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Research Article

Hopf Bifurcation Analysis for a Four-Dimensional Recurrent Neural Network with Two Delays

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A four-dimensional recurrent neural network with two delays is considered. The main result is given in terms of local stability and Hopf bifurcation. Sufficient conditions for local stability of the zero equilibrium and existence of the Hopf bifurcation with respect to both delays are obtained by analyzing the distribution of the roots of the associated characteristic equation. In particular, explicit formulae for determining the direction of the Hopf bifurcation and the stability of the bifurcating periodic solutions are established by using the normal form theory and center manifold theory. Some numerical examples are also presented to verify the theoretical analysis.

1. Introduction

In recent years, neural networks have attracted many scholars' attention all over the world and have been applied in different areas such as signal processing [1], pattern recognition [2–4], optimization [5], and automatic control [6–8]. In particular, the appearance of a cycle bifurcating from an equilibrium of an ordinary or a delayed neural network with a single parameter has been widely investigated [9–17]. In [18], Ruiz et al. studied the following recurrent neural network for the first time:

$$\begin{aligned}
\dot{x}_1(t) &= -x_1(t) + f(x_2(t)), \\
&\vdots \\
\dot{x}_{n-1} &= -x_{n-1}(t) + u(t), \\
\dot{x}_n &= -x_n(t) + w_1 f(x_1(t)) + \cdots + w_{n-1} f(x_{n-1}(t)), \\
y(t) &= f(x_n(t)),
\end{aligned} \tag{1}$$

where $x(t) \in R^n$ is the state, $w_i \in R$, $i = 1, \ldots, n-1$ are the network parameters or weights, $u(t)$ is the input, $y(t)$ is the output, and $f(\cdot)$ is the transfer function of the neurons. The three-node network of system (1) in the feedback configuration, with $u(t) = y(t)$, has been studied in [12, 18, 19]; that is

$$\begin{aligned}
\dot{x}_1(t) &= -x_1(t) + f(x_2(t)), \\
\dot{x}_2(t) &= -x_2(t) + f(x_3(t)), \\
\dot{x}_3(t) &= -x_3(t) + w_1 f(x_1(t)) + w_2 f(x_2(t)).
\end{aligned} \tag{2}$$

It is well known that time delays can play a complicated role on neural networks. They can be the source of instabilities and bifurcation in neural networks. Based on this fact, Hajihosseini et al. [11] considered system (2) with distributed delays and $f(\cdot) = \tanh(\cdot)$. It is shown that a Hopf bifurcation takes place in the delayed system as the mean delay passes a critical value where a family of periodic solutions bifurcate from the equilibrium. The existence and stability of such solutions are determined by the Hopf bifurcation theorem in the frequency domain and the generalized Nyquist stability criterion.

As far as we know, there are some papers on the bifurcations of neural network with two or multiple delays [20–22]. Motivated by the work in [11, 20–22] and considering that when the number of neurons is large, the simplified model can reflect the really large neural networks more closely, we consider the following four-dimensional recurrent neural network with two discrete delays that occur in the interaction between the neurons:

$$\begin{aligned}\dot{x}_1(t) &= -x_1(t) + f(x_2(t-\tau_1)),\\ \dot{x}_2(t) &= -x_2(t) + f(x_3(t-\tau_1)),\\ \dot{x}_3(t) &= -x_3(t) + f(x_4(t-\tau_1)),\\ \dot{x}_4(t) &= -x_4(t) + w_1 f(x_1(t-\tau_2))\\ &\quad + w_2 f(x_2(t-\tau_2)) + w_3 f(x_3(t-\tau_2)),\end{aligned} \tag{3}$$

where $\tau_1 \geq 0$, $\tau_2 \geq 0$ are time delays that occur in the interaction between the neurons.

This paper is organized as follows. In Section 2, the stability of the zero equilibrium of system (3) and the existence of local Hopf bifurcation with respect to possible combinations of the two delays are investigated. In Section 3, the properties of the Hopf bifurcation such as the direction and the stability are determined by using the normal form theory and center manifold theory. Some numerical simulations are also included in Section 4 to illustrate the validity of the main results.

2. Stability of the Zero Equilibrium and Local Hopf Bifurcation

Throughout this paper we make the following assumption on the transfer function $f(\cdot)$:

(H) $f \in C^4(R)$, $f(0) = 0$, and $f'(0) \neq 0$.

Clearly, $E_0 = (0,0,0,0)^T$ is the zero equilibrium of system (3). Linearization of system (3) at the zero equilibrium is

$$\begin{aligned}\dot{x}_1(t) &= -x_1(t) + f'(0)x_2(t-\tau_1),\\ \dot{x}_2(t) &= -x_2(t) + f'(0)x_3(t-\tau_1),\\ \dot{x}_3(t) &= -x_3(t) + f'(0)x_4(t-\tau_1),\\ \dot{x}_4(t) &= -x_4(t) + w_1 f'(0)x_1(t-\tau_2)\\ &\quad + w_2 f'(0)x_2(t-\tau_2) + w_3 f'(0)x_3(t-\tau_2).\end{aligned} \tag{4}$$

The characteristic equation of the linearized system (4) is

$$\begin{aligned}&(\lambda+1)^4 + A(\lambda+1)^2 e^{-\lambda(\tau_1+\tau_2)} + B(\lambda+1)e^{-\lambda(2\tau_1+\tau_2)}\\ &\quad + Ce^{-\lambda(3\tau_1+\tau_2)} = 0,\end{aligned} \tag{5}$$

where

$$A = -w_3 f'^2(0), \qquad B = -w_2 f'^3(0), \qquad C = -w_1 f'^4(0). \tag{6}$$

In order to study the local stability of the zero equilibrium of system (3), we investigate the distribution of the roots of (5) in the following.

Case 1 ($\tau_1 = \tau_2 = 0$). Equation (5) reduces to

$$\lambda^4 + d_1\lambda^3 + d_2\lambda^2 + d_3\lambda + d_4 = 0, \tag{7}$$

where

$$\begin{gathered}d_1 = 4, \qquad d_2 = A + 6, \qquad d_3 = 2A + B + 4,\\ d_4 = B + C + 1.\end{gathered} \tag{8}$$

Obviously, $D_1 = d_1 > 0$. Therefore, by the Routh-Hurwitz criterion, the zero equilibrium $E_0(0,0,0,0)^T$ of system (3) is locally asymptotically stable if the following condition (H_1) holds:

$$\begin{gathered}D_2 = \begin{pmatrix} d_1 & 1\\ d_3 & d_2\end{pmatrix} > 0, \qquad D_3 = \begin{pmatrix} d_1 & 1 & 0\\ d_3 & d_2 & d_1\\ 0 & d_4 & d_3\end{pmatrix} > 0,\\ D_4 = \begin{pmatrix} d_1 & 1 & 0 & 0\\ d_3 & d_2 & d_1 & 1\\ 0 & d_4 & d_3 & d_2\\ 0 & 0 & 0 & d_4\end{pmatrix} > 0.\end{gathered} \tag{9}$$

Case 2 ($\tau_1 > 0$, $\tau_2 = 0$). On substituting $\tau_2 = 0$, (5) becomes

$$(\lambda+1)^4 + A(\lambda+1)^2 e^{-\lambda\tau_1} + B(\lambda+1)e^{-2\lambda\tau_1} + Ce^{-3\lambda\tau_1} = 0. \tag{10}$$

Multiplying $e^{\lambda\tau_1}$ on both sides of (10), it is easy to obtain

$$A(\lambda+1)^2 + (\lambda+1)^4 e^{\lambda\tau_1} + B(\lambda+1)e^{-\lambda\tau_1} + Ce^{-2\lambda\tau_1} = 0. \tag{11}$$

Let $\lambda = i\omega_1$ ($\omega_1 > 0$) be a root of (11). Then, we can get

$$\begin{aligned}A_{11}\cos\tau_1\omega_1 + A_{12}\sin\tau_1\omega_1 + A_{13} &= A_{14},\\ A_{21}\cos\tau_1\omega_1 + A_{22}\sin\tau_1\omega_1 + A_{23} &= A_{24},\end{aligned} \tag{12}$$

where

$$\begin{gathered}A_{11} = \omega_1^4 - 6\omega_1^2 + B - 1, \qquad A_{12} = 4\omega_1^3 + B\omega_1 - 4\omega_1,\\ A_{13} = A\left(1-\omega_1^2\right), \qquad A_{14} = -C\cos 2\tau_1\omega_1,\\ A_{21} = B\omega_1 + 4\omega_1 - 4\omega_1^3, \qquad A_{22} = \omega_1^4 - 6\omega_1^2 - B + 1,\\ A_{23} = 2A\omega_1, \qquad A_{24} = C\sin 2\tau_1\omega_1.\end{gathered} \tag{13}$$

Squaring both sides of the two equations in (12) and adding them up we obtain

$$\begin{aligned}&\left(A_{11}\cos\tau_1\omega_1 + A_{12}\sin\tau_1\omega_1 + A_{13}\right)^2\\ &\quad + \left(A_{21}\cos\tau_1\omega_1 + A_{22}\sin\tau_1\omega_1 + A_{23}\right)^2 = C^2.\end{aligned} \tag{14}$$

According to $\sin\tau_1\omega_1 = \pm\sqrt{1-\cos^2\tau_1\omega_1}$, we consider the two cases.

