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Research Article

Periodic Solutions of Duffing Equation with an Asymmetric Nonlinearity and a Deviating Argument

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We study the existence of periodic solutions of the second-order differential equation $x'' + ax^+ - bx^- + g(x(t-\tau)) = p(t)$, where a, b are two constants satisfying $1/\sqrt{a} + 1/\sqrt{b} = 2/n$, $n \in N$, τ is a constant satisfying $0 \le \tau < 2\pi$, $g, p : R \to R$ are continuous, and p is 2π-periodic. When the limits $\lim_{x\to\pm\infty} g(x) = g(\pm\infty)$ exist and are finite, we give some sufficient conditions for the existence of 2π-periodic solutions of the given equation.

1. Introduction

In this paper, we are concerned with the existence of periodic solutions of the second-order differential equation with an asymmetric nonlinearity and a deviating argument:

$$x'' + ax^+ - bx^- + g(x(t-\tau)) = p(t), \tag{1}$$

where a, b are two constants satisfying $1/\sqrt{a} + 1/\sqrt{b} = 2/n$, $n \in N$, τ is a constant satisfying $0 \le \tau < 2\pi$, $g, p : R \to R$ are continuous, and p is 2π-periodic.

In recent years, the periodic problem of the second-order differential equation with a deviating argument has been widely studied because of its background in applied sciences (see [1–6] and the references cited therein).

In case when $\tau = 0$ and $a = b = n^2$, (1) becomes

$$x'' + n^2 x + g(x) = p(t). \tag{2}$$

Assume that the limits

$$\lim_{x\to\pm\infty} g(x) = g(\pm\infty) \tag{g}$$

exist and are finite. Lazer and Leach [7] proved that (2) has one 2π-periodic solution provided that the function

$$\Psi(\theta) = 2\left[g(+\infty) - g(-\infty)\right] - \int_0^{2\pi} p(t)\sin n(t+\theta)\,dt \tag{3}$$

is of constant sign.

In case when $\tau = 0$ and a, b satisfy the equation $1/\sqrt{a} + 1/\sqrt{b} = 2/n$, $n \in N$, (1) becomes

$$x'' + ax^+ - bx^- + g(x) = p(t). \tag{4}$$

Equation (4) was first introduced by Fučík [8]. Lately, the periodic problem of (4) was widely studied in the literature (see [9–13] and the references cited therein). To deal with the existence of periodic solutions of (4), Dancer [9] introduced a $2\pi/n$-periodic function

$$\Phi(\theta) = 2n\left[\frac{g(+\infty)}{a} - \frac{g(-\infty)}{b}\right] - \int_0^{2\pi} p(t)\,c(t+\theta)\,dt, \tag{5}$$

where $c(t)$ is a $2\pi/n$-periodic function defined by

$$c(t) = \begin{cases} \dfrac{1}{\sqrt{a}}\sin\left(\sqrt{a}t\right), & 0 \le t \le \dfrac{\pi}{\sqrt{a}}, \\ -\sqrt{\dfrac{1}{b}}\sin\left[\sqrt{b}\left(t - \dfrac{\pi}{\sqrt{a}}\right)\right], & \dfrac{\pi}{\sqrt{a}} \le t \le \dfrac{2\pi}{n}. \end{cases} \tag{6}$$

Obviously, $c(t)$ is a periodic solution of the equation $x'' + ax^+ - bx^- = 0$ satisfying the initial value $x(0) = 0$, $x'(0) = 1$. It was proved in [9] that (4) has at least one 2π-periodic solution provided that Φ has a constant sign in $[0, 2\pi/n)$.

In the present paper, we will deal with the periodic solutions of (1) under condition (g). Owing to the appearance of the asymmetric nonlinearity $ax^+ - bx^-$, the methods in [4, 5] are no longer valid. To overcome this difficulty, we embed (1) into an operator equation with the form $Lx = N(x, \lambda)$ instead of $Lx = \lambda Nx$ as in [4, 5]. We first prove a continuation lemma and then apply this continuation lemma to prove the existence of periodic solution of (1).

Let us denote

$$\nu = \tau \left(\bmod \frac{2\pi}{n} \right). \tag{7}$$

Obviously, we have

$$0 \leq \nu < \frac{2\pi}{n}. \tag{8}$$

We obtain the following result.

Theorem 1. *Assume that condition* (g) *holds and* $0 \leq \nu \leq \min\{\pi/\sqrt{a}, \pi/\sqrt{b}\}$. *Then* (1) *has at least one* 2π*-periodic solution provided that either*

$$\begin{aligned} & ng(-\infty)\left(\frac{1-\cos\sqrt{a}\nu}{a} - \frac{1+\cos\sqrt{b}\nu}{b}\right) \\ & \quad + ng(+\infty)\left(\frac{1+\cos\sqrt{a}\nu}{a} - \frac{1-\cos\sqrt{b}\nu}{b}\right) \\ & \neq \int_0^{2\pi} p(t)\, c(t+\theta)\, dt, \quad \forall \theta \in [0, 2\pi] \end{aligned} \tag{9}$$

or

$$\begin{aligned} & n\left(g(-\infty) - g(+\infty)\right)\left(\frac{\sin\sqrt{a}\nu}{\sqrt{a}} + \frac{\sin\sqrt{b}\nu}{\sqrt{b}}\right) \\ & \neq \int_0^{2\pi} p(t)\, s(t+\theta)\, dt, \quad \forall \theta \in [0, 2\pi] \end{aligned} \tag{10}$$

holds, where the function s *is defined by* $s(t) = c'(t)$, $t \in R$.

