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Research Article

Numerical Solution of Higher Order Boundary Value Problems

Shahid S. Siddiqi[1] and Muzammal Iftikhar[1,2]

[1] *Department of Mathematics, University of the Punjab, Lahore 54590, Pakistan*
[2] *Department of Mathematics, University of Education, Okara Campus, Okara 56300, Pakistan*

Correspondence should be addressed to Muzammal Iftikhar; miftikhar@hotmail.com

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The aim of this paper is to use the homotopy analysis method (HAM), an approximating technique for solving linear and nonlinear higher order boundary value problems. Using HAM, approximate solutions of seventh-, eighth-, and tenth-order boundary value problems are developed. This approach provides the solution in terms of a convergent series. Approximate results are given for several examples to illustrate the implementation and accuracy of the method. The results obtained from this method are compared with the exact solutions and other methods (Akram and Rehman (2013), Farajeyan and Maleki (2012), Geng and Li (2009), Golbabai and Javidi (2007), He (2007), Inc and Evans (2004), Lamnii et al. (2008), Siddiqi and Akram (2007), Siddiqi et al. (2012), Siddiqi et al. (2009), Siddiqi and Iftikhar (2013), Siddiqi and Twizell (1996), Siddiqi and Twizell (1998), Torvattanabun and Koonprasert (2010), and Kasi Viswanadham and Raju (2012)) revealing that the present method is more accurate.

1. Introduction

Higher order boundary value problems occur in the study of fluid dynamics, astrophysics, hydrodynamic, hydromagnetic stability, astronomy, beam and long wave theory, induction motors, engineering, and applied physics. The boundary value problems of higher order have been examined due to their mathematical importance and applications in diversified applied sciences.

The seventh-order boundary value problems generally arise in modeling induction motors with two rotor circuits. The induction motor behavior is represented by a fifth-order differential equation model. This model contains two stator state variables, two rotor state variables, and one shaft speed. Normally, two more variables must be added to account for the effects of a second rotor circuit representing deep bars, a starting cage, or rotor distributed parameters. To avoid the computational burden of additional state variables when additional rotor circuits are required, model is often limited to the fifth-order and rotor impedance is algebraically altered as function of rotor speed under the assumption that the frequency of rotor currents depends on rotor speed. This approach is efficient for the steady state response with sinusoidal voltage, but it does not hold up during the transient conditions, when rotor frequency is not a single value. So, the behavior of such models shows up in the seventh order [1].

Chandrasekhar [2] investigated that when an infinite horizontal layer of fluid is heated from below and is subject to rotation, the instability sets in. When this instability sets in as overstability, it is represented by an eighth-order ordinary differential equation. If an infinite horizontal layer of fluid is heated from below, with the assumption that a uniform magnetic field is applied as well across the fluid in the same direction as gravity and the fluid is subject to the action of rotation, the instability sets in. When this instability sets in as ordinary convection, it is modeled by tenth-order boundary value problem.

Siddiqi and Iftikhar used the variation of parameter method for solving the seventh-order boundary value problems in [3]. Liu and Wu [4] give the general differential quadrature rule (GDQR) for the solution of eighth-order differential equation. Explicit weighting coefficients are formulated to implement the GDQR for eighth-order differential equations. Siddiqi and Akram [5] used nonic spline and nonpolynomial spline technique for the numerical solution of eighth-order linear special case boundary value problems. These have also been proven to be second order convergent.

Siddiqi and Twizell [6] presented the solution of eighth-order boundary value problem using octic spline. Inc and Evans [7] presented the solutions of eighth-order boundary value problems using Adomian decomposition method. Golbabai and Javidi [8] used homotopy perturbation method (HPM) to solve eighth-order boundary value problems. Recently, Akram and Rehman presented the numerical solution of eighth-order boundary value problems using the reproducing Kernel space method [9]. Geng and Li [10] construct a reproducing Kernel space and solve a class of linear tenth-order boundary value problems using reproducing Kernel method. Siddiqi et al. [11] used the variational iteration technique for the solution of tenth-order boundary value problem. Siddiqi and Akram [12] presented the numerical solutions of the tenth-order linear special case boundary value problems using eleventh degree spline. Siddiqi and Twizell [13] presented the solutions of tenth-order boundary value problems using tenth degree spline, where some unexpected results, for the solution and higher order derivatives, were obtained near the boundaries of the interval. Lamnii et al. [14] developed a spline collocation method using spline interpolants and analyzed the approximating solutions of some general linear boundary value problems. Domairry and Nadim in [15] compared the HAM and HPM in solving nonlinear heat transfer equation. HAM is employed to compute approximate solution of the system of differential equations governing the problem [16] and also used to detect the fin excellency of convective straight fins with temperature-dependent thermal conductivity in [17]. Moghimi et al. applied HAM to solve MHD Jeffery-Hamel flows in nonparallel walls [18]. Farajeyan and Maleki [19] used nonpolynomial spline in off-step points to solve special tenth order linear boundary value problems. Khan and Hussain in 2011 applied Laplace decomposition method (LDM) to nonlinear Blasius flow equation to obtain series solutions [20]. Khan and Gondal [21] constructed a new method for the solution of Abel's type singular integral equations. The two-step Laplace decomposition algorithm (TSLDA) makes the calculation much simpler.