(I) if $\sin\tau_1\omega_1 = \sqrt{1-\cos^2\tau_1\omega_1}$, then (14) takes the following form:

$$\begin{aligned}&\left(A_{11}\cos\tau_1\omega_1 + A_{12}\sqrt{1-\cos^2\tau_1\omega_1} + A_{13}\right)^2\\&\quad+\left(A_{21}\cos\tau_1\omega_1 + A_{22}\sqrt{1-\cos^2\tau_1\omega_1} + A_{23}\right)^2 = C^2,\end{aligned}\tag{15}$$

which is equivalent to

$$\begin{aligned}&p_1\cos^4\tau_1\omega_1 + p_2\cos^3\tau_1\omega_1 + p_3\cos^2\tau_1\omega_1\\&\quad+p_4\cos\tau_1\omega_1 + p_5 = 0,\end{aligned}\tag{16}$$

where

$$\begin{aligned}p_1 &= \left(A_{11}^2+A_{21}^2-A_{12}^2-A_{22}^2\right)^2 + 4(A_{11}A_{12}+A_{21}A_{22})^2,\\p_2 &= 4\left(A_{11}^2+A_{21}^2-A_{12}^2-A_{22}^2\right)(A_{11}A_{13}+A_{21}A_{23})\\&\quad+8(A_{11}A_{12}+A_{21}A_{22})(A_{12}A_{13}+A_{22}A_{23}),\\p_3 &= 4(A_{11}A_{13}+A_{21}A_{23})^2 + 4(A_{12}A_{13}+A_{22}A_{23})^2\\&\quad-4(A_{11}A_{12}+A_{21}A_{22})^2 + 2\left(A_{11}^2+A_{21}^2-A_{12}^2-A_{22}^2\right)\\&\quad\times\left(A_{12}^2+A_{13}^2+A_{22}^2+A_{23}^2-C^2\right),\\p_4 &= 4(A_{11}A_{13}+A_{21}A_{23})\left(A_{12}^2+A_{13}^2+A_{22}^2+A_{23}^2-C^2\right)\\&\quad-8(A_{11}A_{12}+A_{21}A_{22})(A_{12}A_{13}+A_{22}A_{23}),\\p_5 &= \left(A_{12}^2+A_{13}^2+A_{22}^2+A_{23}^2-C^2\right)^2\\&\quad-(A_{12}A_{13}+A_{22}A_{23})^2.\end{aligned}\tag{17}$$

Let $r = \cos\tau_1\omega_1$, and denote that

$$f(r) = r^4 + \frac{p_2}{p_1}r^3 + \frac{p_3}{p_1}r^2 + \frac{p_4}{p_1}r + \frac{p_5}{p_1}.\tag{18}$$

Thus,

$$f'(r) = 4r^3 + \frac{3p_2}{p_1}r^2 + \frac{2p_3}{p_1}r + \frac{p_4}{p_1}.\tag{19}$$

Let

$$4r^3 + \frac{3p_2}{p_1}r^2 + \frac{2p_3}{p_1}r + \frac{p_4}{p_1} = 0.\tag{20}$$

Let $y = r + (p_2/4p_1)$. Then, (20) becomes

$$y^3 + \gamma_1 y + \gamma_0 = 0,\tag{21}$$

where

$$\gamma_1 = \frac{p_3}{2p_1} - \frac{3p_2^2}{16p_1^2},\qquad \gamma_0 = \frac{p_2^3}{32p_1^3} - \frac{p_2p_3}{8p_1^2} + \frac{p_4}{4p_1}.\tag{22}$$

Define

$$\beta_1 = \left(\frac{\gamma_2}{2}\right)^2 + \left(\frac{\gamma_1}{3}\right)^3,\qquad \beta_2 = \frac{-1+i\sqrt{3}}{2}.\tag{23}$$

Then, we can get

$$\begin{aligned}y_1 &= \sqrt[3]{-\frac{\gamma_2}{2}+\sqrt{\beta_1}} + \sqrt[3]{-\frac{\gamma_2}{2}-\sqrt{\beta_1}},\\y_2 &= \sqrt[3]{-\frac{\gamma_2}{2}+\sqrt{\beta_1}}\beta_2 + \sqrt[3]{-\frac{\gamma_2}{2}-\sqrt{\beta_1}}\overline{\beta}_2,\\y_3 &= \sqrt[3]{-\frac{\gamma_2}{2}+\sqrt{\beta_1}}\overline{\beta}_2 + \sqrt[3]{-\frac{\gamma_2}{2}-\sqrt{\beta_1}}\beta_2.\end{aligned}\tag{24}$$

Then, we can get the expression of $\cos\tau_1\omega_1$ and we denote $f_1(\omega_1) = \cos\tau_1\omega_1$. Substituting $f_1(\omega_1) = \cos\tau_1\omega_1$ into (12), we can get the expression of $\sin\tau_1\omega_1$ and we denote $f_2(\omega_1) = \sin\tau_1\omega_1$. Thus, a function with respect to ω_1 can be established by

$$f_1^2(\omega_1) + f_2^2(\omega_1) = 1.\tag{25}$$

We assume that (H_{21}), (25), has finite positive roots, which are denoted by $\omega_{11},\ldots,\omega_{1k}$. For every fixed ω_{1i} $(1\le i\le k)$, the corresponding critical value of time delay is

$$\begin{aligned}\tau_{1i}^{(j)} &= \frac{1}{\omega_{1i}}\arccos f_1(\omega_{1i}) + \frac{2j\pi}{\omega_{1i}},\\&i = 1,2,\ldots,k;\ j = 0,1,2,\ldots.\end{aligned}\tag{26}$$

Then, $\pm\omega_{1i}$ are a pair of purely imaginary roots of (11) with $\tau_1 = \tau_{1i}^{(j)}$. Let

$$\begin{aligned}&\tau_{10} = \min\left\{\tau_{1i}^{(j)}\right\},\quad \omega_{10} = \omega_{1i_0},\\&i = 1,2,\ldots,k,\ j = 0,1,2\ldots.\end{aligned}\tag{27}$$

(II) If $\sin\tau_1\omega_1 = -\sqrt{1-\cos^2\tau_1\omega_1}$, then (14) can be transformed into the following form:

$$\begin{aligned}&\left(A_{11}\cos\tau_1\omega_1 - A_{12}\sqrt{1-\cos^2\tau_1\omega_1} + A_{13}\right)^2\\&\quad+\left(A_{21}\cos\tau_1\omega_1 - A_{22}\sqrt{1-\cos^2\tau_1\omega_1} + A_{23}\right)^2 = C^2.\end{aligned}\tag{28}$$

Thus, similar as the process in case (I), we can get the expression of $\cos\tau_1\omega_1$ and $\sin\tau_1\omega_1$. Let

$$f_{1*}(\omega_1) = \cos\tau_1\omega_1,\qquad f_{2*} = \sin\tau_1\omega_1.\tag{29}$$

Therefore,

$$f_{1*}^2(\omega_1) + f_{2*}^2(\omega_1) = 1.\tag{30}$$

Then, we can get the critical value of time delay corresponding to every fixed positive root ω_{1i}' of (30):

$$\begin{aligned}\tau_{1i}^{(j)'} &= \frac{1}{\omega_{1i}'}\arccos f_1\left(\omega_{1i}'\right) + \frac{2j\pi}{\omega_{1i}'},\\&i = 1,2,\ldots,k;\ j = 0,1,2\ldots.\end{aligned}\tag{31}$$

Let

$$\tau_{10} = \min\left\{\tau_{1i}^{(j)\prime}\right\}, \quad \omega_{10} = \omega'_{1i_0}, \quad i = 1,2,\ldots,k, \;\; j = 0,1,2,\ldots. \tag{32}$$

Next, we verify the transversality. Taking the derivative of λ with respect to τ_1 in (11), we obtain

$$\left[\frac{d\lambda}{d\tau_1}\right]^{-1} = \frac{2A(\lambda+1) + Be^{-\lambda\tau_1} + 4(\lambda+1)^3 e^{\lambda\tau_1}}{\lambda\left[(\lambda+1)^4 e^{\lambda\tau_1} - B(\lambda+1)e^{-\lambda\tau_1} - 2Ce^{-2\lambda\tau_1}\right]} - \frac{\tau_1}{\lambda}. \tag{33}$$

Thus,

$$\operatorname{Re}\left[\frac{d\lambda}{d\tau_1}\right]^{-1}_{\tau_1=\tau_{10}} = \frac{P_R Q_R + P_I Q_I}{Q_R^2 + Q_I^2}, \tag{34}$$

where

$$\begin{aligned}
P_R &= \left(4 - 12\omega_{10}^2 + B\right)\cos\tau_{10}\omega_{10} - \left(3\omega_{10} - \omega_{10}^3\right)\sin\tau_{10}\omega_{10} + 2A,\\
P_I &= \left(4 - 12\omega_{10}^2 - B\right)\sin\tau_{10}\omega_{10} + \left(3\omega_{10} - \omega_{10}^3\right)\cos\tau_{10}\omega_{10} + 2A\omega_{10},\\
Q_R &= \left(4\omega_{10}^4 + B\omega_{10}^2 - 4\omega_{10}^2\right)\cos\tau_{10}\omega_{10} - \left(B\omega_{10} + \omega_{10} - 6\omega_{10}^3 - \omega_{10}^5\right)\sin\tau_{10}\omega_{10} - 2C\omega_{10}\sin 2\tau_{10}\omega_{10},\\
Q_I &= \left(4\omega_{10}^4 - B\omega_{10}^2 - 4\omega_{10}^2\right)\sin\tau_{10}\omega_{10} + \left(\omega_{10} - B\omega_{10} - 6\omega_{10}^3 - \omega_{10}^5\right)\cos\tau_{10}\omega_{10} - 2C\omega_{10}\cos 2\tau_{10}\omega_{10}.
\end{aligned} \tag{35}$$

Obviously, if the condition (H_{22}): $P_RQ_R + P_IQ_I \neq 0$ holds, then $\operatorname{Re}[d\lambda/d\tau_1]^{-1}_{\tau_1=\tau_{10}} \neq 0$. Namely, if the condition (H_{22}) holds, then the transversality condition is satisfied. By the discussion above and the Hopf bifurcation theorem in [23], it is easy to obtain the following results.

Theorem 1. *If the condition (H_{21}) means that (25) has finite positive roots and (H_{22}) means that $P_RQ_R+P_IQ_I \neq 0$ holds, then the zero equilibrium E_0 of system (3) is asymptotically stable for $\tau_1 \in [0,\tau_{10})$, system (3) undergoes a Hopf bifurcation at E_0 when $\tau_1 = \tau_{10}$, and a branch of periodic solutions bifurcates from the zero equilibrium near $\tau_1 = \tau_{10}$.*

Case 3 ($\tau_2 > 0$, $\tau_1 = 0$). When $\tau_1 = 0$, (5) becomes the following form:

$$(\lambda+1)^4 + \left[A\lambda^2 + (2A+B)\lambda + A + B + C\right]e^{-\lambda\tau_2} = 0. \tag{36}$$

Let $\lambda = i\omega_2$ ($\omega_2 > 0$) be a root of (36). Substituting it into (36) and separating the real and imaginary parts, we obtain

$$\begin{aligned}
&(2A+B)\omega_2\sin\tau_2\omega_2 + \left(A+B+C-A\omega_2^2\right)\cos\omega_2\tau_2 = 6\omega_2^2 - \omega_2^4 - 1,\\
&(2A+B)\omega_2\cos\tau_2\omega_2 - \left(A+B+C-A\omega_2^2\right)\sin\omega_2\tau_2 = 4\omega_2^3 - 4\omega_2.
\end{aligned} \tag{37}$$

It follows that

$$\omega_2^8 + c_3\omega_2^6 + c_2\omega_2^2 + c_1\omega_2 + c_0 = 0, \tag{38}$$

where

$$\begin{gathered}
c_0 = 1 - (A+B+C)^2, \qquad c_2 = 6 - A^2, \qquad c_3 = 4.\\
c_1 = 2A(A+B+C) - (2A+B)^2 + 4.
\end{gathered} \tag{39}$$

Let $\omega_2^2 = z$, then (38) can be transformed into

$$z^4 + c_3 z^3 + c_2 z^2 + c_1 z + c_0 = 0. \tag{40}$$

Next, we make the following assumption.

