b}X_4, P_3, -P_1, P_2, G+\frac{1}{b}L_3+\frac{k}{b}X_3, b>0, k<0 \right\rangle$	$\frac{X_4^2}{P_1^2+P_2^2}, X_4^{-2/b}\left(\frac{P_1+iP_2}{P_1-iP_2}\right), X_4\exp\left(\frac{-bP_3}{2kX_4}\right)$
$p=0, q=\frac{1}{d}: \left\langle 2\frac{\alpha_3}{d}X_4, P_3, X_1, X_2, G+\frac{1}{d}L_3+\frac{\alpha_3}{d}X_3, \alpha_3<0, d>0 \right\rangle$	$\frac{1}{X_1^2+X_2^2}, X_4^{2/d}\left(\frac{X_1-iX_2}{X_1+iX_2}\right), X_4\exp\left(\frac{-dP_3}{2\alpha_3X_4}\right)$

isomorphic to Lie algebras of the type $A^{p}_{5,14}$, five of which depend on parameters and hence constitute continua of subalgebras. This isomorphism is possible only if $p=0$.

Among the invariant operators of these subalgebras there are Casimir operators as well as general invariants.

The nonconjugate subalgebras of the type $A^{p}_{5,14}$ $(p = 0)$ of the Lie algebra of the group P(1,4) and their invariant operators are given in Table 22.

5.7. Lie Algebras of the Type $A^{pq}_{5,16}$. The nonzero commutation relations for algebra $A^{pq}_{5,16}$ have the following form:

$$[e_1,e_5]=e_1, \qquad [e_2,e_5]=e_1+e_2,$$
$$[e_3,e_5]=pe_3-qe_4, \qquad [e_4,e_5]=qe_3+pe_4, \quad (q\neq 0). \tag{30}$$

This Lie algebra is solvable.

There exist two classes of five-dimensional nonconjugate subalgebras of the Lie algebra of the group P(1,4) which are isomorphic to Lie algebras of the type $A^{pq}_{5,16}$. They correspond to two different values of parameters p, q.

Among the invariant operators of these subalgebras there are rational invariants as well as general invariants.

The nonconjugate subalgebras of the type $A^{pq}_{5,16}$ of the Lie algebra of the group P(1,4) and their invariant operators are given in Table 23.

5.8. Lie Algebras of the Type $A^{spq}_{5,17}$. The nonzero commutation relations for algebra $A^{spq}_{5,17}$ have the following form:

$$[e_1,e_5]=pe_1-e_2, \qquad [e_2,e_5]=e_1+pe_2,$$
$$[e_3,e_5]=qe_3-se_4, \qquad [e_4,e_5]=se_3+qe_4, \quad (s\neq 0). \tag{31}$$

This Lie algebra is solvable.

There exist three five-dimensional nonconjugate subalgebras of the Lie algebra of the group P(1,4) which are isomorphic to Lie algebras of the type $A^{spq}_{5,17}$, two of which depend on parameters and hence constitute continua of subalgebras. They correspond to two different values of parameters s, p, q.

Among the invariant operators of these subalgebras there are Casimir operators as well as general invariants.

The nonconjugate subalgebras of the type $A^{spq}_{5,17}$ of the Lie algebra of the group P(1,4) and their invariant operators are given in Table 24.

5.9. Lie Algebras of the Type $A^{ab}_{5,19}$. The nonzero commutation relations for algebra $A^{ab}_{5,19}$ have the following form:

$$[e_2,e_3]=e_1, \qquad [e_1,e_5]=ae_1,$$
$$[e_2,e_5]=e_2, \qquad [e_3,e_5]=(a-1)e_3, \tag{32}$$
$$[e_4,e_5]=be_4, \quad (b\neq 0).$$

This Lie algebra is solvable.

There exist four five-dimensional nonconjugate subalgebras of the Lie algebra of the group P(1,4) which are isomorphic to Lie algebras of the type $A^{ab}_{5,19}$, three of which depend on parameters and hence constitute continua of subalgebras. Isomorphism is possible only if $a=1$, $b=1$.

Invariant operators of all subalgebras are rational invariants. Moreover, all subalgebras have the same invariant operator.

The nonconjugate subalgebras of the type $A^{ab}_{5,19}$ $(a = 1, b = 1)$ of the Lie algebra of the group P(1,4) and their invariant operators are given in Table 25.