Remark 2. In the case when $\min\{\pi/\sqrt{a}, \pi/\sqrt{b}\} \leq \nu < 2\pi/n$, we can obtain the similar sufficient conditions. For brevity, we omit the detailed description.

Remark 3. Obviously, if $\nu = 0$ or $\tau = 2k\pi/n$, $k = 0, 1, 2, \ldots, n-1$, then the first inequality of Theorem 1 reduces to the condition as in [8]; namely,

$$\begin{aligned} & 2n\left[\frac{g(+\infty)}{a} - \frac{g(-\infty)}{b}\right] \neq \int_0^{2\pi} p(t)\, c(t+\theta)\, dt, \\ & \qquad \forall \theta \in [0, 2\pi]. \end{aligned} \tag{11}$$

Throughout this paper, we always use $\mathbf{R}$ to denote the real number set. For a multivariate function ζ depending on r, the notation $\zeta = o(1)$ always means that, for $r \to \infty$, $\zeta \to 0$ holds uniformly with respect to other variables, whereas $\zeta = O(1)$ (or $\zeta = O(r^{-1})$) always means that ζ (or $r \cdot \zeta$) is bounded for r large enough. For any continuous 2π-periodic function $\phi(t)$, we always set $\|\phi\|_\infty = \max_{0 \leq t \leq 2\pi} |\phi(t)|$.

2. Preliminary Lemmas

We now embed (1) into a family of equations with one parameter $\lambda \in [0, 1]$,

$$x'' + ax^+ - bx^- + (1-\lambda)\psi\left(x'\right) + \lambda g(x(t-\tau)) = \lambda p(t), \tag{12}$$

where $\psi : R \to R$ is continuous and satisfies the sign condition as follows:

$$\psi(x)\, x > 0, \quad \forall x \in R, \ x \neq 0. \tag{13}$$

Lemma 4. *Suppose that there exist two positive constants* M_1 *and* M_2 *such that, for any* 2π*-periodic solution* $x(t)$ *of* (12), *the following conditions hold:*

$$\|x\|_\infty < M_1, \qquad \left\|x'\right\|_\infty < M_2. \tag{14}$$

Then (1) *has at least one* 2π*-periodic solution.*

Proof. We follow an argument in [14] to prove Lemma 4. At first, we introduce some notations. Let X and Y be two Banach spaces defined by

$$\begin{aligned} X &= \left\{x \in C^1(\mathbf{R}, \mathbf{R}) : x(t+2\pi) = x(t), \forall t \in \mathbf{R}\right\}, \\ Y &= \left\{y \in C(\mathbf{R}, \mathbf{R}) : y(t+2\pi) = y(t), \forall t \in \mathbf{R}\right\}, \end{aligned} \tag{15}$$

with the norms

$$\|x\|_X = \max\left\{\|x\|_\infty, \left\|x'\right\|_\infty\right\}, \qquad \|y\|_Y = \|y\|_\infty. \tag{16}$$

Define a linear operator by

$$L : D(L) \subset X \longrightarrow Y, \qquad Lx = x'', \tag{17}$$

where $D(L) = \{x \in X : x'' \in C(\mathbf{R}, \mathbf{R})\}$, and a nonlinear operator $N : X \times [0, 1] \to Y$,

$$\begin{aligned} N(x, \lambda)(t) = & -\left(ax^+ - bx^-\right) - (1-\lambda)\psi\left(x'\right) \\ & - \lambda g(x(t-\tau)) + \lambda p(t). \end{aligned} \tag{18}$$

It is easy to see that

$$\operatorname{Ker} L = \mathbf{R}, \qquad \operatorname{Im} L = \left\{y \in Y : \int_0^T y(t)\, dt = 0\right\}. \tag{19}$$

It follows that L is a Fredholm mapping of index zero.

Let us define two continuous projectors $P : X \to \operatorname{Ker} L$ and $Q : Y \to Y$ by setting

$$Px = x(0), \qquad Qy = \frac{1}{T}\int_0^T y(t)\, dt. \tag{20}$$

Set $L_P = L|_{D(L) \cap \operatorname{Ker} P} \to \operatorname{Im} L$. Then L_P is an algebraic isomorphism, and we define $K_P : \operatorname{Im} L \to D(L)$ by

$$K_P = L_P^{-1}. \tag{21}$$

Clearly, we have that, for any $y \in \operatorname{Im} L$,

$$(K_P y)(t) = -\frac{t}{T}\int_0^T (t-s)\, y(s)\, ds + \int_0^t (t-s)\, y(s)\, ds. \tag{22}$$

For any open bounded set $\Omega \subset X$, we can prove by standard arguments that $K_P(I-Q)N$ and QN are relatively compact on the closure $\overline{\Omega}$. Therefore, N is L-compact on $\overline{\Omega}$.