(H_{31}) means that (40) has at least one positive root.

Without loss of generality, we assume that (40) has four positive roots, which are denoted by z_1, z_2, z_3, and z_4. Thus, (38) has four positive roots $\omega_{2k} = \sqrt{z_k}$, $k = 1,2,3,4$. The corresponding critical value of time delay is

$$\begin{aligned}
\tau_{2k}^{(j)} = \frac{1}{\omega_{2k}}\arccos\Bigg(&\left(A\omega_{2k}^6 + (A+3B-C)\omega_{2k}^4 + (2B+6C-A)\omega_{2k}^2 - (A+B+C)\right)\\
&\times\left((2A+B)^2\omega_{2k}^2 + \left(A+B+C-A\omega_{2k}^2\right)^2 \times\left(6\omega_{2k}^2 - \omega_{2k}^4 - 1\right)^2\right)^{-1}\Bigg) + \frac{2j\pi}{\omega_{2k}},\\
&k = 1,2,3,4;\;\; j = 0,1,2\ldots.
\end{aligned} \tag{41}$$

Then, $\pm i\omega_{2k}$ are a pair of purely imaginary roots of (36) with $\tau_2 = \tau_{2k}^{(j)}$. Let

$$\tau_{20} = \min\left\{\tau_{2k}^{(0)}\right\}, \quad k = 1,2,3,4, \quad \omega_{20} = \omega_{2k_0}. \tag{42}$$

Taking the derivative of λ with respect to τ_2 in (36), we can get

$$\left[\frac{d\lambda}{d\tau_2}\right]^{-1} = -\frac{4\lambda^3 + 12\lambda^2 + 12\lambda + 4}{\lambda\left(\lambda^4 + 4\lambda^3 + 6\lambda^2 + 4\lambda + 1\right)} + \frac{2A\lambda + 2A + B}{\lambda\left[A\lambda^2 + (2A+B)\lambda + A + B + C\right]} - \frac{\tau_2}{\lambda}. \tag{43}$$

Then, we can get

$$\operatorname{Re}\left[\frac{d\lambda}{d\tau_2}\right]_{\tau=\tau_{20}}^{-1} = \frac{4\omega_{20}^6 + 12\omega_{20}^4 + 12\omega_{20}^2 + 4}{\omega_{20}^8 + 4\omega_{20}^6 + 6\omega_{20}^4 + 4\omega_{20}^2 + 1} - \left(2A^2\omega_{20}^2 + (2A+B)^2 - 2A(A+B+C)\right) \times \left(A^2\omega_{20}^4 + \left[(2A+B)^2 - 2A(A+B+C)\right] \times \omega_{20}^2 + (A+B+C)^2\right)^{-1}. \tag{44}$$

From (38), we have

$$\omega_{20}^8 + 4\omega_{20}^6 + 6\omega_{20}^4 + 4\omega_{20}^2 + 1 = A^2\omega_{20}^4 + \left[(2A+B)^2 - 2A(A+B+C)\right]\omega_{20}^2 + (A+B+C)^2. \tag{45}$$

Thus,

$$\operatorname{Re}\left[\frac{d\lambda}{d\tau_2}\right]_{\tau=\tau_{20}}^{-1} = \frac{g'(z_0)}{\omega_{20}^8 + 4\omega_{20}^6 + 6\omega_{20}^4 + 4\omega_{20}^2 + 1}, \tag{46}$$

where

$$g(z) = z^4 + c_3z^3 + c_2z^2 + c_1z + c_0, \qquad z_0 = \omega_{20}^2. \tag{47}$$

Therefore, if the condition (H_{32}): $g'(z_0) \neq 0$, then $\operatorname{Re}[d\lambda/d\tau_2]_{\tau=\tau_{20}}^{-1} \neq 0$. From the analysis above and by the Hopf bifurcation theorem in [23], we have the following results.

Theorem 2. *If the condition (H_{31}) means that (40) has at least one positive root and (H_{32}) means that $g'(z_0) \neq 0$ holds, then the zero equilibrium E_0 of system (3) is asymptotically stable for $\tau_2 \in [0, \tau_{20})$, system (3) undergoes a Hopf bifurcation at E_0 when $\tau_2 = \tau_{20}$, and a branch of periodic solutions bifurcates from the zero equilibrium near $\tau_2 = \tau_{20}$.*

Case 4 ($\tau_1 = \tau_2 = \tau > 0$). For $\tau_1 = \tau_2 = \tau > 0$, (5) can be transformed into the following form:

$$(\lambda+1)^4 + A(\lambda+1)^2e^{-2\lambda\tau} + B(\lambda+1)e^{-3\lambda\tau} + Ce^{-4\lambda\tau} = 0. \tag{48}$$

Multiplying $e^{2\lambda\tau}$ on both sides of (48), we obtain

$$A(\lambda+1)^2 + (\lambda+1)^4e^{2\lambda\tau} + Ce^{-2\lambda\tau} + B(\lambda+1)e^{-\lambda\tau} = 0. \tag{49}$$

Let $\lambda = i\omega$ be a root of (49); then we have

$$\begin{aligned} &\overline{A}_{11}\cos 2\tau_1\omega_1 - \overline{A}_{12}\sin 2\tau_1\omega_1 + \overline{A}_{13} = \overline{A}_{14}, \\ &\overline{A}_{21}\sin 2\tau_1\omega_1 + \overline{A}_{22}\cos 2\tau_1\omega_1 + \overline{A}_{23} = \overline{A}_{24}, \end{aligned} \tag{50}$$

where

$$\begin{aligned} &\overline{A}_{11} = \omega^4 - 6\omega^2 + C + 1, \qquad \overline{A}_{12} = 4\omega - \omega^3, \\ &\overline{A}_{13} = A - A\omega^2, \qquad \overline{A}_{14} = B\cos\tau\omega - B\omega\sin\tau\omega, \\ &\overline{A}_{21} = \omega^4 - 6\omega^2 - C + 1, \qquad \overline{A}_{22} = 4\omega - \omega^3, \\ &\overline{A}_{23} = 2A\omega, \qquad \overline{A}_{24} = B\cos\tau\omega + B\omega\sin\tau\omega. \end{aligned} \tag{51}$$

Then, we get

$$\left(\overline{A}_{11}\cos 2\tau_1\omega_1 - \overline{A}_{12}\sin 2\tau_1\omega_1 + \overline{A}_{13}\right)^2 + \left(\overline{A}_{21}\sin 2\tau_1\omega_1 + \overline{A}_{22}\cos 2\tau_1\omega_1 + \overline{A}_{23}\right)^2 = B^2\left(1+\omega^2\right). \tag{52}$$

Similar as in Case 2, we can obtain the expression of $\cos 2\tau\omega$ and $\sin 2\tau\omega$, which is denoted as $g_1(\omega)$ and $g_2(\omega)$, respectively. Further we can get a function with respect to ω

$$g_1^2(\omega) + g_2^2(\omega) = 1. \tag{53}$$

Next, we make the following assumption. (H_{41}): Equation (53) has finite positive real roots, which are denoted by $\omega_1, \ldots, \omega_k$, respectively. For every fixed positive root of (53), the corresponding critical value of time delay is

$$\tau_i^{(j)} = \frac{1}{2\omega_i}\arccos g_1(\omega_i) + \frac{2j\pi}{2\omega_i}, \qquad i = 1, \ldots, k;\ j = 0, 1, 2, \ldots. \tag{54}$$

Then, $\pm i\omega_i$ are a pair of purely imaginary roots of (49) with $\tau = \tau_i^{(j)}$. Let

$$\tau_0 = \min\left\{\tau_i^{(j)}\right\}, \qquad \omega_0 = \omega_{i_0}, \qquad i = 1, 2, \ldots, k,\ j = 0, 1, 2, \ldots. \tag{55}$$

Differentiating both sides of (49) with respect to t, we can obtain

$$\left[\frac{d\lambda}{d\tau}\right]^{-1} = \frac{2A(\lambda+1) + 4(\lambda+1)^3e^{2\lambda\tau} + Be^{-\lambda\tau}}{B\lambda(\lambda+1)e^{-\lambda\tau} + 2C\lambda e^{-2\lambda\tau} - 2\lambda(\lambda+1)^4e^{2\lambda\tau}} - \frac{\tau}{\lambda}. \tag{56}$$

Thus,

$$\operatorname{Re}\left[\frac{d\lambda}{d\tau_2}\right]_{\tau=\tau_0}^{-1} = \frac{P_R'Q_R' + P_I'Q_I'}{Q_R'^2 + Q_I'^2}, \tag{57}$$

where

$$\begin{aligned} P_R' &= \left(1-3\omega_0^2\right)\cos 2\tau_0\omega_0 - \left(3\omega_0-\omega_0^3\right)\sin 2\tau_0\omega_0 \\ &\quad + B\cos\tau_0\omega_0 + 2A, \\ P_I' &= \left(1-3\omega_0^2\right)\sin 2\tau_0\omega_0 + \left(3\omega_0-\omega_0^3\right)\cos 2\tau_0\omega_0 \\ &\quad - B\cos\tau_0\omega_0 + 2A\omega_0, \\ Q_R' &= B\omega_0\sin\tau_0\omega_0 - B\omega_0^2\cos\tau_0\omega_0 \\ &\quad + \left(\omega_0^5 + (1+2C)\,\omega_0 - 6\omega_0^6\right)\sin 2\tau_0\omega_0 \\ &\quad - 8\left(\omega_0^4-\omega_0^2\right)\cos 2\tau_0\omega_0, \\ Q_I' &= B\omega_0\cos\tau_0\omega_0 + B\omega_0^2\sin\tau_0\omega_0 \\ &\quad - \left(\omega_0^5 + (1+2C)\,\omega_0 - 6\omega_0^6\right)\cos 2\tau_0\omega_0 \\ &\quad - 8\left(\omega_0^4-\omega_0^2\right)\sin 2\tau_0\omega_0. \end{aligned} \tag{58}$$

Obviously, if the condition (H_{42}): $P_R'Q_R' + P_I'Q_I' \neq 0$ holds, then $\mathrm{Re}[d\lambda/d\tau_2]^{-1}_{\tau=\tau_0} \neq 0$. Namely, if the condition (H_{42}) holds, the transversality condition is satisfied. Thus, by the Hopf bifurcation theorem in [23] we have the following results.