5.10. Lie Algebras of the Type $A^{a}_{5,20}$. The nonzero commutation relations for algebra $A^{a}_{5,20}$ have the following form:

$$[e_2,e_3]=e_1, \qquad [e_1,e_5]=ae_1,$$
$$[e_2,e_5]=e_2, \qquad [e_3,e_5]=(a-1)e_3, \tag{33}$$
$$[e_4,e_5]=e_1+ae_4.$$

This Lie algebra is solvable.

There exist four classes of five-dimensional nonconjugate subalgebras of the Lie algebra of the group P(1,4) which are isomorphic to Lie algebras of the type $A^{a}_{5,20}$. Isomorphism is possible if $a=1$.

Invariant operators of these subalgebras are general invariants.

The nonconjugate subalgebras of the type $A^{a}_{5,20}$ $(a = 1)$ of the Lie algebra of the group P(1,4) and their invariant operators are given in Table 26.

5.11. Lie Algebras of the Type $A^{p\varepsilon}_{5,26}$. The nonzero commutation relations for algebra $A^{p\varepsilon}_{5,26}$ have the following form:

$$[e_2,e_3]=e_1, \qquad [e_1,e_5]=2pe_1,$$
$$[e_2,e_5]=pe_2+e_3, \qquad [e_3,e_5]=pe_3-e_2, \tag{34}$$
$$[e_4,e_5]=\varepsilon e_1+2pe_4, \quad (\varepsilon=\pm 1).$$

This Lie algebra is solvable.

Table 24: Invariant operators of subalgebras of the type $A_{5,17}^{spq}$.

Basis elements of subalgebras	Invariant operators
$s=1, p=0, q=0: \left\langle X_1, X_2, -X_3, X_0-X_4, L_3+\frac{1}{2}P_3+\frac{1}{2}C_3+X_0-X_4 \right\rangle$	$X_1^2+X_2^2, X_3^2+(X_0-X_4)^2, \arcsin\frac{X_1}{\sqrt{X_1^2+X_2^2}}+\arcsin\frac{X_0-X_4}{\sqrt{X_3^2+(X_0-X_4)^2}}$
$s=-\frac{2}{e}, p=0, q=0: \left\langle -X_2, X_1, X_4-X_0, X_3, L_3+\frac{1}{e}P_3+\frac{1}{e}C_3+X_0-X_4, e>2 \right\rangle$	$X_1^2+X_2^2, X_3^2+(X_0-X_4)^2, \arcsin\frac{-X_2}{\sqrt{X_1^2+X_2^2}}-\frac{e}{2}\arcsin\frac{X_3}{\sqrt{X_3^2+(X_0-X_4)^2}}$
$s=-1, p=0, q=0: \left\langle X_3, X_0-X_4, X_1, X_2, -L_3-\frac{1}{2}P_3-\frac{1}{2}C_3-\frac{\alpha}{2}(X_0+X_4), \alpha<0 \right\rangle$	$X_3^2+(X_0-X_4)^2, X_1^2+X_2^2, \arcsin\frac{X_3}{\sqrt{X_3^2+(X_0-X_4)^2}}-\arcsin\frac{X_2}{\sqrt{X_1^2+X_2^2}}$

Table 25: Invariant operators of subalgebras of the type $A_{5,19}^{ab}$ $(a = 1, b = 1)$.

Basis elements of subalgebras	Invariant operators
$\langle 2X_4, P_1, X_1, P_2, G\rangle$	$\frac{X_4}{P_2}$
$\langle 2X_4, P_1, X_1 + eX_3, P_2, G\rangle$	$\frac{X_4}{P_2}$
$\langle 2X_4, P_1, X_1, P_2, G + a_3X_3, a_3 < 0\rangle$	$\frac{X_4}{P_2}$
$\langle 2X_4, P_1, X_1 + \mu X_3, P_2, G - \alpha\mu X_3, \alpha < 0, \mu > 0\rangle$	$\frac{X_4}{P_2}$

Table 26: Invariant operators of subalgebras of the type $A_{5,20}^{a}$ $(a = 1)$.