Khan et al. [22] proposed a method which efficiently finds exact solution and is used to solve nonlinear Volterra integral equations. Khan et al. [23] proposed the coupling of homotopy perturbation and Laplace transformation for solving system of partial differential equations. Nadeem et al. [24] described the stagnation point flow of a viscous fluid towards a stretching sheet and obtained an analytical solution of the boundary layer equation by HAM.

Recently, Shaban et al. [25] presented modification of the HAM for solving nonlinear boundary value problems. Arqub and El-Ajou [26] investigated the accuracy of the HAM for solving the fractional order problem of the spread of a disease in a population. In [27], Russo and Van Gorder discussed the application of HAM to general nonlinear Klein-Gordon type equations.

In the present paper, the seventh-, eighth-, and tenth-order boundary value problems are solved using the homotopy analysis method (HAM). The following seventh-, eighth-, and tenth-order boundary value problems are considered:

$$u^{(m)}(x) = f(x, u(x)), \quad a \le x \le b,$$
$$u^{(i)}(a) = A_i,$$
$$u^{(j)}(b) = B_j, \tag{1}$$

where for $m = 7$, $i = 0, 1, 2, \ldots, m-4$ and $j = 0, 1, \ldots, m-5$; for $m = 8$, $i = j = 0, 1, 2, \ldots, m-5$, and for $m = 10$, $i = j = 0, 1, 2, \ldots, m-6$. A_i's and B_j's are finite real constants. Also, $f(x, u(x))$ is a continuous function on $[a, b]$.

2. Homotopy Analysis Method

Liao was the first to apply homotopy analysis method (HAM) [28–31]. This is a general analytic approach to get series solutions of nonlinear equations, including algebraic equations, ordinary differential equations, partial differential equations, differential-integral equations, differential-difference equation, and coupled equations of them. For a given nonlinear differential equation

$$N[u(x)] = 0, \quad x \in \Theta, \tag{2}$$

where N is a nonlinear operator and $u(x)$ is an unknown function, Liao constructed a one parameter family of equations in the embedding parameter $q \in [0, 1]$, called the zeroth-order deformation equation

$$(1-q) L[U(x,q) - u_0(x)] - qhH(x) N[U(x,q)] = 0, \quad x \in \Theta,\ q \in [0,1], \tag{3}$$

where h is a nonzero auxiliary parameter, $H(x)$ is an auxiliary function, L is an auxiliary linear operator, $u_0(x)$ is an initial guess, and $U(x,q)$ is an unknown function. The homotopy provides us larger freedom to choose both the auxiliary linear operator L and the initial guess than the traditional nonperturbation methods, as pointed out by Liao [29, 31]. At $q = 0$ and $q = 1$, we have $U(x,0) = u_0(x)$ and $U(x,1) = u(x)$, respectively. Thus, as q increases from 0 to 1, the solution $U(x,q)$ varies from the initial guess $u_0(x)$ to the solution $u(x)$. Expanding $U(x,q)$ by Taylor series with respect to q, (2) becomes

$$U(x,q) = u_0(x) + \sum_{m=1}^{\infty} u_m(x) q^m, \tag{4}$$

where

$$u_m(x) = \frac{1}{m!} \left. \frac{\partial^m U(x,q)}{\partial q^m} \right|_{q=0}. \tag{5}$$

If the auxiliary linear operator, the initial guess, the auxiliary parameter h, and the auxiliary function are properly chosen, series (4) converges at $q = 1$, and then the homotopy series solution

$$u(x) = u_0 + \sum_{m=1}^{\infty} u_m(x) \tag{6}$$

must be one of solutions of original equations $N[u(x)] = 0$ [29]. Here, $u_m(x)$ is governed by a linear differential equation related to the auxiliary linear operator L. According to definition (6), the governing equation can be deduced from the zeroth-order deformation (4). Define the vector

$$\vec{u}_N(x) = \{u_0(x), u_1(x), \ldots, u_N(x)\}. \quad (7)$$

Differentiating equation (3) m times with respect to the embedded homotopy parameter q, then setting $q = 0$, and then finally dividing them by $m!$, the mth-order deformation equation is obtained as

$$L[u_m(x) - \chi_m u_{m-1}(x)] - hH(x) R_m(\vec{u}_{m-1}) = 0, \quad x \in \Theta, \; q \in [0,1], \quad (8)$$

where

$$R_m(\vec{u}_{m-1}) = \frac{1}{(m-1)!} \left.\frac{\partial^{m-1} N(U(x,q))}{\partial q^{m-1}}\right|_{q=0}, \qquad \chi_m = \begin{cases} 0, & m \le 1, \\ 1, & m \ge 1. \end{cases} \quad (9)$$

Finally, an N-th order approximate solution is given by

$$u(x) = u_0 + \sum_{m=1}^{N} u_m(x). \quad (10)$$

To implement the HAM, several numerical examples are considered in the following section.