It is noted that (12), together with the 2π-periodic boundary condition, is equivalent to the operator equation

$$Lx = N(x,\lambda). \tag{23}$$

Let $\Omega \subset X$ be the open bounded set defined by

$$\Omega = \left\{x \in X : \|x\|_\infty < M_1, \left\|x'\right\|_\infty < M_2\right\}. \tag{24}$$

From (14), we have

$$Lx \neq N(x,\lambda), \quad \text{for } x \in \partial\Omega \cap D(L),\ \lambda \in [0,1]. \tag{25}$$

Since L is a Fredholm operator with index zero and N is L-compact on $\overline{\Omega} \times [0,1]$, we get from the homotopic invariance of the coincidence degree that

$$D_L(L - N(\cdot,1),\Omega) = D_L(L - N(\cdot,0),\Omega). \tag{26}$$

Next, we will compute $D_L(L - N(\cdot,0),\Omega)$. To this end, we introduce an auxiliary operator $S : \overline{\Omega} \times [0,1] \to Y$ defined by

$$S(x,\mu) = -\left(ax^+ - bx^-\right) - \psi\left(x'\right) - \mu x'. \tag{27}$$

Clearly, S is L-compact on $\overline{\Omega} \times [0,1]$ and

$$S(x,0) = N(x,0), \quad \text{for } x \in \overline{\Omega}. \tag{28}$$

Now, we will prove that

$$Lx \neq S(x,\mu), \quad \text{for } x \in \partial\Omega \cap \operatorname{dom} L,\ \mu \in [0,1]. \tag{29}$$

Obviously, it follows from (25) and (28) that

$$Lx \neq S(x,0), \quad \text{for } x \in \partial\Omega. \tag{30}$$

On the other hand, if $x \in \operatorname{dom} L$ is a solution of $Lx = S(x,\mu)$, then x satisfies the equation as follows:

$$x'' + \mu x' + \left(ax^+ - bx^-\right) + \psi\left(x'\right) = 0. \tag{31}$$

Multiplying both sides of (31) by x' and integrating over $[0,2\pi]$, we get

$$\mu \int_0^{2\pi} x'^2(t)\, dt + \int_0^{2\pi} \psi\left(x'\right) x'\, dt = 0. \tag{32}$$

If $\mu > 0$, then we infer from (13) and (32) that $x'(t) \equiv 0$ for every $t \in [0,2\pi]$. Furthermore, $x(t) \equiv c$ for every $t \in [0,2\pi]$, where c is a constant. Consequently, we have $x(t) \equiv 0$, and then $x \in \Omega$.

From the homotopic invariance of the coincidence degree, we have

$$D_L(L - S(\cdot,0),\Omega) = D_L(L - S(\cdot,1),\Omega). \tag{33}$$

In the following, we will compute $D_L(L - S(\cdot,1),\Omega)$. To this end, we use the equality [15] as follows:

$$\left|D_L(L - S(\cdot,1),\Omega)\right| = \left|d_B\left(-QS(\cdot,1)|_{\operatorname{Ker} L}, \Omega \cap \operatorname{Ker} L, 0\right)\right|, \tag{34}$$

which holds provided that the following conditions are satisfied,

$$Lx \neq \lambda S(x,1), \quad \forall x \in \partial\Omega \cap \operatorname{dom} L,\ \lambda \in (0,1], \tag{35}$$

$$QS(x,1) \neq 0, \quad \forall x \in \partial\Omega \cap \operatorname{Ker} L. \tag{36}$$

In what follows, we will prove that conditions (35) and (36) are satisfied. In fact, if $x \in \partial\Omega \cap \operatorname{dom} L$ is a solution of $Lx = \lambda S(x,1)$, then $x(t)$ satisfies the equation as follows:

$$x''(t) + \lambda x' + \lambda\left(ax^+ - bx^-\right) + \lambda\psi\left(x'\right) = 0. \tag{37}$$

Using the same method as before, we can get $x \in \Omega$. This is a contradiction. To check condition (36), we notice that if $x \in \partial\Omega \cap \operatorname{Ker} L$, then $x(t) = c'$ with $|c'| = M_1$. Hence, we have that, for $x \in \partial\Omega \cap \operatorname{Ker} L$,

$$QS(x,1) = \frac{1}{T}\int_0^T \left(-ax^+ + bx^-\right) dt = -ac' \quad \text{or} \quad -bc' \neq 0. \tag{38}$$

Finally, we can easily calculate the Brouwer degree $d_B(-QS(\cdot,1)|_{\operatorname{Ker} L}, \Omega \cap \operatorname{Ker} L, 0)$ and obtain

$$d_B\left(-QS(\cdot,1)|_{\operatorname{Ker} L}, \Omega \cap \operatorname{Ker} L, 0\right) = 1. \tag{39}$$

Therefore, we have

$$D_L(L - N(\cdot,1),\Omega) \neq 0. \tag{40}$$

Consequently, the equation

$$Lx = N(x,1) \tag{41}$$

has at least one 2π-periodic solution. Equivalently, (1) has at least one 2π-periodic solution. □

Remark 5. In (12), if ψ satisfies the following condition,

$$x\psi(x) < 0, \quad \forall x \in \mathbf{R},\ x \neq 0, \tag{42}$$

then the conclusion of Lemma 4 still holds. This claim can be proved by using the same method as the one used for proving Lemma 4. In fact, we only need to modify the term $-\mu x'$ in the auxiliary operator $S(x,\mu)$ to the term $\mu x'$.

3. Periodic Solutions of Duffing Equation with a Deviating Argument

At first, we choose a continuous function $\psi : R \rightarrow R$ satisfying

$$\lim_{x\rightarrow\pm\infty}\psi(x)=\psi(\pm\infty), \tag{43}$$

where $\psi(\pm\infty) \in R$ are constants. Moreover, ψ satisfies condition (13).

Considering the equivalent system of (12),

$$\begin{aligned} x' &= y, \\ y' &= -\left(ax^{+}-bx^{-}\right)-\lambda g\left(x\left(t-\tau\right)\right)-\left(1-\lambda\right)\psi\left(x'\right) \\ &\quad +\lambda p\left(t\right). \end{aligned} \tag{44}$$

Let $x(t)$ be any (possible) 2π-periodic solution of (12). Write $y(t) = x'(t)$. Then, $(x(t), y(t))$ is a 2π-periodic solution of system (44).