Theorem 3. *If the condition (H_{41}) means that (53) has finite positive real roots and (H_{42}) means that $P_R'Q_R' + P_I'Q_I' \neq 0$ holds, then the zero equilibrium E_0 of system (3) is asymptotically stable for $\tau \in [0,\tau_0)$, system (3) undergoes a Hopf bifurcation at E_0 when $\tau = \tau_0$, and a branch of periodic solutions bifurcates from the zero equilibrium near $\tau = \tau_0$.*

Case 5 ($\tau_1 > 0$ and $\tau_2 > 0$). We consider (5) with τ_1 in its stable interval and τ_2 is considered as a parameter. Without loss of generality, we consider (5) under Case 2.

Let $\lambda = i\omega_{2*}$ ($\omega_{2*} > 0$) be a root of (5). Then, we can get

$$\begin{aligned} &\omega^8 + 4\omega^6 + 6\omega^4 + 4\omega^2 + 1 + 2B\left(A\omega^3 + C\omega\right)\sin\tau_1\omega \\ &\quad - 2B(A+C)\cos\tau_1\omega + 2AC\left(\omega^2-1\right)\cos 2\tau_1\omega \\ &\quad + 2AC\omega\sin 2\tau_1\omega = 0. \end{aligned} \tag{59}$$

Suppose that (H_{51}) means that (59) has finite positive real roots, which are denoted as $\omega_{21*}, \omega_{22*}, \ldots, \omega_{2k*}$. For every positive real root ω_{2i*} ($i = 1,2,\ldots,k$), their exists a sequence $\{\tau_{2i*}^{(j)} \mid j = 0,1,2,\ldots\}$, such that (59) has a pair of purely imaginary roots $\pm i\omega_{2i*}$ when $\tau_2 = \tau_{2i*}^{(j)}$.

Let $\tau_2^* = \min\{\tau_{2i*}^{(j)} \mid j = 0,1,2,\ldots\}$, and when $\tau_2 = \tau_2^*$ (59) has a pair of purely imaginary roots $\pm i\omega_2^*$. In the following, we make the following assumption.

(H_{52}) : $\mathrm{Re}\,[d\lambda/d\tau_2]^{-1}_{\tau_2=\tau_2^*} \neq 0$.

Through the analysis above and by the Hopf bifurcation theorem in [23], we have the following results.

Theorem 4. *If the condition (H_{51}) means that (59) has finite positive real roots and (H_{52}) means that $\mathrm{Re}[d\lambda/d\tau_2]^{-1}_{\tau_2=\tau_2^*} \neq 0$ holds, and $\tau_1 \in (0,\tau_{10})$, then the zero equilibrium E_0 of system (3) is asymptotically stable for $\tau_2 \in [0,\tau_2^*)$, system (3) undergoes a Hopf bifurcation at E_0 when $\tau_2 = \tau_2^*$, and a branch of periodic solutions bifurcates from the zero equilibrium near $\tau_2 = \tau_2^*$.*

3. Stability of Bifurcated Periodic Solutions

In this section, the formulae for determining the direction of Hopf bifurcation and the stability of bifurcating periodic solutions of system (3) with respect to τ_2 for $\tau_1 \in (0,\tau_{10})$ are derived by using the normal form method and center manifold theorem introduced by Hassard et al. [23]. Throughout this section, it is considered that system (3) undergoes Hopf bifurcation at $\tau_2 = \tau_2^*$ and $\tau_1 \in (0,\tau_{10})$. Without loss of generality, we assume that $\tau_1^* < \tau_2^*$, where $\tau_1^* \in (0,\tau_{10})$.

For convenience, let $t = s\tau_2$, $\overline{u}_i(t) = u_i(\tau_2 t)$, ($i = 1,2,3,4$). Drop the bars for simplification of notations. Then system (3) becomes

$$\dot{u}(t) = L_\mu u_t + F(\mu, u_t), \tag{60}$$

where $u(t) = (u_1(t), u_2(t), u_3(t), u_4(t))^T \in C = C([-1,0], R^4)$ and $L_\mu : C \to R^4$, $F : R\times C \to R^4$ are given, respectively, by

$$\begin{aligned} L_\mu\phi &= (\tau_2^* + \mu)\left(A'\phi(0) + B'\phi\left(-\frac{\tau_1^*}{\tau_2^*}\right) + C'\phi(-1)\right), \\ F(\mu,\phi) &= (\tau_2^* + \mu)(F_1, F_2, F_3, F_4)^T, \end{aligned} \tag{61}$$

with

$$\phi(\theta) = \left(\phi_1(\theta), \phi_2(\theta), \phi_3(\theta), \phi_4(\theta)\right)^T \in C\left([-1,0],\, R^4\right),$$

$$A' = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

$$B' = \begin{pmatrix} 0 & f'(0) & 0 & 0 \\ 0 & 0 & f'(0) & 0 \\ 0 & 0 & 0 & f'(0) \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$C' = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ w_1 f'(0) & w_2 f'(0) & w_3 f'(0) & 0 \end{pmatrix},$$

$$F_1 = \frac{f''}{2!}\phi_2^2\left(-\frac{\tau_1^*}{\tau_2^*}\right) + \frac{f'''}{3!}\phi_2^3\left(-\frac{\tau_1^*}{\tau_2^*}\right) + \cdots,$$

$$F_2 = \frac{f''}{2!}\phi_3^2\left(-\frac{\tau_1^*}{\tau_2^*}\right) + \frac{f'''}{3!}\phi_3^3\left(-\frac{\tau_1^*}{\tau_2^*}\right) + \cdots,$$

$$F_3 = \frac{f''}{2!}\phi_4^2\left(-\frac{\tau_1^*}{\tau_2^*}\right) + \frac{f'''}{3!}\phi_4^3\left(-\frac{\tau_1^*}{\tau_2^*}\right) + \cdots,$$

$$\begin{aligned} F_4 = {} & \frac{w_1 f''(0)}{2!}\phi_1^2(-1) + \frac{w_1 f'''(0)}{3!}\phi_1^3(-1) \\ & + \frac{w_2 f''(0)}{2!}\phi_2^2(-1) + \frac{w_2 f'''(0)}{3!}\phi_2^3(-1) \\ & + \frac{w_3 f''(0)}{2!}\phi_1^2(-1) + \frac{w_3 f'''(0)}{3!}\phi_3^3(-1) + \cdots. \end{aligned} \tag{62}$$

Therefore, according to the Riesz representation theorem, there exists a 4×4 matrix function $\eta(\theta, \mu) : [-1, 0] \rightarrow R^4$ whose elements are of bounded variation such that

$$L_\mu \phi = \int_{-1}^{0} d\eta(\theta, \mu)\,\phi(\theta), \quad \phi \in C\left([-1, 0], R^4\right). \tag{63}$$

In fact, we choose

$$\eta(\theta, \mu) = \begin{cases} \left(\tau_2^* + \mu\right)\left(A' + B' + C'\right), & \theta = 0, \\ \left(\tau_2^* + \mu\right)\left(B' + C'\right), & \theta \in \left[-\dfrac{\tau_1^*}{\tau_2^*}, 0\right), \\ \left(\tau_2^* + \mu\right)C', & \theta \in \left(-1, -\dfrac{\tau_1^*}{\tau_2^*}\right), \\ 0, & \theta = -1. \end{cases} \tag{64}$$

For $\phi \in C([-1, 0], R^4)$, we define

$$A(\mu)\phi = \begin{cases} \dfrac{d\phi(\theta)}{d\theta}, & -1 \le \theta < 0, \\ \displaystyle\int_{-1}^{0} d\eta(\theta, \mu)\,\phi(\theta), & \theta = 0, \end{cases}$$
$$R(\mu)\phi = \begin{cases} 0, & -1 \le \theta < 0, \\ F(\mu, \phi), & \theta = 0. \end{cases} \tag{65}$$

Then system (60) can be transformed into the following operator equation:

$$\dot{u}(t) = A(\mu)u_t + R(\mu)u_t, \tag{66}$$

where $u_t = u(t + \theta)$ for $\theta \in [-1, 0]$.

For $\varphi \in C'([0, 1], (R^4)^*)$, where $(R^4)^*$ is the 4-dimensional space of row vector, we define the adjoint operator A^* of A:

$$A^*(\varphi) = \begin{cases} -\dfrac{d\varphi(s)}{ds}, & 0 < s \le 1, \\ \displaystyle\int_{-1}^{0} d\eta^T(s, 0)\,\varphi(-s), & s = 0, \end{cases} \tag{67}$$

and a bilinear inner product

$$\begin{aligned} & \langle \varphi(s), \phi(\theta) \rangle \\ & = \overline{\varphi}(0)\phi(0) - \int_{\theta=-1}^{0}\int_{\xi=0}^{\theta} \overline{\varphi}(\xi - \theta)\,d\eta(\theta)\,\phi(\xi)\,d\xi, \end{aligned} \tag{68}$$

where $\eta(\theta) = \eta(\theta, 0)$.