3. Numerical Examples

Example 1. Consider the following seventh-order boundary value problem:

$$\begin{gathered} u^{(7)}(x) = -u(x) - e^x(35 + 12x + 2x^2), \quad 0 \le x \le 1, \\ u(0) = 0, \quad u(1) = 0, \\ u^{(1)}(0) = 1, \quad u^{(1)}(1) = -e, \\ u^{(2)}(0) = 0, \quad u^{(2)}(1) = -4e, \\ u^{(3)}(0) = -3. \end{gathered} \quad (11)$$

The exact solution of Example 1 is $u(x) = x(1-x)e^x$ [3].

Using the HAM (3), the zeroth-order deformation is given by

$$\begin{aligned} &(1-q) L[U(x,q) - u_0(x)] \\ &= qhH(x)\left(\frac{\partial^7 U(x,q)}{\partial q^7} + U(x,q) + e^x(35 + 12x + 2x^2)\right). \end{aligned} \quad (12)$$

Now, the initial approximation, $u_0(x)$, is the solution of $(\partial^7/\partial x^7)u = 0$ subject to boundary conditions in (11); that is,

$$\begin{aligned} u_0(x) = {} & x - \frac{x^3}{2} + \frac{1}{2}(-17 + 6e)x^4 \\ & + \frac{1}{2}(27 - 10e)x^5 + \frac{1}{2}(-11 + 4e)x^6. \end{aligned} \quad (13)$$

The linear operator L normally consists of the homogeneous part of nonlinear operator N, whereas parameter h and function $H(x)$ are introduced in order to optimize the initial guess. Try to choose h in such a way that they get a convergent series. Under the rule of solution expression (4), the auxiliary function $H(x)$ can be chosen as $H(x) = 1$. In this way, good approximations of such problems can be obtained without having to go up to high order of approximation and without requiring a small parameter.

Hence, the mth-order deformation can be given by

$$L[u_m(x) - \chi_m u_{m-1}(x)] = hH(x) R_m(\vec{u}_{m-1}), \quad (14)$$

where

$$\begin{aligned} R_1(\vec{u}_0) = {} & \frac{\partial^7 u_{m-1}(x,q)}{\partial q^7} \\ & + u_{m-1}(x,q) + e^x(35 + 12x + 2x^2), \end{aligned} \quad (15)$$

$$R_m(\vec{u}_{m-1}) = \frac{\partial^7 u_{m-1}(x,q)}{\partial q^7} + u_{m-1}(x,q), \quad m \ge 2. \quad (16)$$

Now, the solution of the mth-order deformation equations (15) and (16) for $m \ge 1$ becomes

$$u_m(x) = \chi_m u_{m-1}(x) + hL^{-1}[R_m(\vec{u}_{m-1})]. \quad (17)$$

Consequently, the first few terms of the homotopy series solution are as follows:

$$\begin{aligned} u_0(x) = {} & x - \frac{x^3}{2} + \frac{1}{2}(-17 + 6e)x^4 \\ & + \frac{1}{2}(27 - 10e)x^5 + \frac{1}{2}(-11 + 4e)x^6, \end{aligned} \quad (18)$$

$$\begin{aligned} u_1(x) = {} & \frac{h}{518918400} \\ & \times (-32691859200 + 32691859200e^x \\ & - 24389164800x - 8302694400e^x x \\ & - 9081072000x^2 + 1037836800e^x x^2 \\ & - 2335132800x^3 + 795242426295x^4 \\ & - 292929440550ex^4 - 1230096003906x^5 \\ & + 452496853920ex^5 + 502831877247x^6 \\ & - 184994415376ex^6 + 518918400x^7 \end{aligned}$$

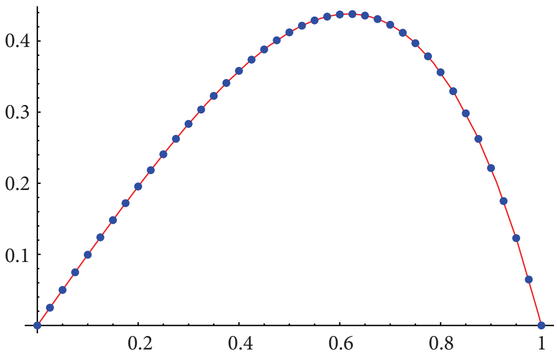

Figure 1: Comparison of the approximate solution with the exact solution for problem (11). Dotted line: approximate solution; solid line: the exact solution.