In what follows, we will introduce a transformation. To this end, let us denote by $c(t)$ a solution of equation $x''+ax^{+}-bx^{-} = 0$ satisfying the initial condition $c(0) = 0$, $c'(0) = 1$. Obviously, $c(t)$ is $2\pi/n$-periodic. The derivative of $c(t)$ will be denoted by $s(t) = c'(t)$. It is easy to check that the following properties are satisfied:

(1) $c(t + 2\pi/n) = c(t)$, $s(t + 2\pi/n) = s(t)$.

(2) $c'(t) = s(t)$, $s'(t) = -(ac^{+}(t) - bc^{-}(t))$.

(3) $s(t)^2 + ac^{+}(t)^2 + bc^{-}(t)^2 = 1$, $\forall t \in R$.

Let us define a mapping $\Phi : (\theta, \rho) \in S^1 \times (0, +\infty) \rightarrow (x, y) \in R^2 \setminus \{0\}$ as follows:

$$x=\rho^{1/2}c\left(\frac{\theta}{n}\right),\qquad y=\rho^{1/2}s\left(\frac{\theta}{n}\right), \tag{45}$$

where $S^1 = R/2\pi Z$.

Under the transformation Φ, if $|x(t)| + |y(t)| \neq 0$, $\forall t \in [0, 2\pi]$, then the 2π-periodic solution $(x(t), y(t))$ of system (44) can be expressed in the form $(\rho(t), \theta(t))$ satisfying the equations as follows:

$$\begin{aligned} \frac{d\rho}{dt} &= -2\lambda\rho^{1/2} \\ &\quad\times\left(g\left(\rho^{1/2}\left(t-\tau\right)c\left(\frac{\theta\left(t-\tau\right)}{n}\right)\right)s\left(\frac{\theta}{n}\right)-p\left(t\right)s\left(\frac{\theta}{n}\right)\right) \\ &\quad-2\left(1-\lambda\right)\rho^{1/2}\psi\left(\rho^{1/2}s\left(\frac{\theta}{n}\right)\right)s\left(\frac{\theta}{n}\right), \\ \frac{d\theta}{dt} &= n+n\lambda\rho^{-1/2} \\ &\quad\times\left(g\left(\rho^{1/2}\left(t-\tau\right)c\left(\frac{\theta\left(t-\tau\right)}{n}\right)\right)c\left(\frac{\theta}{n}\right)-p\left(t\right)c\left(\frac{\theta}{n}\right)\right) \\ &\quad+n\left(1-\lambda\right)\rho^{-1/2}\psi\left(\rho^{1/2}s\left(\frac{\theta}{n}\right)\right)c\left(\frac{\theta}{n}\right). \end{aligned} \tag{46}$$

Let us denote $(\rho_0, \theta_0) = (\rho(0), \theta(0))$. From now on, we always assume that g is bounded. From the first equation of (46) we get that

$$\begin{aligned} \frac{d\rho^{1/2}}{dt} &= -\lambda g\left(\rho^{1/2}\left(t-\tau\right)c\left(\frac{\theta\left(t-\tau\right)}{n}\right)\right)s\left(\frac{\theta}{n}\right) \\ &\quad+\lambda p\left(t\right)s\left(\frac{\theta}{n}\right)-\left(1-\lambda\right)\psi\left(\rho^{1/2}s\left(\frac{\theta}{n}\right)\right)s\left(\frac{\theta}{n}\right). \end{aligned} \tag{47}$$

Therefore, we have

$$\rho(t)^{1/2}=\rho_0^{1/2}+O\left(1\right). \tag{48}$$

Furthermore, we get

$$\rho(t)^{-1/2}=\rho_0^{-1/2}+O\left(\rho_0^{-1}\right). \tag{49}$$

From the second equation of (46), we have

$$\frac{d\theta}{dt}=n+O\left(\rho_0^{-1/2}\right). \tag{50}$$

As a result,

$$\theta\left(t\right)=\theta_0+nt+O\left(\rho_0^{-1/2}\right). \tag{51}$$

Substituting (51) in (47), we obtain that, for $t \in [0, 2\pi]$,

$$\begin{aligned} \frac{d\rho^{1/2}}{dt} &= -\lambda g\left(\rho_0^{1/2}c\left(t-\tau+\frac{\theta_0}{n}\right)+O\left(1\right)\right)s\left(t+\frac{\theta_0}{n}\right) \\ &\quad+\lambda p\left(t\right)s\left(t+\frac{\theta_0}{n}\right)-\left(1-\lambda\right)\psi\left(\rho^{1/2}s\left(\frac{\theta}{n}\right)\right)s\left(\frac{\theta}{n}\right) \\ &\quad+O\left(\rho_0^{-1/2}\right). \end{aligned} \tag{52}$$

Consequently,

$$\begin{aligned} &\rho^{1/2}\left(2\pi\right) \\ &= \rho_0^{1/2}-\lambda\int_0^{2\pi}g\left(\rho_0^{1/2}c\left(t-\tau+\frac{\theta_0}{n}\right)+O\left(1\right)\right)s\left(t+\frac{\theta_0}{n}\right)dt \\ &\quad-\left(1-\lambda\right)\int_0^{2\pi}\psi\left(\rho_0^{1/2}s\left(t+\frac{\theta_0}{n}\right)+O\left(1\right)\right)s\left(t+\frac{\theta_0}{n}\right)dt \\ &\quad+\lambda\int_0^{2\pi}p\left(t\right)s\left(t+\frac{\theta_0}{n}\right)dt+O\left(\rho_0^{-1/2}\right). \end{aligned} \tag{53}$$