Basis elements of subalgebras	Invariant operators
$\langle 2a_2X_4, P_1, a_2X_1, P_2, G + a_2X_2 + a_3X_3, a_2 < 0, a_3 < 0\rangle$	$X_4 \exp\left(-\frac{P_2}{2a_2X_4}\right)$
$\langle 2a_2X_4, P_1, a_2X_1, P_2, G + a_2X_2, a_2 < 0\rangle$	$X_4 \exp\left(-\frac{P_2}{2a_2X_4}\right)$
$\langle 2\alpha X_4, P_1, \alpha\left(X_1 + \mu X_3\right), P_2, G + \alpha X_2, \alpha < 0, \mu > 0\rangle$	$X_4 \exp\left(-\frac{P_2}{2\alpha X_4}\right)$
$\langle 2\beta X_4, P_1, \beta\left(X_1 + \mu X_3\right), P_2, G + \beta X_2 - \alpha\mu X_3, \beta > 0, \mu > 0, \alpha > 0\rangle$	$X_4 \exp\left(-\frac{P_2}{2\beta X_4}\right)$

Table 27: Invariant operators of subalgebras of the type $A_{5,26}^{p\varepsilon}$ $(p = 0, \varepsilon = -1)$.

Basis elements of subalgebras	Invariant operators
$\left\langle -4X_4, P_1 + X_2, P_2 - X_1, -\frac{2}{d_3}\left(P_3 + \tilde{h}_3X_3\right), -L_3 - d_3X_3, d_3 < 0\right\rangle$	X_4
$\langle -4\mu X_4, P_1 + \mu X_2, P_2 - \mu X_1, -2\mu X_3, P_3 - L_3, \mu > 0\rangle$	X_4

There exist two classes of five-dimensional nonconjugate subalgebras of the Lie algebra of the group P(1,4) which are isomorphic to Lie algebras of the type $A_{5,26}^{p\varepsilon}$. Isomorphism is possible if $p = 0$, $\varepsilon = -1$.

Invariant operators of all subalgebras are Casimir operators. Moreover, all subalgebras have the same invariant operator.

The nonconjugate subalgebras of the type $A_{5,26}^{p\varepsilon}(p = 0, \varepsilon = -1)$ of the Lie algebra of the group P(1,4) and their invariant operators are given in Table 27.

5.12. Lie Algebras of the Type $A_{5,30}$. The nonzero commutation relations for algebra $A_{5,30}$ have the following form:

$$\begin{gathered}
\left[e_2, e_4\right] = e_1, \qquad \left[e_3, e_4\right] = e_2, \\
\left[e_1, e_5\right] = (a+1)\, e_1, \qquad \left[e_2, e_5\right] = ae_2, \\
\left[e_3, e_5\right] = (a-1)\, e_3, \qquad \left[e_4, e_5\right] = e_4.
\end{gathered} \tag{35}$$

This Lie algebra is solvable.

There exist three five-dimensional nonconjugate subalgebras of the Lie algebra of the group P(1,4) which are isomorphic to Lie algebras of the type $A_{5,30}$, two of which depend on parameters and hence constitute continua of subalgebras. Isomorphism is possible only if $a = 0$.

Invariant operators of all subalgebras are Casimir operators. Moreover, all subalgebras have the same invariant operator.

The nonconjugate subalgebras of the type $A_{5,30}\,(a = 0)$ of the Lie algebra of the group P(1,4) and their invariant operators are given in Table 28.

5.13. Lie Algebras of the Type $A_{5,35}^{ab}$. The nonzero commutation relations for algebra $A_{5,35}^{ab}$ have the following form:

$$\begin{gathered}
\left[e_1, e_4\right] = be_1, \qquad \left[e_2, e_4\right] = e_2, \\
\left[e_3, e_4\right] = e_3, \qquad \left[e_1, e_5\right] = ae_1, \\
\left[e_2, e_5\right] = -e_3, \qquad \left[e_3, e_5\right] = e_2, \quad \left(a^2 + b^2 \neq 0\right).
\end{gathered} \tag{36}$$

This Lie algebra is solvable.

There exist five five-dimensional nonconjugate subalgebras of the Lie algebra of the group P(1,4) which are isomorphic to Lie algebras of the type $A_{5,35}^{ab}$, three of which depend on parameters and hence constitute continua of subalgebras. Isomorphism is possible only if $a = 0$, $b = 1$.

Invariant operators of all subalgebras are rational invariants.

Table 28: Invariant operators of subalgebras of the type $A_{5,30}$ $(a = 0)$.