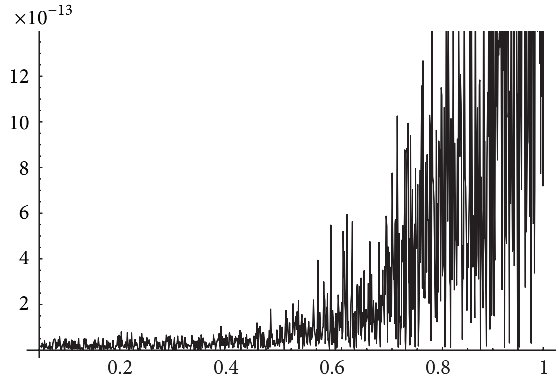

Figure 2: Absolute errors for problem (11).

$$\begin{aligned}&+12870x^{8}-429x^{10}-2652x^{11}\\&+936ex^{11}+1755x^{12}-650ex^{12}\\&-330x^{13}+120ex^{13}\Big).\end{aligned}\tag{19}$$

The Nth-order approximation of $u(x)$ can be expressed by

$$u_N(x)=u_0(x)+\sum_{m=1}^{N}u_m(x).\tag{20}$$

Equation (20) is a family of the approximate solutions to problem (11) in terms of the convergence control parameter h.

We choose the value of auxiliary parameter as $h=-1$ to ensure that the solution series converges. The errors in absolute values obtained using the present method are compared with those obtained using the variation of parameter method [3] for Example 1 given in Table 1, which shows that the present method is quite accurate. Figures 1 and 2 show the comparison of exact with approximate solution and absolute errors for Example 1 solution respectively.

Convergence Theorem. *In this subsection, one proves that if the solution series* (6) *given by HAM is convergent, it must be an exact solution of the considered problem.*

If the series $u_0(x)+\sum_{m=1}^{\infty}u_m(x)$ converges, where $u_m(x)$ is governed by (17) *under the definitions* (15) *and* (16), *it must be an exact solution of problem* (11).

Proof. Let the series

$$\sum_{m=0}^{\infty}u_m(x)\tag{21}$$

be convergent. Then,

$$u(x)=\sum_{m=0}^{\infty}u_m(x),\tag{22}$$

$$\lim_{n\to\infty}u_n(x)=0.\tag{23}$$

We have

$$\begin{aligned}&\sum_{m=1}^{n}\left[u_m(x)-\chi_m u_{m-1}(x)\right]\\&\quad=u_1+(u_2-u_1)+(u_3-u_2)+\cdots+(u_n-u_{n-1}).\end{aligned}\tag{24}$$

Using (22),

$$\sum_{m=1}^{\infty}\left[u_m(x)-\chi_m u_{m-1}(x)\right]=\lim_{n\to\infty}u_n(x)=0,\tag{25}$$

and applying the operator L, we can write

$$\begin{aligned}&L\sum_{m=1}^{\infty}\left[u_m(x)-\chi_m u_{m-1}(x)\right]\\&\quad=\sum_{m=1}^{\infty}L\left[u_m(x)-\chi_m u_{m-1}(x)\right]=0;\end{aligned}\tag{26}$$

using the definition (14),

$$\sum_{m=1}^{\infty}L\left[u_m(x)-\chi_m u_{m-1}(x)\right]=hH(x)\sum_{m=1}^{\infty}R_m(\vec{u}_{m-1}),\tag{27}$$

and since $h\neq 0$, $H(x)\neq 0$, we have

$$\sum_{m=1}^{\infty}R_m(\vec{u}_{m-1})=0.\tag{28}$$

From (15) and (16), the following holds:

$$\begin{aligned}&\sum_{m=1}^{\infty}R_m(\vec{u}_{m-1})\\&\quad=\sum_{m=1}^{\infty}\left(\frac{\partial^7u_{m-1}(x,q)}{\partial q^7}+u_{m-1}(x,q)\right.\\&\qquad\left.+(1-\chi_m)e^{x}\left(35+12x+2x^2\right)\right)\\&\quad=0.\end{aligned}\tag{29}$$

This completes the proof. □

Table 1: Comparison of numerical results for Example 1.

x	Exact solution	Approximate series solution	Absolute error present method	Absolute error Siddiqi and Iftikhar [3]
0.0	0.0000	0.0000	0.0000	0.0000
0.1	0.9946	0.9946	$5.39291E-14$	$8.55607E-13$
0.2	0.1954	0.1954	$4.85167E-14$	$9.94041E-12$
0.3	0.2835	0.2835	$3.92464E-14$	$3.52244E-11$
0.4	0.3580	0.3580	$2.21489E-14$	$7.3224E-10$
0.5	0.4122	0.4122	$3.84137E-14$	$1.08769E-10$
0.6	0.4373	0.4373	$2.10831E-13$	$1.29035E-10$
0.7	0.4229	0.4229	$1.99785E-13$	$1.51466E-10$
0.8	0.3561	0.3561	$3.29736E-13$	$2.717974E-10$
0.9	0.2214	0.2214	$1.77622E-12$	$7.48179E-10$
1.0	0.0000	$-1.65159E-12$	$1.65159E-12$	$2.1729E-09$

Example 2. The following seventh-order nonlinear boundary value problem is considered:

$$\begin{gathered} u^{(7)}(x)=u(x)u'(x)+e^{-2x}\left(2+e^{x}(x-8)-3x+x^{2}\right), \\ 0\leq x\leq 1, \\ u(0)=1,\quad u(1)=0, \\ u^{(1)}(0)=-2,\quad u^{(1)}(1)=-\frac{1}{e}, \\ u^{(2)}(0)=3,\quad u^{(2)}(1)=\frac{2}{e}, \\ u^{(3)}(0)=-4. \end{gathered} \tag{30}$$

The exact solution of Example 2 is $u(x)=(1-x)e^{-x}$.