Then $A(0)$ and $A^*(0)$ are adjoint operators. From the discussion above, we know that $\pm i\omega_2^*\tau_2^*$ are eigenvalues of $A(0)$ and they are also eigenvalues of $A^*(0)$. Let $q(\theta) = (1, q_2, q_3, q_4)^T e^{i\omega_2^*\tau_2^*\theta}$ be the eigenvector of $A(0)$ corresponding to the eigenvalue $+i\omega_2^*\tau_2^*$, and let $q^*(s) = D(1, q_2^*, q_3^*, q_4^*)e^{i\omega_2^*\tau_2^* s}$ be the eigenvector of $A^*(0)$ corresponding to the eigenvalue $-i\omega_2^*\tau_2^*$. Then, we have

$$A(0)q(\theta) = i\omega_2^*\tau_2^* q(\theta), \qquad A^*(0)q^*(0) = -i\omega_2^*\tau_2^* q^*(\theta). \tag{69}$$

By a simple computation, we can obtain

$$\begin{aligned} & q_2 = \frac{i\omega_2^* + 1}{f'(0)e^{-i\omega_2^*\tau_1^*}}, \qquad q_3 = \frac{q_2}{f'(0)e^{-i\omega_2^*\tau_1^*} - i\omega_2^*}, \\ & q_4 = \frac{f'(0)\left(w_1 + w_2 q_2 + w_3 q_3\right)}{\left(i\omega_2^* + 1\right)e^{i\omega_2^*\tau_2^*}}, \\ & q_4^* = \frac{1 - i\omega_2^*}{w_1 f'(0)e^{i\omega_2^*\tau_2^*}}, \qquad q_2^* = \frac{e^{i\omega_2^*\tau_1^*} + w_2 q_4^* e^{i\omega_2^*\tau_2^*}}{1 - i\omega_2^*}f'(0), \\ & q_3^* = \frac{q_2^* e^{i\omega_2^*\tau_1^*} + w_3 q_4^* e^{i\omega_2^*\tau_2^*}}{1 - i\omega_2^*}f'(0) \end{aligned} \tag{70}$$

and $\langle q^*, q\rangle = 1$, $\langle q^*, \overline{q}\rangle = 0$.

From (68), we can get

$$\begin{aligned} \overline{D} = \Big[& 1 + q_2\overline{q}_2^* + q_3\overline{q}_3^* + q_4\overline{q}_4^* \\ & + \tau_1^* f'(0)\left(q_2 + \overline{q}_2^* q_3 + \overline{q}_3^* q_4\right)e^{-i\omega_2^*\tau_1^*} \\ & + \tau_2^* f'(0)\,\overline{q}_4^*\left(w_1 + w_2 q_2 + w_3 q_3\right)e^{-i\omega_2^*\tau_2^*}\Big]^{-1}. \end{aligned} \tag{71}$$

Following the algorithms given in [23] and using similar computation process in [24], we can get the coefficients which can be used to determine direction of the Hopf bifurcation and stability of the bifurcating periodic solutions:

$$\begin{aligned} g_{20} = f''(0)\overline{D}\Big[& e^{-2i\omega_2^*\tau_1^*}\left(q_2^2 + \overline{q}_2^* q_3^2 + \overline{q}_3^* q_4^2\right) \\ & + \overline{q}_4^* e^{-2i\omega_2^*\tau_2^*}\left(w_1 + w_2 q_2^2 + w_3 q_3^2\right)\Big], \end{aligned}$$

$$\begin{aligned} g_{11} = f''(0)\overline{D}\big[& q_2\overline{q}_2 + \overline{q}_2^* q_3\overline{q}_3 + \overline{q}_3^* q_4\overline{q}_4 \\ & + \overline{q}_4^*\left(w_1 + w_2 q_2\overline{q}_2 + w_3 q_3\overline{q}_3\right)\big], \end{aligned}$$

$$g_{02} = f''(0)\overline{D}\Big[e^{2i\omega_2^*\tau_1^*}\left(\overline{q}_2^2 + \overline{q}_2^*\overline{q}_3^2 + \overline{q}_3^*\overline{q}_4^2\right) + \overline{q}_4^* e^{2i\omega_2^*\tau_2^*}\left(w_1 + w_2\overline{q}_2^2 + w_3\overline{q}_3^2\right)\Big],$$

$$\begin{aligned} g_{21} = \overline{D}\Big[& f''(0)\left(2W_{11}^{(2)}\left(-\frac{\tau_1^*}{\tau_2^*}\right)q_2 e^{-i\omega_2^*\tau_1^*} + W_{20}^{(2)}\left(-\frac{\tau_1^*}{\tau_2^*}\right)\overline{q}_2 e^{i\omega_2^*\tau_1^*}\right) \\ & + f'''(0)\, q_2^2\overline{q}_2 e^{-i\omega_2^*\tau_1^*} \\ & + \overline{q}_2^*\Big(f''(0)\left(2W_{11}^{(3)}\left(-\frac{\tau_1^*}{\tau_2^*}\right)q_3 e^{-i\omega_2^*\tau_1^*} + W_{20}^{(3)}\left(-\frac{\tau_1^*}{\tau_2^*}\right)\overline{q}_3 e^{i\omega_2^*\tau_1^*}\right) \\ & \quad + f'''(0)\, q_3^2\overline{q}_3 e^{-i\omega_2^*\tau_1^*}\Big) \\ & + \overline{q}_3^*\Big(f''(0)\left(2W_{11}^{(4)}\left(-\frac{\tau_1^*}{\tau_2^*}\right)q_2 e^{-i\omega_2^*\tau_1^*} + W_{20}^{(4)}\left(-\frac{\tau_1^*}{\tau_2^*}\right)\overline{q}_2 e^{i\omega_2^*\tau_1^*}\right) \\ & \quad + f'''(0)\, q_4^2\overline{q}_4 e^{-i\omega_2^*\tau_1^*}\Big) \\ & + \overline{q}_4^*\Big(w_1 f''(0)\left(2W_{11}^{(1)}(-1)\, e^{-i\omega_2^*\tau_2^*} + W_{20}^{(1)}(-1)\, e^{i\omega_2^*\tau_2^*}\right) \\ & \quad + w_1 f'''(0)\, e^{-i\omega_2^*\tau_2^*} \\ & \quad + w_2 f''(0)\left(2W_{11}^{(2)}(-1)\, q_2 e^{-i\omega_2^*\tau_2^*} + W_{20}^{(2)}(-1)\,\overline{q}_2 e^{i\omega_2^*\tau_2^*}\right) \\ & \quad + w_2 f'''(0)\, q_2^2\overline{q}_2 e^{-i\omega_2^*\tau_2^*} \\ & \quad + w_3 f''(0)\left(2W_{11}^{(3)}(-1)\, q_3 e^{-i\omega_2^*\tau_2^*} + W_{20}^{(3)}(-1)\,\overline{q}_3 e^{i\omega_2^*\tau_2^*}\right) \\ & \quad + w_3 f'''(0)\, q_3^2\overline{q}_3 e^{-i\omega_2^*\tau_2^*}\Big)\Big], \end{aligned} \tag{72}$$

with

$$\begin{aligned} W_{20}(\theta) &= \frac{ig_{20}q(0)}{\omega_2^*\tau_2^*}e^{i\omega_2^*\tau_2^*\theta} + \frac{i\overline{g}_{02}\overline{q}(0)}{3\omega_2^*\tau_2^*}e^{-i\omega_2^*\tau_2^*\theta} + E_1 e^{2i\omega_2^*\tau_2^*\theta}, \\ W_{11}(\theta) &= -\frac{ig_{11}q(0)}{\omega_2^*\tau_2^*}e^{i\omega_2^*\tau_2^*\theta} + \frac{i\overline{g}_{11}\overline{q}(0)}{\omega_2^*\tau_2^*}e^{-i\omega_2^*\tau_2^*\theta} + E_2, \end{aligned} \tag{73}$$

where E_1 and E_2 can be computed by the following equations, respectively:

$$\begin{gathered} \begin{pmatrix} 2i\omega_2^* + 1 & \alpha_{12} & 0 & 0 \\ 0 & 2i\omega_2^* + 1 & \alpha_{23} & 0 \\ 0 & 0 & 2i\omega_2^* + 1 & \alpha_{34} \\ \alpha_{41} & \alpha_{42} & \alpha_{43} & 2i\omega_2^* + 1 \end{pmatrix} E_1 = \begin{pmatrix} E_1^{(1)} \\ E_1^{(2)} \\ E_1^{(3)} \\ E_1^{(4)} \end{pmatrix}, \\ \begin{pmatrix} 1 & -f'(0) & 0 & 0 \\ 0 & 1 & -f'(0) & 0 \\ 0 & 0 & 1 & -f'(0) \\ -w_1 f'(0) & -w_2 f'(0) & -w_3 f'(0) & 1 \end{pmatrix} E_2 = -\begin{pmatrix} E_2^{(1)} \\ E_2^{(2)} \\ E_2^{(3)} \\ E_2^{(4)} \end{pmatrix}, \end{gathered} \tag{74}$$

with

$$\begin{aligned} \alpha_{12} &= \alpha_{23} = \alpha_{34} = -f'(0)\, e^{-2i\omega_2^*\tau_1^*}, \\ \alpha_{41} &= -w_1 f'(0)\, e^{-2i\omega_2^*\tau_2^*}, \\ \alpha_{42} &= -w_2 f'(0)\, e^{-2i\omega_2^*\tau_2^*}, \\ \alpha_{43} &= -w_3 f'(0)\, e^{-2i\omega_2^*\tau_2^*}. \\ E_1^{(1)} &= f''(0)\, q_2^2 e^{-2i\omega_2^*\tau_1^*}, \\ E_1^{(2)} &= f''(0)\, q_3^2 e^{-2i\omega_2^*\tau_1^*}, \\ E_1^{(3)} &= f''(0)\, q_4^2 e^{-2i\omega_2^*\tau_1^*}, \\ E_1^{(4)} &= f''(0)\left(w_1 + w_2 q_2^2 + w_3 q_3^2\right) e^{-2i\omega_2^*\tau_2^*}, \\ E_2^{(1)} &= f''(0)\, q_2\overline{q}_2, \\ E_2^{(2)} &= f''(0)\, q_3\overline{q}_3, \\ E_2^{(3)} &= f''(0)\, q_4\overline{q}_4, \\ E_2^{(4)} &= f''(0)\left(w_1 + w_2 q_2\overline{q}_2 + w_3 q_3\overline{q}_3\right). \end{aligned} \tag{75}$$

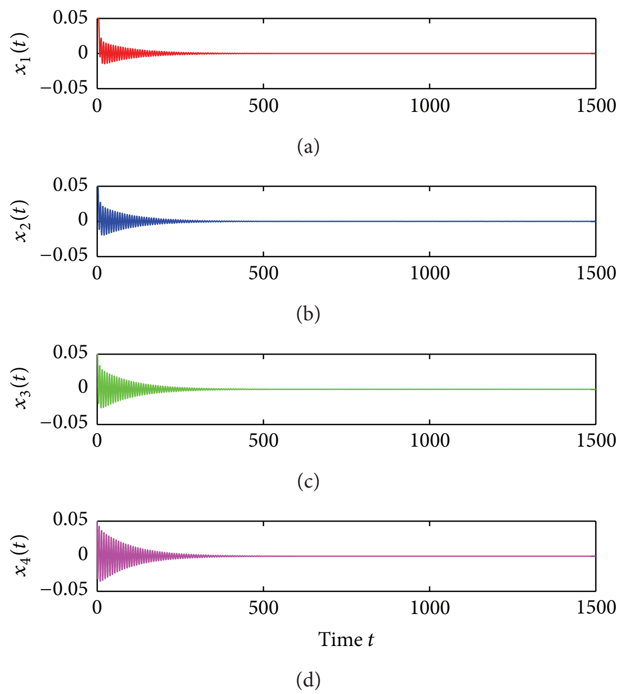

FIGURE 1: The trajectory of x_1, x_2, x_3, and x_4 when $\tau_1 = 1.3 < 1.4022 = \tau_{10}$.