Basis elements of subalgebras	Invariant operators
$\langle 2X_4, -X_3, X_0, P_3, G\rangle$	$X_3^2 - 4X_4X_0$
$\left\langle 2X_4, -X_3, X_0, P_3, G + \frac{1}{e}L_3, e > 0\right\rangle$	$X_3^2 - 4X_4X_0$
$\langle 2X_4, -X_3, X_0, P_3, G + aX_1, a < 0\rangle$	$X_3^2 - 4X_4X_0$

Table 29: Invariant operators of subalgebras of the type $A^{ab}_{5,35}$ $(a = 0, b = 1)$.

Basis elements of subalgebras	Invariant operators
$\langle P_3, P_1, P_2, G, L_3\rangle$	$\dfrac{P_3^2}{P_1^2 + P_2^2}$
$\langle X_4, P_2, P_1, G, -L_3\rangle$	$\dfrac{X_4^2}{P_1^2 + P_2^2}$
$\langle X_4, P_2, P_1, G + a_3X_3, -L_3 - d_3X_3, d_3 < 0, a_3 < 0\rangle$	$\dfrac{X_4^2}{P_1^2 + P_2^2}$
$\langle X_4, P_2, P_1, G + a_3X_3, -L_3, a_3 < 0\rangle$	$\dfrac{X_4^2}{P_1^2 + P_2^2}$
$\langle X_4, P_2, P_1, G, -L_3 - d_3X_3, d_3 < 0\rangle$	$\dfrac{X_4^2}{P_1^2 + P_2^2}$

The nonconjugate subalgebras of the type $A^{ab}_{5,35}$ $(a = 0, b = 1)$ of the Lie algebra of the group P(1,4) and their invariant operators are given in Table 29.

As we write above, our investigation of isomorphism of the five-dimensional nonconjugate subalgebras of the Lie algebra of the group P(1,4) to Lie algebras from the classification of five-dimensional Lie algebras given by Mubarakzyanov [43, 44] is based on the Theorem from Section 2. Direct application of this Theorem gives us that there are no five-dimensional nonconjugate subalgebras of the Lie algebra of the group P(1,4) which are isomorphic to the Lie algebras of the following types: $A_{5,1}$, $A_{5,2}$, $A_{5,3}$, $A_{5,6}$, $A^c_{5,8}$, $A_{5,10}$, $A^c_{5,11}$, $A_{5,12}$, $A^a_{5,15}$, $A^p_{5,18}$, $A_{5,21}$, $A_{5,22}$, $A^b_{5,23}$, $A^\varepsilon_{5,24}$, $A^{bp}_{5,25}$, $A_{5,27}$, $A^a_{5,28}$, $A_{5,29}$, $A_{5,31}$, $A^a_{5,32}$, $A^{ab}_{5,33}$, $A^a_{5,34}$, $A_{5,36}$, $A_{5,37}$, $A_{5,38}$, $A_{5,39}$, and $A_{5,40}$. Commutation relations for those types of five-dimensional real Lie algebras as well as their invariant operators have been described in the paper by Patera et al. [7].

6. Conclusions

The aim of this study was to construct invariant operators (generalized Casimir operators) for all five-dimensional nonconjugate subalgebras of the Lie algebra of the Poincaré group P(1,4). To realize this aim, we at first perform the classification of those subalgebras into classes of isomorphic subalgebras by using a complete classification of real structures of Lie algebras of dimension ≤5 obtained by Mubarakzyanov [43, 44]. Next, we construct invariant operators for all five-dimensional nonconjugate subalgebras of the Lie algebra of the group P(1,4) by using invariant operators of all real Lie algebras of dimension ≤5 constructed by Patera et al. [7]. The results obtained are summarized in Tables 1–29.

Let us give a few comments on the results of this paper.

(i) Invariant operators for nonconjugate subalgebras of the Lie algebra of the Poincaré group P(1,4) which are isomorphic to five-dimensional nilpotent Lie algebras of the types $A_{5,4}$ and $A_{5,5}$ are Casimir operators.

(ii) Invariant operators for nonconjugate subalgebras of the Lie algebra of the group P(1,4) which are isomorphic to five-dimensional solvable Lie algebras from the Mubarakzyanov's list are Casimir operators, rational invariants, and general invariants.