Using the HAM (3), the zeroth-order deformation is given by

$$\begin{aligned} (1-q)L[U(x,q)-u_0(x)] &= qhH(x)\left(\frac{\partial^7 U(x,q)}{\partial q^7}+U(x,q)\frac{\partial U(x,q)}{\partial q}\right. \\ &\quad \left.+e^{-2x}\left(2+e^{x}(-8+x)-3x+x^{2}\right)\right). \end{aligned} \tag{31}$$

Now, the initial approximation, $u_0(x)$, is the solution of $(\partial^7/\partial x^7)u=0$ subject to boundary conditions in (30); that is,

$$\begin{aligned} u_0(x) &= 1-2x+\frac{3x^2}{2}-\frac{2x^3}{3}+\frac{(36-12e)x^4}{6e} \\ &\quad +\frac{(-66+24e)x^5}{6e}+\frac{(30-11e)x^6}{6e}. \end{aligned} \tag{32}$$

The linear operator L normally consists of the homogeneous part of nonlinear operator N, whereas parameter h and function $H(x)$ are introduced in order to optimize the initial guess. Try to choose h in such a way that they get a convergent series. Under the rule of solution expression (4), the auxiliary function $H(x)$ can be chosen as $H(x)=1$. In this way, good approximations of such problems can be obtained without having to go up to high order of approximation and without requiring a small parameter.

Hence, the mth-order deformation can be given by

$$L[u_m(x)-\chi_m u_{m-1}(x)]=hH(x)R_m(\vec{u}_{m-1}), \tag{33}$$

where

$$\begin{aligned} R_1(\vec{u}_0) &= \frac{\partial^7 u_{m-1}(x,q)}{\partial q^7}+u_{m-1}(x,q)\frac{\partial u_{m-1}(x,q)}{\partial q} \\ &\quad +e^{-2x}\left(2+e^{x}(-8+x)-3x+x^{2}\right), \\ R_m(\vec{u}_{m-1}) &= \frac{\partial^7 u_{m-1}(x,q)}{\partial q^7} \\ &\quad +u_{m-1}(x,q)\frac{\partial u_{m-1}(x,q)}{\partial q},\quad m\geq 2. \end{aligned} \tag{34}$$

Now, the solution of the mth-order deformation equations (34) for $m\geq 1$ becomes

$$u_m(x)=\chi_m u_{m-1}(x)+hL^{-1}[R_m(\vec{u}_{m-1})]. \tag{35}$$

Consequently, the first few terms of the homotopy series solution are as follows:

$$\begin{aligned} u_0(x) &= 1-2x+\frac{3x^2}{2}-\frac{2x^3}{3}+\frac{(36-12e)x^4}{6e} \\ &\quad +\frac{(-66+24e)x^5}{6e}+\frac{(30-11e)x^6}{6e}, \end{aligned} \tag{36}$$

$$\begin{aligned} u_1(x) &= \frac{he^{-2-2x}}{52929676800} \\ &\quad \times\left(2274322050e^{2}-52929676800e^{2+x}\right. \\ &\quad +50655354750e^{2+2x}+1654052400e^{2}x \\ &\quad +52929676800e^{2+x}x-102964761900e^{2+2x}x \end{aligned}$$

$$+ 413513100e^{2}x^{2} + 77740462800e^{2+2x}x^{2} - 34735100400e^{2+2x}x^{3} - 100276694820e^{2x}x^{4} + 317578525200e^{1+2x}x^{4} - 92426387280e^{2+2x}x^{4} + 168299259192e^{2x}x^{5} - 582227931918e^{1+2x}x^{5} + 188997561564e^{2+2x}x^{5} - 72364427100e^{2x}x^{6} + 264649706520e^{1+2x}x^{6} - 87281523925e^{2+2x}x^{6} + 21003840e^{2+2x}x^{7} - 9189180e^{2+2x}x^{8} + 3208920e^{2+2x}x^{9} - 2100384e^{1+2x}x^{10} + 160446e^{2+2x}x^{10} + 3659760e^{1+2x}x^{11} - 1113840e^{2+2x}x^{11} - 2864160e^{1+2x}x^{12} + 1003340e^{2+2x}x^{12} + 1306620e^{1+2x}x^{13} - 471240e^{2+2x}x^{13} - 440640e^{2x}x^{14} + 69360e^{1+2x}x^{14} + 83640e^{2+2x}x^{14} + 969408e^{2x}x^{15} - 626688e^{1+2x}x^{15} + 99552e^{2+2x}x^{15} - 830790e^{2x}x^{16} + 596700e^{1+2x}x^{16} - 107100e^{2+2x}x^{16} + 326700e^{2x}x^{17} - 238590e^{1+2x}x^{17} + 43560e^{2+2x}x^{17} - 49500e^{2x}x^{18} + 36300e^{1+2x}x^{18} - 6655e^{2+2x}x^{18} \Big). \quad (37)$$

The Nth-order approximation can be expressed by

$$u(x) = u_0(x) + \sum_{m=1}^{N} u_M(x). \quad (38)$$

Equation (38) is a family of the approximate solutions to problem (12) in terms of the convergence control parameter h.