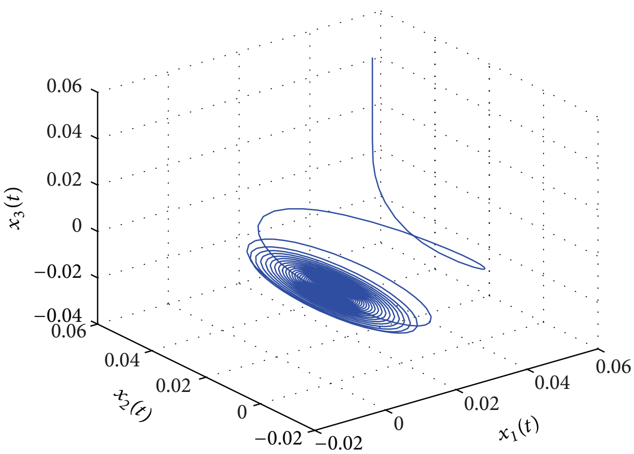

FIGURE 2: The phase plot of x_1, x_2, and x_3 when $\tau_1 = 1.3 < 1.4022 = \tau_{10}$.

Therefore, we can calculate the following values:

$$
\begin{gathered}
C_1(0) = \frac{i}{2\omega_2^* \tau_2^*}\left(g_{11}g_{20} - 2|g_{11}|^2 - \frac{|g_{02}|^2}{3}\right) + \frac{g_{21}}{2}, \\
\mu_2 = -\frac{\operatorname{Re}\{C_1(0)\}}{\operatorname{Re}\{\lambda'(\tau_2^*)\}}, \\
\beta_2 = 2\operatorname{Re}\{C_1(0)\}, \\
T_2 = -\frac{\operatorname{Im}\{C_1(0)\} + \mu_2 \operatorname{Im}\{\lambda'(\tau_2^*)\}}{\omega_2^* \tau_2^*}.
\end{gathered}
\tag{76}
$$

Based on the discussion above, we can obtain the following results.

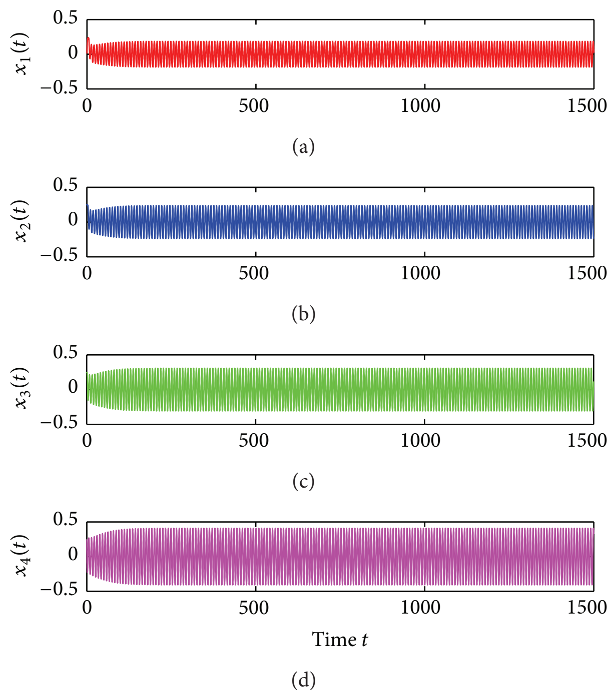

FIGURE 3: The trajectory of x_1, x_2, x_3, and x_4 when $\tau_1 = 1.5 > 1.4022 = \tau_{10}$.

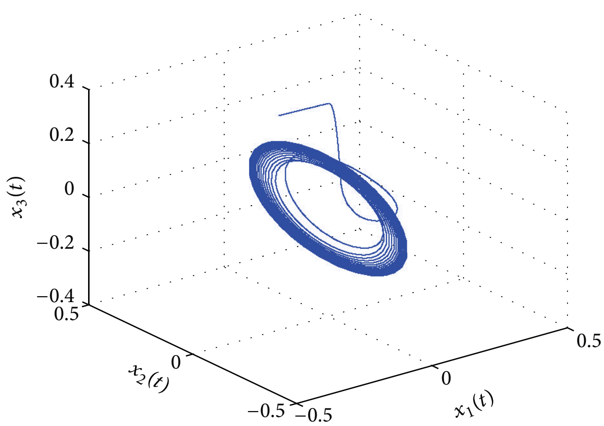

FIGURE 4: The phase plot of x_1, x_2, and x_3 when $\tau_1 = 1.5 > 1.4022 = \tau_{10}$.

Theorem 5. *For system* (3),

(i) μ_2 *determines the direction of the Hopf bifurcation. If* $\mu_2 > 0$ $(\mu_2 < 0)$*; then the Hopf bifurcation is supercritical (subcritical);*

(ii) β_2 *determines the stability of the bifurcating periodic solutions. If* $\beta_2 < 0$ $(\beta_2 > 0)$*; then the bifurcating periodic solutions are stable (unstable);*

(iii) T_2 *determines the period of the bifurcating periodic solutions. If* $T_2 > 0$ $(T_2 < 0)$*; then the period of the bifurcating periodic solutions increases (decreases).*

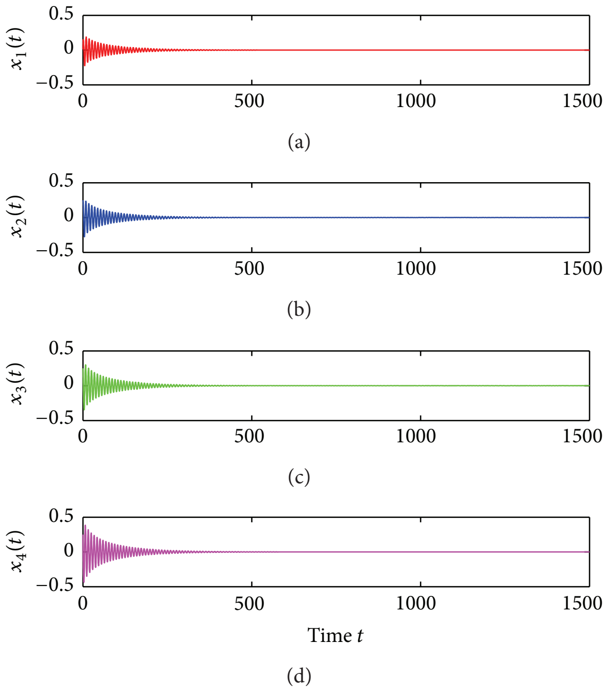

FIGURE 5: The trajectory of x_1, x_2, x_3, and x_4 when $\tau_2 = 2.75 < 3.1610 = \tau_{20}$.

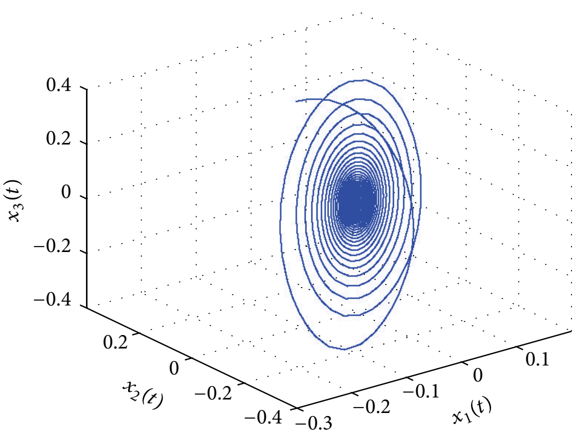

FIGURE 6: The phase plot of x_1, x_2, and x_3 when $\tau_2 = 2.75 < 3.1610 = \tau_{20}$.

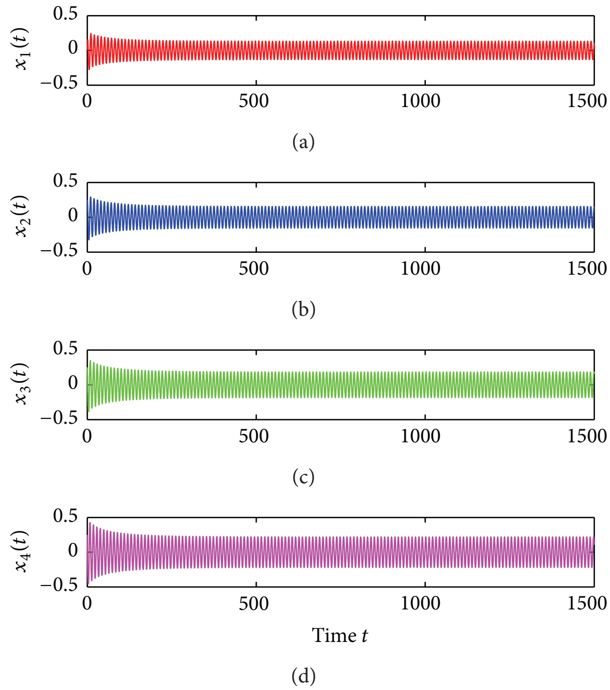

FIGURE 7: The trajectory of x_1, x_2, x_3, and x_4 when $\tau_2 = 3.5 > 3.1610 = \tau_{20}$.