Similarly, substituting (51) in the second equality of (44), we get that, for $t \in [0, 2\pi]$,

$$\begin{aligned}\frac{d\theta}{dt} = n + n\lambda\rho_0^{-1/2} g\left(\rho_0^{1/2} c\left(t - \tau + \frac{\theta_0}{n}\right) + O(1)\right) \\ \times c\left(t + \frac{\theta_0}{n}\right) - n\lambda\rho_0^{-1/2} p(t) c\left(t + \frac{\theta_0}{n}\right) \\ + n(1-\lambda)\rho_0^{-1/2}\psi\left(\rho_0^{1/2} s\left(t + \frac{\theta_0}{n}\right) + O(1)\right) \\ \times c\left(t + \frac{\theta_0}{n}\right) + O\left(\rho_0^{-1}\right).\end{aligned} \tag{54}$$

Therefore, we have

$$\begin{aligned}&\theta(2\pi) \\ &= \theta_0 + 2n\pi + n\lambda\rho_0^{-1/2} \\ &\quad \times \int_0^{2\pi} g\left(\rho_0^{1/2} c\left(t - \tau + \frac{\theta_0}{n}\right) + O(1)\right) c\left(t + \frac{\theta_0}{n}\right) dt \\ &\quad + n(1-\lambda)\rho_0^{-1/2} \\ &\quad \times \int_0^{2\pi} \psi\left(\rho_0^{1/2} s\left(t + \frac{\theta_0}{n}\right) + O(1)\right) c\left(t + \frac{\theta_0}{n}\right) dt \\ &\quad - n\lambda\rho_0^{-1/2} \int_0^{2\pi} p(t) c\left(t + \frac{\theta_0}{n}\right) dt + O\left(\rho_0^{-1}\right).\end{aligned} \tag{55}$$

Write

$$\begin{aligned}\psi_1(\theta_0) &= \int_0^{2\pi} g\left(\rho_0^{1/2} c\left(t - \tau + \frac{\theta_0}{n}\right) + O(1)\right) s\left(t + \frac{\theta_0}{n}\right) dt, \\ \psi_2(\theta_0) &= \int_0^{2\pi} g\left(\rho_0^{1/2} c\left(t - \tau + \frac{\theta_0}{n}\right) + O(1)\right) c\left(t + \frac{\theta_0}{n}\right) dt, \\ \psi_3(\theta_0) &= \int_0^{2\pi} \psi\left(\rho_0^{1/2} s\left(t + \frac{\theta_0}{n}\right) + O(1)\right) s\left(t + \frac{\theta_0}{n}\right) dt, \\ \psi_4(\theta_0) &= \int_0^{2\pi} \psi\left(\rho_0^{1/2} s\left(t + \frac{\theta_0}{n}\right) + O(1)\right) c\left(t + \frac{\theta_0}{n}\right) dt.\end{aligned} \tag{56}$$

Recalling that $\nu = \tau(\mathrm{mod}(2\pi/n))$ and $0 \le \nu < 2\pi/n$, we have the following estimates.

Lemma 6. *Assume that condition (g) holds. Then, for $\rho_0 \to +\infty$,*

$$\psi_1(\theta_0) = \begin{cases} -n\left(g(+\infty) - g(-\infty)\right)\left(\dfrac{\sin\sqrt{a}\nu}{\sqrt{a}} + \dfrac{\sin\sqrt{b}\nu}{\sqrt{b}}\right) \\ \quad + o(1), \\ \quad \text{for } \nu \le \dfrac{\pi}{\sqrt{a}} \le \dfrac{\pi}{\sqrt{b}}, \text{ or } \nu \le \dfrac{\pi}{\sqrt{b}} \le \dfrac{\pi}{\sqrt{a}}, \\ n\left(g(+\infty) - g(-\infty)\right) \\ \quad \times \left[\dfrac{\sin\sqrt{b}\left(\nu - \pi/\sqrt{a}\right)}{\sqrt{b}} - \dfrac{\sin\sqrt{b}\nu}{\sqrt{b}}\right] + o(1), \\ \quad \text{for } \dfrac{\pi}{\sqrt{a}} < \nu \le \dfrac{\pi}{\sqrt{b}}, \\ n\left(g(+\infty) - g(-\infty)\right) \\ \quad \times \left[\dfrac{\sin\sqrt{a}\left(\nu - \pi/\sqrt{b}\right)}{\sqrt{a}} - \dfrac{\sin\sqrt{a}\nu}{\sqrt{a}}\right] + o(1), \\ \quad \text{for } \dfrac{\pi}{\sqrt{b}} < \nu \le \dfrac{\pi}{\sqrt{a}}, \\ n\left(g(+\infty) - g(-\infty)\right) \\ \quad \times \left[\dfrac{\sin\sqrt{a}\left(\nu - \pi/\sqrt{b}\right)}{\sqrt{a}} + \dfrac{\sin\sqrt{b}\left(\nu - \pi/\sqrt{a}\right)}{\sqrt{b}}\right] \\ \quad + o(1), \\ \quad \text{for } \dfrac{\pi}{\sqrt{a}} \le \dfrac{\pi}{\sqrt{b}} < \nu \text{ or } \dfrac{\pi}{\sqrt{b}} \le \dfrac{\pi}{\sqrt{a}} < \nu. \end{cases} \tag{57}$$