We choose the value of auxiliary parameter as $h = -1$ to ensure that the solution series converges. In Table 2, the exact solution and the series solution of Example 2 are compared, which shows that the method is quite accurate. Figure 3 shows the comparison of exact solution with approximate solution, and Figure 4 shows the absolute errors for Example 2.

Example 3. The following seventh-order linear boundary value problem is considered:

$$u^{(7)}(x) = xu(x) + e^{x}\left(x^2 - 2x - 6\right), \quad 0 \le x \le 1,$$
$$u(0) = 1, \quad u(1) = 0,$$
$$u^{(1)}(0) = 0, \quad u^{(1)}(1) = -e,$$
$$u^{(2)}(0) = -1, \quad u^{(2)}(1) = -2e,$$
$$u^{(3)}(0) = -2. \quad (39)$$

The exact solution of Example 3 is $u(x) = (1-x)e^{x}$ [32].

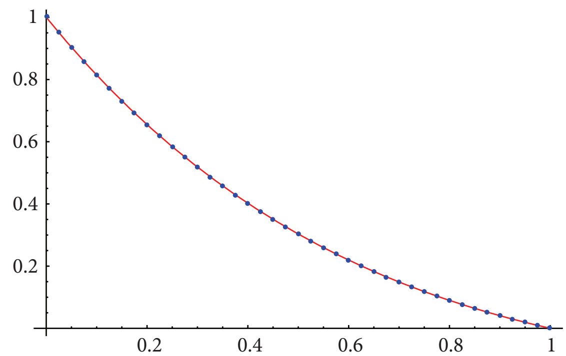

FIGURE 3: Comparison of the approximate solution with the exact solution for problem (12). Dotted line: approximate solution; solid line: the exact solution.

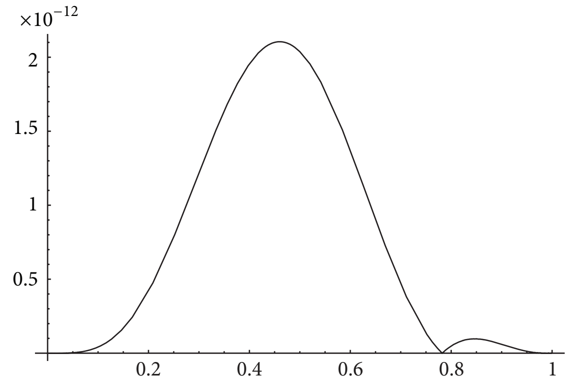

FIGURE 4: Absolute errors for problem (12).

TABLE 2: Comparison of numerical results for Example 2.

x	Exact solution	Approximate series solution	Absolute error
0.0	1.0000	1.0000	0.0000
0.1	0.814354	0.814354	4.15223E − 14
0.2	0.654985	0.654985	4.18332E − 13
0.3	0.518573	0.518573	1.21736E − 12
0.4	0.402192	0.402192	1.95471E − 12
0.5	0.303265	0.303265	2.03731E − 12
0.6	0.219525	0.219525	1.37063E − 12
0.7	0.148976	0.148976	4.66988E − 13
0.8	0.0898658	0.0898658	4.8378E − 14
0.9	0.040657	0.040657	6.00561E − 14
1.0	0.0000	−1.29172E − 15	1.29172E − 15

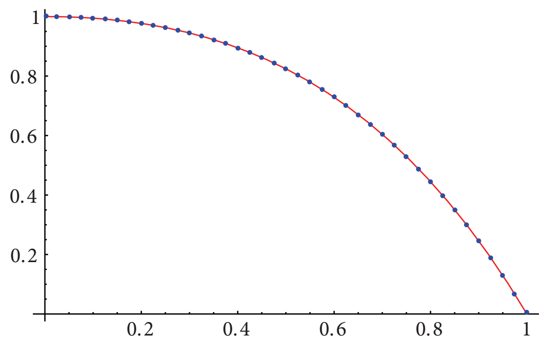

FIGURE 5: Comparison of the approximate solution with the exact solution for problem (13). Dotted line: approximate solution; solid line: the exact solution.