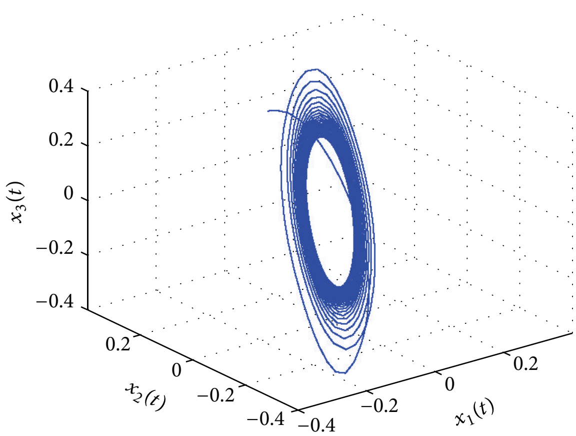

FIGURE 8: The phase plot of x_1, x_2, and x_3 when $\tau_2 = 3.5 > 3.1610 = \tau_{20}$.

4. Numerical Simulation

In this section, we present some numerical simulations to support the theoretical analysis in Sections 2 and 3. As an example, we consider the following special case of system (3) with the parameters $w_1 = 1$, $w_2 = -1$, $w_3 = -1$, and $f(x) = \tanh(x)$. Then $f(0) = 0$, $f'(0) = 1$, and system (3) becomes

$$\begin{aligned}
\dot{x}_1(t) &= -x_1(t) + \tanh(x_2(t-\tau_1)),\\
\dot{x}_2(t) &= -x_2(t) + \tanh(x_3(t-\tau_1)),\\
\dot{x}_3(t) &= -x_3(t) + \tanh(x_4(t-\tau_1)),\\
\dot{x}_4(t) &= -x_4(t) + \tanh(x_1(t-\tau_2))\\
&\quad - \tanh(x_2(t-\tau_2)) - \tanh(x_3(t-\tau_2)).
\end{aligned} \tag{77}$$

Obviously, $E_0(0,0,0,0)$ is the equilibrium of system (77). By a simple computation, we get $D_2 = 21 > 0$, $D_3 = 131 > 0$, and $D_4 = 131 > 0$. That is, the condition (H_1) holds.

For $\tau_1 > 0$, $\tau_2 = 0$. We can obtain $\omega_{10} = 1.7216$, $\tau_{10} = 1.4022$ by some complicated computations. From Theorem 1, we know that $E_0(0,0,0,0)$ is asymptotically stable when $\tau_1 < \tau_{10}$ as illustrated by Figures 1 and 2. When τ_1 passes through, the critical value τ_{10}, $E_0(0,0,0,0)$ becomes unstable and a

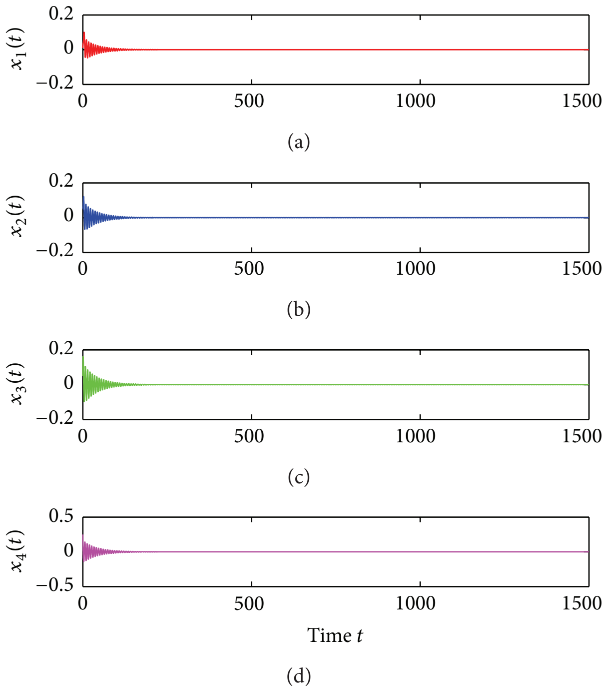

FIGURE 9: The trajectory of x_1, x_2, x_3, and x_4 when $\tau = 0.7 < 0.7915 = \tau_0$.

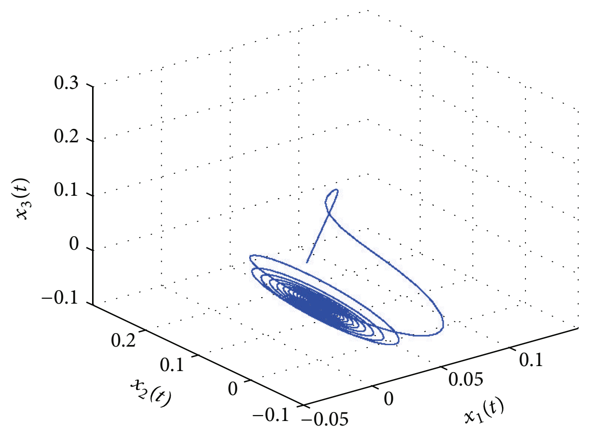

FIGURE 10: The phase plot of x_1, x_2, and x_3 when $\tau = 0.7 < 0.7915 = \tau_0$.

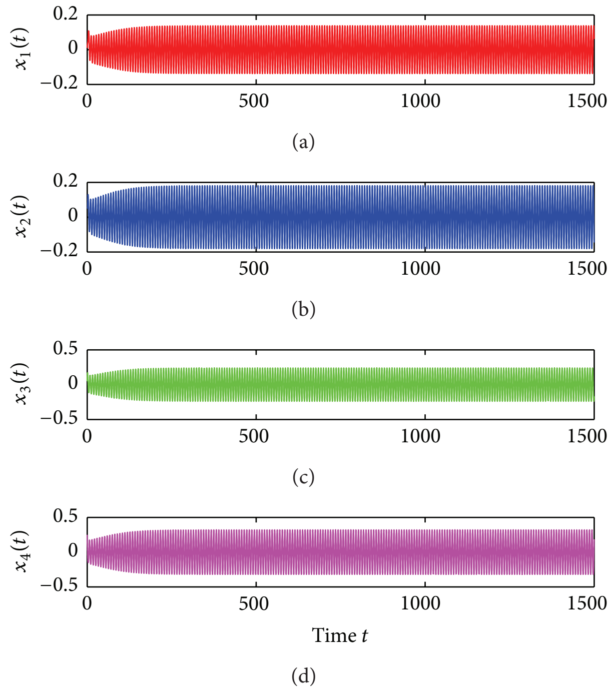

FIGURE 11: The trajectory of x_1, x_2, x_3, and x_4 when $\tau = 0.85 > 0.7915 = \tau_0$.

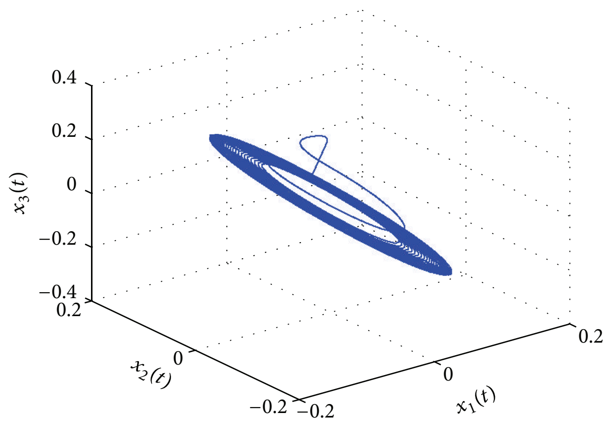

FIGURE 12: The phase plot of x_1, x_2, and x_3 when $\tau = 0.85 > 0.7915 = \tau_0$.

Hopf bifurcation occurs and a branch of periodic solutions bifurcate from $E_0(0,0,0,0)$, which can be seen from Figures 3 and 4. Similarly, we have $\omega_{20} = 0.5194$, $\tau_{20} = 3.1610$ for $\tau_1 = 0$, $\tau_2 > 0$. The corresponding waveforms and the phase plots are shown in Figures 5, 6, 7, and 8.

For $\tau_1 = \tau_2 = \tau > 0$, we obtain $\omega_0 = 2.0967$, $\tau_0 = 0.7915$. From Theorem 3, when τ increases from zero to the critical value τ_0, $E_0(0,0,0,0)$ is asymptotically stable, then it will lose its stability and a Hopf bifurcation occurs once $\tau > \tau_0$. These properties can be shown in Figures 9, 10, 11, and 12.

Lastly, for $\tau_2 > 0$ and $\tau_1^* = 0.35 \in (0, \tau_{10})$, we get $\omega_2^* = 1.3743$, $\tau_2^* = 1.7488$. By Theorem 4, $E_0(0,0,0,0)$ is asymptotically stable when $\tau_2 \in [0, \tau_2^*)$, and $E_0(0,0,0,0)$ is unstable when $\tau_2 > \tau_2^*$ and a Hopf bifurcation occurs, which can be illustrated by Figures 13, 14, 15, and 16.

5. Conclusion

In this paper, we have investigated a four-dimensional recurrent neural network with two discrete delays. Compared with the literature [11], we consider the neural network model which can reflect the really large neural networks more closely. By regarding the possible combinations of the two delays as the bifurcation parameter, sufficient conditions for the local stability of the zero equilibrium and the existence of Hopf bifurcation are obtained. If the conditions are satisfied,

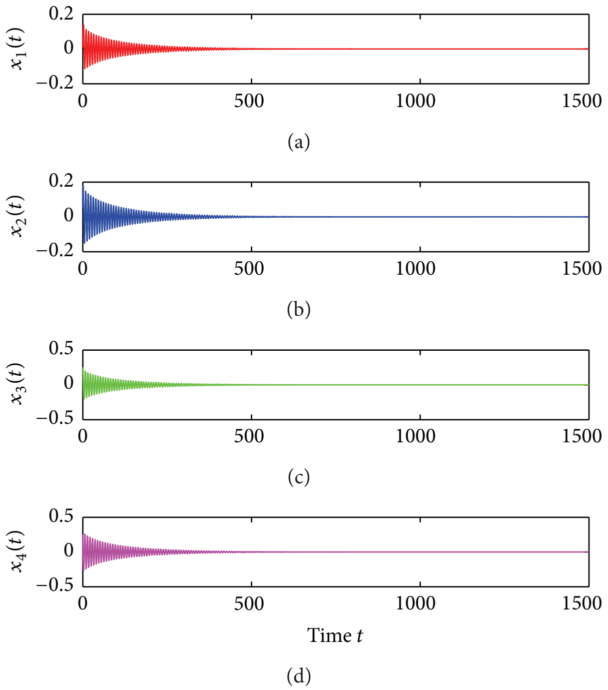

FIGURE 13: The trajectory of x_1, x_2, x_3, and x_4 when $\tau_2 = 1.65 < 1.7488 = \tau_2^*$ and $\tau_1^* = 0.35$.