Proof. We only give the proof for the case $0 \le \nu \le \pi/\sqrt{a} \le \pi/\sqrt{b} < 2\pi/n$. The other cases can be treated similarly. Since $s(t)$ is $2\pi/n$-periodic, it follows from the expression of $\psi_1(\theta_0)$ that

$$\begin{aligned}\psi_1(\theta_0) &= \int_0^{2\pi} g\left(\rho_0^{1/2} c\left(t - \tau + \frac{\theta_0}{n}\right) + O(1)\right) s\left(t + \frac{\theta_0}{n}\right) dt \\ &= \int_0^{2\pi} g\left(\rho_0^{1/2} c(u) + O(1)\right) s(u + \tau)\, du \\ &= \int_0^{2\pi} g\left(\rho_0^{1/2} c(u) + O(1)\right) s(u + \nu)\, du.\end{aligned} \tag{58}$$

From the dominated convergent theorem, we have that, for $\rho_0 \to \infty$,

$$\begin{aligned}&\psi_1(\theta_0) \\ &= ng(+\infty)\int_0^{\pi/\sqrt{a}} s(u+\nu)\, du + ng(-\infty) \\ &\quad \times \int_{\pi/\sqrt{a}}^{2\pi/n} s(u+\nu)\, du + o(1) \\ &= ng(+\infty)\left[\int_0^{\pi/\sqrt{a}-\nu} s(u+\nu)\, du + \int_{\pi/\sqrt{a}-\nu}^{\pi/\sqrt{a}} s(u+\nu)\, du\right]\end{aligned}$$

$$
\begin{aligned}
&+ ng(-\infty)\left[\int_{\pi/\sqrt{a}}^{2\pi/n-\nu} s(u+\nu)\,du\right.\\
&\qquad\left.+\int_{2\pi/n-\nu}^{2\pi/n} s(u+\nu)\,du\right]+o(1)\\
&= -ng(+\infty)\left(\frac{\sin\sqrt{a}\nu}{\sqrt{a}}+\frac{\sin\sqrt{b}\nu}{\sqrt{b}}\right)\\
&\quad + ng(-\infty)\left(\frac{\sin\sqrt{a}\nu}{\sqrt{a}}+\frac{\sin\sqrt{b}\nu}{\sqrt{b}}\right)+o(1)\\
&= -n\left(g(+\infty)-g(-\infty)\right)\left(\frac{\sin\sqrt{a}\nu}{\sqrt{a}}+\frac{\sin\sqrt{b}\nu}{\sqrt{b}}\right)+o(1).
\end{aligned}
\tag{59}
$$

□

Lemma 7. *Assume that condition* (g) *holds. Then, for* $\rho_0 \to +\infty$,

$$
\psi_2(\theta_0) = \begin{cases}
ng(+\infty)\left(\dfrac{1+\cos\sqrt{a}\nu}{a}-\dfrac{1-\cos\sqrt{b}\nu}{b}\right)\\
\quad +ng(-\infty)\left(\dfrac{1-\cos\sqrt{a}\nu}{a}-\dfrac{1+\cos\sqrt{b}\nu}{b}\right)+o(1),\\
\quad \text{for } \nu\le\dfrac{\pi}{\sqrt{a}}\le\dfrac{\pi}{\sqrt{b}}, \text{ or } \nu\le\dfrac{\pi}{\sqrt{b}}\le\dfrac{\pi}{\sqrt{a}},\\
ng(+\infty)\left[\dfrac{\cos\sqrt{b}\nu-\cos\sqrt{b}\left(\nu-\pi/\sqrt{a}\right)}{b}\right]+ng(-\infty)\\
\quad\times\left[\left(\dfrac{2}{a}-\dfrac{2}{b}\right)-\dfrac{\cos\sqrt{b}\nu-\cos\sqrt{b}\left(\nu-\pi/\sqrt{a}\right)}{b}\right]\\
\quad +o(1),\\
\quad \text{for } \dfrac{\pi}{\sqrt{a}}<\nu\le\dfrac{\pi}{\sqrt{b}},\\
ng(+\infty)\left[\left(\dfrac{2}{a}-\dfrac{2}{b}\right)-\dfrac{\cos\sqrt{a}\left(\nu-\pi/\sqrt{b}\right)-\cos\sqrt{a}\nu}{a}\right]\\
\quad +ng(-\infty)\left[\dfrac{\cos\sqrt{a}\left(\nu-\pi/\sqrt{b}\right)-\cos\sqrt{a}\nu}{a}\right]\\
\quad +o(1),\\
\quad \text{for } \dfrac{\pi}{\sqrt{b}}<\nu\le\dfrac{\pi}{\sqrt{a}},\\
ng(+\infty)\left[\dfrac{1-\cos\sqrt{a}\left(\nu-\pi/\sqrt{b}\right)}{a}\right.\\
\qquad\left.-\dfrac{1+\cos\sqrt{b}\left(\nu-\pi/\sqrt{a}\right)}{b}\right]\\
\quad +ng(-\infty)\left[\dfrac{1+\cos\sqrt{a}\left(\nu-\pi/\sqrt{b}\right)}{a}\right.\\
\qquad\left.-\dfrac{1-\cos\sqrt{b}\left(\nu-\pi/\sqrt{a}\right)}{b}\right]\\
\quad +o(1),\\
\quad \text{for } \dfrac{\pi}{\sqrt{a}}\le\dfrac{\pi}{\sqrt{b}}<\nu \text{ or } \dfrac{\pi}{\sqrt{b}}\le\dfrac{\pi}{\sqrt{a}}<\nu.
\end{cases}
\tag{60}
$$