(iii) All nonconjugate subalgebras of the Lie algebra of the group P(1,4) from classes of ones which are isomorphic to the following types of Lie algebras

$$A_{3,6} \oplus A_2, A_{5,4}, A_{5,5}, A^{ab}_{5,19}, A^{p\varepsilon}_{5,26}, A_{5,30}. \tag{37}$$

have the same invariant operators.

In particular, the results obtained can be used

(i) in the representation theory of the group P(1,4) and its nonconjugate subgroups;

(ii) to solve the problems of reduction of the irreducible representations of the group P(1,4) (or the Lie algebra of the group P(1,4)) by the irreducible representations of its subgroups (or its subalgebras); it should be noted that the realisations of all classes of unitary irreducible representations of the Poincaré group P(1,4) on a basis in which the Casimir operators of its important subgroup, that is, the Galilei group G(1,3), are of diagonal form, have been found by Fushchich and Nĭkĭtĭn [37];

(iii) for the construction of differential equations invariant with respect to nonconjugate subgroups of the Poincaré group P(1,4);

(iv) for the construction of systems of coordinates, in which differential equations, invariant with respect to the group P(1,4) (or its nonconjugate subgroups), admit partial or full separation of variables.

Since the Lie algebra of the group P(1,4) contains, as subalgebras, the Lie algebra of the Poincaré group P(1,3) and the Lie algebra of the extended Galilei group G(1,3) (Fushchich and Nĭkĭtĭn [37]), the results obtained will be useful to solve the problems of relativistic and nonrelativistic physics.

References

[1] H. B. G. Casimir, "Ueber die Konstruktion einer zu den irreduzibelen Darstellungen halbeinfacher kontinuierlicher Gruppen gehorigen Differentialgleichung," *Proceedings of the Royal Academy Amsterdam*, vol. 34, pp. 844–846, 1931.

[2] E. G. Beltrametti and A. Blasi, "On the number of Casimir operators associated with any Lie group," *Physics Letters B*, vol. 20, pp. 62–64, 1966.

[3] M. Pauri and G. M. Prosperi, "On the construction of the invariant operators for any finite-parameter lie group," *Nuovo Cimento A*, vol. 43, no. 2, pp. 533–537, 1966.

[4] W. I. Fushchich and I. Yu. Krivsky, *On Wave Equations in the Minkowski Five-Space*, Preprint No. ITF-68-72, Institute for Theoretical Physics, Ukrainian Academy of Sciences, Kiev, Ukraine, 1968 (Russian).

[5] V. I. Fushchich and I. Yu. Krivsky, "On representations of the inhomogeneous de Sitter group and equations in five-dimensional Minkowski space," *Nuclear Physics B*, vol. 14, pp. 573–585, 1969.

[6] L. Abellanas and L. Martínez Alonso, "A general setting for Casimir invariants," *Journal of Mathematical Physics*, vol. 16, pp. 1580–1584, 1975.

[7] J. Patera, R. T. Sharp, P. Winternitz, and H. Zassenhaus, "Invariants of real low dimension Lie algebras," *Journal of Mathematical Physics*, vol. 17, no. 6, pp. 986–994, 1976.

[8] J. Patera, R. T. Sharp, P. Winternitz, and H. Zassenhaus, "Subgroups of the Poincaré group and their invariants," *Journal of Mathematical Physics*, vol. 17, no. 6, pp. 977–985, 1976.

[9] H. Zassenhaus, "On the invariants of a Lie group. I: computers in nonassociative rings and algebras," in *Special Session of the 82nd Annual Meeting of the American Mathematical Society (San Ontario, 1976)*, R. E. Beck and B. Kolman, Eds., pp. 139–155, Academic Press, New York, NY, USA, 1977.

[10] M. Perroud, "The fundamental invariants of inhomogeneous classical groups," *Journal of Mathematical Physics*, vol. 24, no. 6, pp. 1381–1391, 1983.

[11] J. Lemke, Y. Neeman, and J. Pecina-Cruz, "Wigner analysis and Casimir operators of SA(4, R)," *Journal of Mathematical Physics*, vol. 33, no. 8, pp. 2656–2659, 1992.

[12] "The Casimir effect 50 years later," in *Proceedings of the Fourth Workshop on Quantum Field Theory under the Influence of External Conditions (Leipzig, September 14–18, 1998)*, M . Bordag, Ed., World Scientific, River Edge, NJ, USA, 1999.