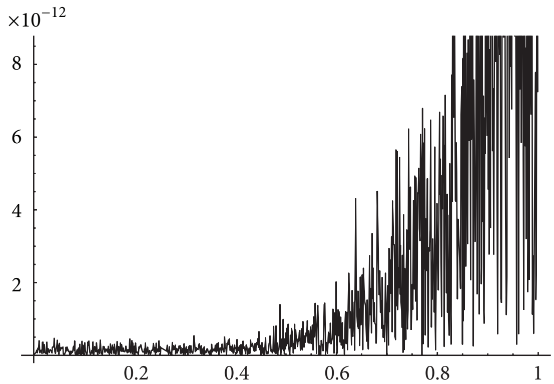

FIGURE 6: Absolute errors for problem (13).

TABLE 3: Comparison of numerical results for Example 3.

x	Exact solution	Approximate series solution	Absolute error present method	Absolute error Siddiqi et al. [32]
0.0	1.0000	1.0000	0.0000	0.0000
0.1	0.0994	0.0994	$3.41727E-13$	$4.6585E-13$
0.2	0.9771	0.9771	$6.25056E-14$	$5.7126E-12$
0.3	0.9449	0.9449	$1.42442E-13$	$2.1299E-11$
0.4	0.8950	0.8950	$8.83738E-14$	$4.6995E-11$
0.5	0.8243	0.8243	$6.43929E-14$	$7.4307E-11$
0.6	0.7288	0.7288	$1.51812E-12$	$8.9219E-11$
0.7	0.6041	0.6041	$1.47904E-12$	$7.9767E-11$
0.8	0.4451	0.4451	$4.94338E-12$	$4.6686E-11$
0.9	0.2459	0.2459	$5.3817E-12$	$1.0960E-11$
1.0	0.0000	$1.20811E-11$	$1.20811E-11$	$6.9252E-16$

Following the procedure of the previous example, this problem is solved using the convergence control parameter $h = -1$.

The comparison of the absolute errors obtained by the present method and the absolute errors obtained by the method in [32] is given in Table 3, which shows that the present method is quite accurate. In Figure 5, the comparison of exact solution with approximate solution is shown, and Figure 6 shows absolute errors for Example 3.

Example 4. Consider the following eighth-order boundary value problem:

$$\begin{gathered} u^{(8)}(x) = -xu(x) - e^{x}\left(48 + 15x + 2x^{3}\right), \quad 0 \le x \le 1, \\ u(0) = 0, \quad u(1) = 0, \\ u^{(1)}(0) = 1, \quad u^{(1)}(1) = -e, \\ u^{(2)}(0) = 0, \quad u^{(2)}(1) = -4e, \\ u^{(3)}(0) = -3, \quad u^{(3)}(1) = -9e. \end{gathered} \tag{40}$$

The exact solution of Example 4 is $u(x) = x(1-x)e^{x}$ [5, 7, 9].

Using the HAM (3), the zeroth-order deformation is given by

$$\begin{aligned} &(1-q) L\left[U(x,q) - u_0(x)\right] \\ &\quad = qhH(x)\left(\frac{\partial^8 U(x,q)}{\partial q^8} + xU(x,q)\right. \\ &\qquad \left. + e^{x}\left(48 + 15x + 2x^{3}\right)\right). \end{aligned} \tag{41}$$

Now, the initial approximation, $u_0(x)$, is the solution of $(\partial^8/\partial x^8)u = 0$ subject to boundary conditions in (40); that is,

$$\begin{aligned} u_0(x) &= x - \frac{x^3}{2} + \frac{1}{2}(-36 + 13e)x^4 + \frac{1}{2}(84 - 31e)x^5 \\ &\quad + \frac{1}{2}(-68 + 25e)x^6 + \frac{1}{2}(19 - 7e)x^7. \end{aligned} \tag{42}$$

The linear operator L normally consists of the homogeneous part of nonlinear operator N, whereas parameter h and function $H(x)$ are introduced in order to optimize the initial guess. Try to choose h in such a way that they get a convergent series. Under the rule of solution expression (4), the auxiliary function $H(x)$ can be chosen as $H(x) = 1$. In this way, good approximations of such problems can be obtained without having to go up to high order of approximation and without requiring a small parameter.

Hence, the mth-order deformation can be given by

$$L\left[u_m(x) - \chi_m u_{m-1}(x)\right] = hH(x) R_m\left(\vec{u}_{m-1}\right), \tag{43}$$

where

$$\begin{aligned} R_1(\vec{u}_0) &= \frac{\partial^8 u_{m-1}(x,q)}{\partial q^8} + xu_{m-1}(x,q) \\ &\quad + e^{x}\left(48 + 15x + 2x^{3}\right), \end{aligned} \tag{44}$$

$$R_m(\vec{u}_{m-1}) = \frac{\partial^8 u_{m-1}(x,q)}{\partial q^8} + u_{m-1}(x,q), \quad m \ge 2.$$

Table 4: Comparison of absolute errors for Example 4.