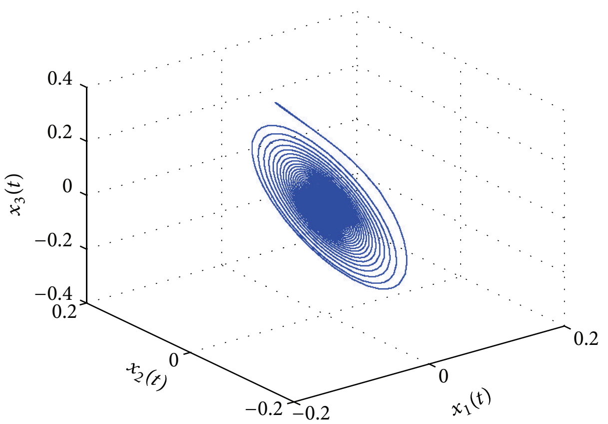

FIGURE 14: The phase plot of x_1, x_2, and x_3 when $\tau_2 = 1.65 < 1.7488 = \tau_2^*$ and $\tau_1^* = 0.35$.

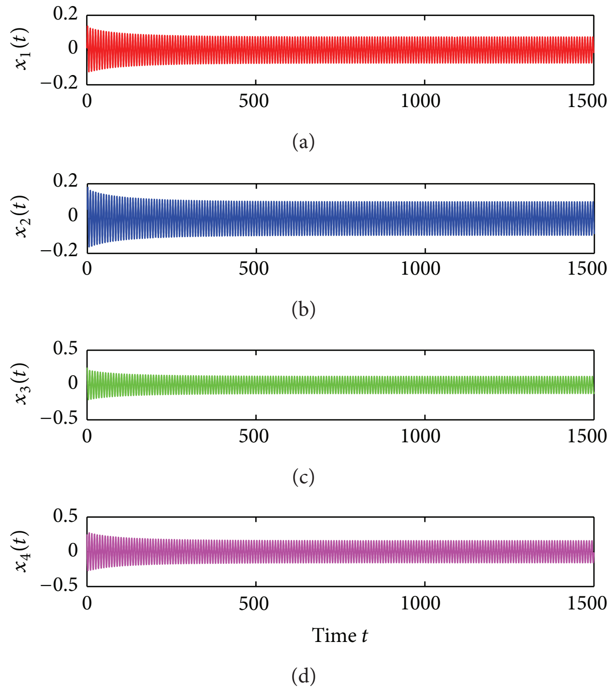

FIGURE 15: The trajectory of x_1, x_2, x_3, and x_4 when $\tau_2 = 1.85 > 1.7488 = \tau_2^*$ and $\tau_1^* = 0.35$.

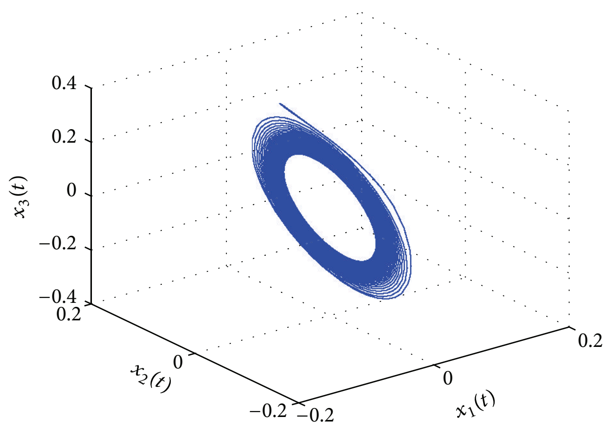

FIGURE 16: The phase plot of x_1, x_2, and x_3 when $\tau_2 = 1.85 > 1.7488 = \tau_2^*$ and $\tau_1^* = 0.35$.

then there exists a critical value of the time delay below which the system is stable and above which the system is unstable. The results have shown that the two delays can play a complicated role on the model. And from the numerical simulations, we find that τ_1 is marked in the model because the critical value of τ_1 is much smaller than that of τ_2 when we only consider them, respectively. Furthermore, the direction of the Hopf bifurcation and the stability of the bifurcating periodic solutions are discussed by the normal form theory and center manifold theory. Finally, some numerical simulations are also presented to support the theoretical analysis.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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References

[1] C. Yao, H. J. Chen, and J. P. Li, "Analysis of dynamic behaviors of improved pulse coupled neural network in image processing," *Acta Automatica Sinica*, vol. 34, no. 10, pp. 1291–1297, 2008 (Chinese).

[2] R. S. Toqeer and N. S. Bayindir, "Speed estimation of an induction motor using Elman neural network," *Neurocomputing*, vol. 55, no. 3-4, pp. 727–730, 2003.

[3] W.-M. Qi, Y.-C. Cheng, Q.-L. Ji, and W.-Y. Cai, "PID Elman neural network and its application to dynamical system identification," *Control and Decision*, vol. 20, no. 10, pp. 1197–1200, 2005 (Chinese).

[4] J. Yang, Y. Wen, and P. Li, "Feature extraction and identification of leak acoustic signal in water distribution pipelines using correlation analysis and approximate entropy," *Chinese Journal of Scientific Instrument*, vol. 30, no. 2, pp. 272–279, 2009 (Chinese).

[5] B. Luitel and G. K. Venayagamoorthy, "Quantum inspired PSO for the optimization of simultaneous recurrent neural networks as MIMO learning systems," *Neural Networks*, vol. 23, no. 5, pp. 583–586, 2010.

[6] L. Y. Wang, T. Y. Chai, and Z. Fang, "Neural network control and application of robotic manipulators including actuator dynamics," *Acta Automatica Sinica*, vol. 35, no. 5, pp. 622–626, 2009 (Chinese).

[7] Y. X. Wu and C. Wang, "Deterministic learning based adaptive network control of robot in task space," *Acta Automatica Sinica*, vol. 39, no. 1, pp. 1–10, 2013 (Chinese).

[8] J. Chen, Z. P. Li, and G. Z. Zhang, "Variable structure neural network adaptive robust control," *Acta Automatica Sinica*, vol. 36, no. 1, pp. 174–178, 2010 (Chinese).

[9] J. Wei and S. Ruan, "Stability and bifurcation in a neural network model with two delays," *Physica D*, vol. 130, no. 3-4, pp. 255–272, 1999.

[10] B. Wang and J. Jian, "Stability and Hopf bifurcation analysis on a four-neuron BAM neural network with distributed delays," *Communications in Nonlinear Science and Numerical Simulation*, vol. 15, no. 2, pp. 189–204, 2010.

[11] A. Hajihosseini, G. R. Rokni Lamooki, B. Beheshti, and F. Maleki, "The Hopf bifurcation analysis on a time-delayed recurrent neural network in the frequency domain," *Neurocomputing*, vol. 73, no. 4–6, pp. 991–1005, 2010.

[12] F. Maleki, B. Beheshti, A. Hajihosseini, and G. R. R. Lamooki, "The bogdanov-takens bifurcation analysis on a three dimensional recurrent neural network," *Neurocomputing*, vol. 73, no. 16–18, pp. 3066–3078, 2010.

[13] J. Wei and C. Zhang, "Bifurcation analysis of a class of neural networks with delays," *Nonlinear Analysis*, vol. 9, no. 5, pp. 2234–2252, 2008.

[14] C. Xu and X. He, "Stability and bifurcation analysis in a class of two-neuron networks with resonant bilinear terms," *Abstract and Applied Analysis*, vol. 2011, Article ID 697630, 21 pages, 2011.

[15] M. Xiao, W. X. Zheng, and J. D. Cao, "Bifurcation and control in a neural network with small and large delays," *Neural Networks*, vol. 44, pp. 132–142, 2013.

[16] C. Feng and R. Plamondon, "An oscillatory criterion for a time delayed neural ring network model," *Neural Networks*, vol. 29-30, pp. 70–79, 2012.

[17] X. Yang, M. Yang, H. Liu, and X. Liao, "Bautin bifurcation in a class of two-neuron networks with resonant bilinear terms," *Chaos, Solitons & Fractals*, vol. 38, no. 2, pp. 575–589, 2008.

[18] A. Ruiz, D. H. Owens, and S. Townley, "Existence, learning, and replication of periodic motions in recurrent neural networks," *IEEE Transactions on Neural Networks*, vol. 9, no. 4, pp. 651–661, 1998.

[19] S. Townley, A. Ilchmann, M. G. Weiss et al., "Existence and learning of oscillations in recurrent neural networks," *IEEE Transactions on Neural Networks*, vol. 11, no. 1, pp. 205–214, 2000.

[20] H. Hu and L. Huang, "Stability and Hopf bifurcation analysis on a ring of four neurons with delays," *Applied Mathematics and Computation*, vol. 213, no. 2, pp. 587–599, 2009.

[21] E. Javidmanesh, Z. Afsharnezhad, and S. Effati, "Existence and stability analysis of bifurcating periodic solutions in a delayed five-neuron BAM neural network model," *Nonlinear Dynamics*, vol. 72, no. 1-2, pp. 149–164, 2013.

[22] C. Xu, X. Tang, and M. Liao, "Stability and bifurcation analysis of a six-neuron BAM neural network model with discrete delays," *Neurocomputing*, vol. 74, no. 5, pp. 689–707, 2011.

[23] B. D. Hassard, N. D. Kazarinoff, and Y. H. Wan, *Theory and Applications of Hopf Bifurcation*, vol. 41, Cambridge University Press, Cambridge, UK, 1981.

[24] X.-Y. Meng, H.-F. Huo, and H. Xiang, "Hopf bifurcation in a three-species system with delays," *Journal of Applied Mathematics and Computing*, vol. 35, no. 1-2, pp. 635–661, 2011.