[13] S. Tremblay and P. Winternitz, "Invariants of the nilpotent and solvable triangular Lie algebras," *Journal of Physics A*, vol. 34, no. 42, pp. 9085–9099, 2001.

[14] J. C. Ndogmo, "Invariants of a semi-direct sum of Lie algebras," *Journal of Physics A*, vol. 37, no. 21, pp. 5635–5647, 2004.

[15] F. J. Echarte, J. Núñez, and F. Ramírez, "Relations among invariants of complex filiform Lie algebras," *Applied Mathematics and Computation*, vol. 147, no. 2, pp. 365–376, 2004.

[16] E. G. Kalnins, Z. Thomova, and P. Winternitz, "Subgroup type coordinates and the separation of variables in Hamilton-Jacobi and Schrödinger equations," *Journal of Nonlinear Mathematical Physics*, vol. 12, no. 2, pp. 178–208, 2005.

[17] J. N. Pecina-Cruz, "On the Casimir of the group $ISL(n, R)$ and its algebraic decomposition," *Journal of Mathematical Physics*, vol. 46, no. 6, Article ID 063503, 2005.

[18] L. Šnobl and P. Winternitz, "A class of solvable Lie algebras and their Casimir invariants," *Journal of Physics A*, vol. 38, no. 12, pp. 2687–2700, 2005.

[19] J. M. Ancochea, R. Campoamor-Stursberg, and L. Garcia Vergnolle, "Solvable Lie algebras with naturally graded nilradicals and their invariants," *Journal of Physics A*, vol. 39, no. 6, pp. 1339–1355, 2006.

[20] V. Boyko, J. Patera, and R. Popovych, "Computation of invariants of Lie algebras by means of moving frames," *Journal of Physics A*, vol. 39, no. 20, pp. 5749–5762, 2006.

[21] V. Boyko, J. Patera, and R. Popovych, "Invariants of Lie algebras with fixed structure of nilradicals," *Journal of Physics A*, vol. 40, no. 1, pp. 113–130, 2007.

[22] V. Boyko, J. Patera, and R. Popovych, "Invariants of triangular Lie algebras," *Journal of Physics A*, vol. 40, no. 27, pp. 7557–7572, 2007.

[23] V. Boyko, J. Patera, and R. Popovych, "Invariants of triangular Lie algebras with one nil-independent diagonal element," *Journal of Physics A*, vol. 40, no. 32, pp. 9783–9792, 2007.

[24] V. Boyko, J. Patera, and R. Popovych, "Invariants of solvable Lie algebras with triangular nilradicals and diagonal nilindependent elements," *Linear Algebra and Its Applications*, vol. 428, no. 4, pp. 834–854, 2008.

[25] V. Boyko, J. Patera, and R. Popovych, "Invariants of Lie algebras via moving frames," in *Group Analysis of Differential Equations and Integrable Systems*, pp. 36–44, [s.n.], [s.l.], 2009.

[26] R. Campoamor-Stursberg, *Structural Data and Invariants of Nine Dimensional Real Lie Algebras with Nontrivial Levi Decomposition*, Nova Science, New York, NY, USA, 2009.

[27] R. Campoamor-Stursberg and S. G. Low, "Virtual copies of semisimple Lie algebras in enveloping algebras of semidirect products and Casimir operators," *Journal of Physics A*, vol. 42, no. 6, Article ID 065205, 2009.

[28] V. I. Fuščič, "Representations of the total inhomogeneous de Sitter group, and equations in the five-dimensional approach. I," *Teoreticheskaya i Matematicheskaya Fizika*, vol. 4, no. 3, pp. 360–382, 1970 (Russian).

[29] V. G. Kadyševskiĭ, "A new approach to the theory of electromagnetic interactions," *Fizika Elementarnykh Chastits i Atomnogo Yadra*, vol. 11, no. 1, pp. 5–39, 1980 (Russian).

[30] V. I. Fushchich and A. G. Nikitin, *Symmetry of Equations of Quantum Mechanics*, Nauka, Moscow, Russia, 1990 (Russian).

[31] W. I. Fushchych and I. Yu. Krivsky, "On a possible approach to the variable-mass problem," *Nuclear Physics B*, vol. 7, no. 1, pp. 79–82, 1968.