Proof. We also only give the proof for the case $0 \le \nu \le \pi/\sqrt{a} \le \pi/\sqrt{b} < 2\pi/n$. The other cases can be treated similarly. Since $c(t)$ is $2\pi/n$-periodic, it follows from the expression of $\psi_2(\theta_0)$ and the dominated convergent theorem that, for $\rho_0 \to \infty$,

$$
\begin{aligned}
&\psi_2(\theta_0)\\
&\quad = \int_0^{2\pi} g\left(\rho_0^{1/2}c\left(t-\tau+\frac{\theta_0}{n}\right)+O(1)\right)c\left(t+\frac{\theta_0}{n}\right)dt\\
&\quad = \int_0^{2\pi} g\left(\rho_0^{1/2}c(u)+O(1)\right)c(u+\tau)\,du\\
&\quad = \int_0^{2\pi} g\left(\rho_0^{1/2}c(u)+O(1)\right)c(u+\nu)\,du\\
&\quad = ng(+\infty)\int_0^{\pi/\sqrt{a}} c(u+\nu)\,du\\
&\qquad + ng(-\infty)\int_{\pi/\sqrt{a}}^{2\pi/n} c(u+\nu)\,du+o(1)\\
&\quad = ng(+\infty)\\
&\qquad\times\left[\int_0^{\pi/\sqrt{a}-\nu} c(u+\nu)\,du+\int_{\pi/\sqrt{a}-\nu}^{\pi/\sqrt{a}} c(u+\nu)\,du\right]\\
&\qquad + ng(-\infty)\\
&\qquad\times\left[\int_{\pi/\sqrt{a}}^{2\pi/n-\nu} c(u+\nu)\,du+\int_{2\pi/n-\nu}^{2\pi/n} c(u+\nu)\,du\right]+o(1)\\
&\quad = ng(+\infty)\left(\frac{1+\cos\sqrt{a}\nu}{a}-\frac{1-\cos\sqrt{b}\nu}{b}\right)\\
&\qquad + ng(-\infty)\left(\frac{1-\cos\sqrt{a}\nu}{a}-\frac{1+\cos\sqrt{b}\nu}{b}\right)+o(1).
\end{aligned}
\tag{61}
$$

□

Lemma 8. *Assume that condition* (43) *holds. Then, for* $\rho_0 \to +\infty$,

$$
\begin{aligned}
\psi_3(\theta_0) &= 2\left[\psi(+\infty)-\psi(-\infty)\right]+o(1),\\
\psi_4(\theta_0) &= n\left[\psi(+\infty)+\psi(-\infty)\right]\left(\frac{1}{a}-\frac{1}{b}\right)+o(1).
\end{aligned}
\tag{62}
$$

Proof. From the expression of $\psi_3(\theta_0)$ and the dominated convergent theorem we have that, for $\rho_0 \to \infty$,

$$
\begin{aligned}
&\psi_3(\theta_0)\\
&\quad = \int_0^{2\pi} \psi\left(\rho_0^{1/2}s\left(t+\frac{\theta_0}{n}\right)+O(1)\right)s\left(t+\frac{\theta_0}{n}\right)dt\\
&\quad = \int_0^{2\pi} \psi\left(\rho_0^{1/2}s(u)+O(1)\right)s(u)\,du
\end{aligned}
$$

$$= n\psi(+\infty)\left(\int_0^{\pi/2\sqrt{a}} s(u)\,du + \int_{\pi/\sqrt{a}+\pi/2\sqrt{b}}^{2\pi/n} s(u)\,du\right)$$
$$+ n\psi(-\infty)\left(\int_{\pi/2\sqrt{a}}^{\pi/\sqrt{a}} s(u)\,du + \int_{\pi/\sqrt{a}}^{\pi/\sqrt{a}+\pi/2\sqrt{b}} s(u)\,du\right)$$
$$+ o(1)$$
$$= 2[\psi(+\infty) - \psi(-\infty)] + o(1). \tag{63}$$

Similarly, we have that, for $\rho_0 \to +\infty$,

$$\psi_4(\theta_0)$$
$$= \int_0^{2\pi} \psi\left(\rho_0^{1/2} s\left(t + \frac{\theta_0}{n}\right) + O(1)\right) c\left(t + \frac{\theta_0}{n}\right) dt$$
$$= \int_0^{2\pi} \psi\left(\rho_0^{1/2} s(u) + O(1)\right) c(u)\,du$$
$$= n\psi(+\infty)\left(\int_0^{\pi/2\sqrt{a}} c(u)\,du + \int_{(\pi/\sqrt{a})+(\pi/2\sqrt{b})}^{2\pi/n} c(u)\,du\right)$$
$$+ n\psi(-\infty)\left(\int_{\pi/2\sqrt{a}}^{\pi/\sqrt{a}} c(u)\,du + \int_{\pi/\sqrt{a}}^{(\pi/\sqrt{a})+(\pi/2\sqrt{b})} c(u)\,du\right)$$
$$+ o(1) = n[\psi(+\infty) + \psi(-\infty)]\left(\frac{1}{a} - \frac{1}{b}\right) + o(1). \tag{64}$$

□

Proof of Theorem 1. We proceed to prove Theorem 1 in two different cases.