[32] V. M. Fedorčuk, "Continuous subgroups of the inhomogeneous de Sitter group P(1, 4)," Preprint No. 78.18, Institute of Mathematics, Ukrainian Academy of Sciences, Kiev, Ukraine, 1978 (Russian).

[33] V. M. Fedorčuk, "Split subalgebras of the Lie algebra of the generalized Poincaré group P(1,4)," *Ukrainian Mathematical Journal*, vol. 31, no. 6, pp. 554–558, 1979.

[34] V. M. Fedorchuk and W. I. Fushchich, "On subgroups of the generalized Poincaré group," in *Proceedings of the International Seminar on Group Theoretical Methods in Physics (Zvenigorod, 1979)*, vol. 1, pp. 61–66, Nauka, Moscow, Russia, 1980 (Russian).

[35] V. M. Fedorchuk, "Nonsplit subalgebras of the Lie algebra of the generalized Poincaré group P(1,4)," *Ukrainian Mathematical Journal*, vol. 33, no. 5, pp. 535–538, 1981.

[36] W. I. Fushchich, A. F. Barannik, L. F. Barannik, and V. M. Fedorchuk, "Continuous subgroups of the Poincaré group P(1,4)," *Journal of Physics A*, vol. 18, no. 15, pp. 2893–2899, 1985.

[37] V. I. Fushchich and A. G. Nĭkĭtĭn, "Reduction of the representations of the generalized Poincaré algebra by the Galilei algebra," *Journal of Physics A*, vol. 13, no. 7, pp. 2319–2330, 1980.

[38] V. M. Fedorchuk, "Invariant operators of splittable subgroups of the generalized Poincaré group P(1, 4)," in *Symmetry and Solutions of Equations of Mathematical Physics*, pp. 90–92, Institute of Mathematics, Ukrainian Academy of Sciences, Kiev, Ukraine, 1989 (Russian).

[39] V. M. Fedorchuk, "Operators invariant under nonsplittable subgroups of the generalized Poincaré group P(1, 4)," in *Algebra-Theoretic Analysis of Equations of Mathematical Physics*, pp. 98–100, Institute of Mathematics, Ukrainian Academy of Sciences, Kiev, Ukraine, 1990 (Russian).

[40] M. Léveillé, "Casimir invariants for the eight-dimensional subgroups of the Poincaré group P(1, 4)," *Journal of Mathematical Physics*, vol. 25, no. 11, pp. 3331–3333, 1984.

[41] V. M. Fedorchuk and V. Ī. Fedorchuk, "On invariant operators of low-dimension nonconjugate subalgebras of the Lie algebra of the Poincaré group P(1, 4)," *Matematicheskie Metody i Fiziko-Mekhanicheskie Polya*, vol. 50, no. 1, pp. 16–23, 2007 (Ukrainian).

[42] V. M. Fedorchuk and V. Ī. Fedorchuk, "Invariant operators for four-dimensional nonconjugate subalgebras of the Lie algebra of the Poincaré group P(1, 4)," *Matematicheskie Metody i Fiziko-Mekhanicheskie Polya*, vol. 53, no. 4, pp. 17–27, 2010 (Ukrainian), English translation in *Journal of Mathematical Sciences*, vol. 181, no. 3, pp. 305–319, 2012.

[43] G. M. Mubarakzjanov, "On solvable Lie algebras," *Izvestija Vysših Učebnyh Zavedeniĭ Matematika*, vol. 32, no. 1, pp. 114–123, 1963 (Russian).

[44] G. M. Mubarakzjanov, "Classification of real structures of Lie algebras of fifth order," *Izvestija Vysših Učebnyh Zavedeniĭ Matematika*, vol. 34, no. 3, pp. 99–106, 1963 (Russian).

[45] L. V. Ovsiannikov, *Group Analysis of Differential Equations*, Academic Press, New York, NY, USA, 1982.

[46] J. Patera, P. Winternitz, and H. Zassenhaus, "Continuous subgroups of the fundamental groups of physics. I. General method and the Poincaré group," *Journal of Mathematical Physics*, vol. 16, no. 8, pp. 1597–1614, 1975.

[47] V. I. Fushchich, L. F. Barannik, and A. F. Barannik, *Subgroup Analysis of Galilei and Poincaré Groups and the Reduction of Nonlinear Equations*, Naukova Dumka, Kiev, Ukraine, 1991 (Russian).