(1) Assume that the first inequality of Theorem 1 holds. Without loss of generality, we assume

$$ng(-\infty)\left(\frac{1-\cos\sqrt{a}\nu}{a} - \frac{1+\cos\sqrt{b}\nu}{b}\right)$$
$$+ ng(+\infty)\left(\frac{1+\cos\sqrt{a}\nu}{a} - \frac{1-\cos\sqrt{b}\nu}{b}\right) \tag{65}$$
$$> \int_0^{2\pi} p(t)\,c(t+\theta)\,dt, \quad \forall\theta \in [0, 2\pi].$$

Let us set

$$\eta(\theta) = ng(-\infty)\left(\frac{1-\cos\sqrt{a}\nu}{a} - \frac{1+\cos\sqrt{b}\nu}{b}\right)$$
$$+ ng(+\infty)\left(\frac{1+\cos\sqrt{a}\nu}{a} - \frac{1-\cos\sqrt{b}\nu}{b}\right) \tag{66}$$
$$- \int_0^{2\pi} p(t)\,c(t+\theta)\,dt > 0, \quad \theta \in [0, 2\pi].$$

We now choose a function ψ satisfying (43) and (13). Moreover, $\psi(\pm\infty)$ satisfy

$$\mu = [\psi(+\infty) + \psi(-\infty)]\left(\frac{1}{a} - \frac{1}{b}\right) > 0. \tag{67}$$

Then we infer from Lemmas 7 and 8 that, for $\rho_0 \to \infty$,

$$\theta(2\pi) = \theta_0 + 2n\pi + n\rho_0^{-1/2}\left[\lambda\eta\left(\frac{\theta_0}{n}\right) + n(1-\lambda)\mu\right]$$
$$+ o\left(\rho_0^{-1/2}\right). \tag{68}$$

Since $\eta(\theta) > 0$, $\theta \in [0, 2\pi]$, and $\mu > 0$, there exists a constant $\gamma > 0$ such that, for $\theta \in [0, 2\pi]$ and $\lambda \in [0, 1]$,

$$\lambda\eta(\theta) + n(1-\lambda)\mu \geq \gamma. \tag{69}$$

From (68) and (69) we have that, for $\rho_0 \to \infty$,

$$\theta(2\pi) = \theta_0 + 2n\pi + o(1), \qquad \theta(2\pi) > \theta_0 + 2n\pi. \tag{70}$$

Consequently, there exists a constant $M > 0$ such that if $(\rho(t), \theta(t))$ is a 2π-periodic solution of system (46), then $\rho(t) \leq M$, $t \in [0, 2\pi]$. Furthermore, there exist constants $M_1 > 0$ and $M_2 > 0$ such that if $x(t)$ is a 2π-periodic solution of (12), then

$$\|x\|_\infty < M_1, \qquad \left\|x'\right\|_\infty < M_2. \tag{71}$$

From Lemma 4, we know that (1) has at least one 2π-periodic solution.

(2) We assume that the second inequality of Theorem 1 holds. Without loss of generality, we assume

$$n(g(-\infty) - g(+\infty))\left(\frac{\sin\sqrt{a}\nu}{\sqrt{a}} + \frac{\sin\sqrt{b}\nu}{\sqrt{b}}\right)$$
$$> \int_0^{2\pi} p(t)\,s(t+\theta)\,dt, \quad \forall\theta \in [0, 2\pi]. \tag{72}$$

Let us set

$$\zeta(\theta) = n(g(-\infty) - g(+\infty))\left(\frac{\sin\sqrt{a}\nu}{\sqrt{a}} + \frac{\sin\sqrt{b}\nu}{\sqrt{b}}\right)$$
$$- \int_0^{2\pi} p(t)\,s(t+\theta)\,dt > 0, \quad \theta \in [0, 2\pi]. \tag{73}$$

Similarly, we choose a continuous function ψ satisfying (43) and (13). Moreover, $\psi(\pm\infty)$ satisfy

$$\mu' = \psi(+\infty) - \psi(-\infty) > 0. \tag{74}$$

Then we infer from Lemmas 7 and 8 that, for $\rho_0 \to \infty$,

$$\rho^{1/2}(2\pi) = \rho_0^{1/2} - \left[\lambda\zeta\left(\frac{\theta_0}{n}\right) + 2(1-\lambda)\mu'\right] + o(1). \tag{75}$$

Since $\zeta(\theta) > 0$, $\theta \in [0, 2\pi]$, and $\mu' > 0$, there exists a constant $\gamma' > 0$ such that, for $\theta \in [0, 2\pi]$ and $\lambda \in [0, 1]$ and $\rho_0 \to \infty$,

$$\lambda\zeta(\theta) + 2(1-\lambda)\mu' \geq \gamma'. \tag{76}$$

From (75) and (76) we have that, for sufficiently large ρ_0,

$$\rho^{1/2}(2\pi) \leq \rho_0^{1/2} - \frac{\gamma'}{2}. \tag{77}$$

Consequently, there exists a constant $M' > 0$ such that if $(\rho(t), \theta(t))$ is a 2π-periodic solution of system (46), then $\rho(t) \leq M'$, $t \in [0, 2\pi]$. Furthermore, there exist constants $M_1' > 0$ and $M_2' > 0$ such that if $x(t)$ is a 2π periodic solution of (12), then

$$\|x\|_\infty < M_1', \qquad \left\|x'\right\|_\infty < M_2'. \tag{78}$$

From Lemma 4 we know that (1) has at least one 2π periodic solution. □

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