x	Exact solution	Approximate series solution	Absolute error present method	Akram and Rehman [9]	Siddiqi and Akram [5]	Inc and Evans [7]
0.1	0.0994654	0.0994654	3.89966$E-15$	1.63$E-10$	5.62$E-10$	3.73$E-09$
0.2	0.195424	0.195424	9.45355$E-14$	1.63$E-09$	4.88$E-09$	6.61$E-09$
0.3	0.28347	0.28347	7.04437$E-14$	4.90$E-09$	1.37$E-08$	2.33$E-08$
0.4	0.358038	0.358038	4.36873$E-13$	8.46$E-09$	2.29$E-08$	5.17$E-08$
0.5	0.41218	0.41218	1.28897$E-13$	1.01$E-08$	2.71$E-08$	9.76$E-08$
0.6	0.437309	0.437309	4.01956$E-13$	8.68$E-09$	2.38$E-08$	1.78$E-06$
0.7	0.422888	0.422888	2.06929$E-12$	5.15$E-09$	1.49$E-08$	4.12$E-06$
0.8	0.356087	0.356087	2.65915$E-12$	1.76$E-09$	5.54$E-09$	1.83$E-04$

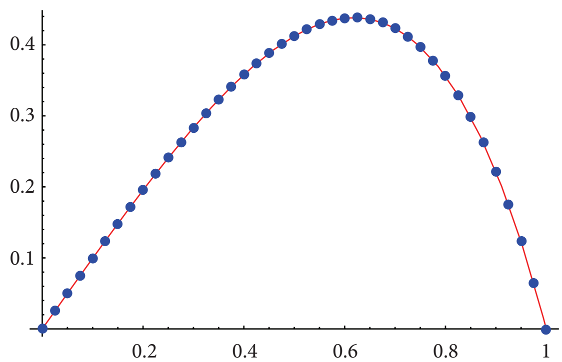

Figure 7: Comparison of the approximate solution with the exact solution for problem (14). Dotted line: approximate solution; solid line: the exact solution.

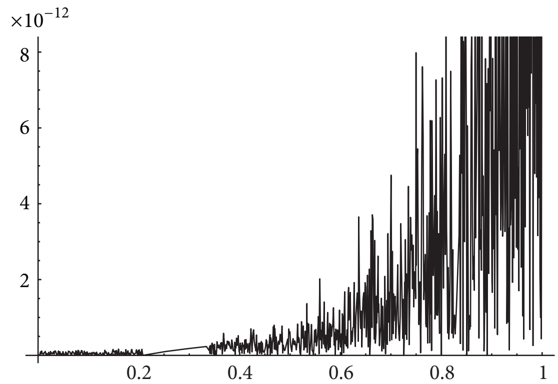

Figure 8: Absolute errors.

Now, the solution of the mth-order deformation equations (44) for $m \geq 1$ becomes

$$u_m(x) = \chi_m u_{m-1}(x) + hL^{-1}\left[R_m\left(\vec{u}_{m-1}\right)\right]. \quad (45)$$

Consequently, the approximations $u_0, u_1, \ldots$ of the homotopy series solution are obtained.

The N-th order approximation can be expressed by

$$u(x) = u_0(x) + \sum_{m=1}^{N} u_m(x). \quad (46)$$

Equation (46) is a family of the approximate solutions to problem (14) in terms of the convergence control parameter h.

We choose the value of auxiliary parameter as $h = -1$ to ensure that the solution series converges. For problem (14), comparison of the results of the present method with the results of Akram and Rehman [9], Inc and Evans [7], and Siddiqi and Akram [5] is shown in Table 4. It is observed that the errors in absolute values of the present method are better. The comparison of exact solution with approximate solution is shown in Figure 7, and absolute errors are shown in Figure 8, respectively.

Example 5. Consider the following boundary value problem:

$$u^{(8)}(x) = u(x) - 8(2x\cos(x) + 7\sin(x)), \quad -1 \leq x \leq 1,$$
$$u(-1) = u(1) = 0,$$

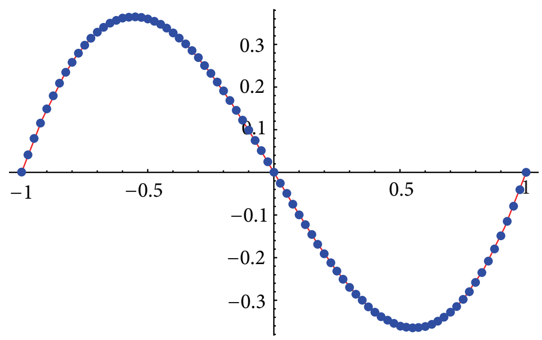

Figure 9: Comparison of the approximate solution with the exact solution for problem (15). Dotted line: approximate solution, solid line: the exact solution.

$$u^{(1)}(-1) = u^{(1)}(1) = 2\sin(1),$$
$$u^{(2)}(-1) = -u^{(2)}(1) = -4\cos(1) - 2\sin(1),$$
$$u^{(3)}(-1) = u^{(3)}(1) = 6\cos(1) - 6\sin(1). \quad (47)$$

The exact solution of problem (15) is $u(x) = (x^2 - 1)\sin(x)$ [5, 6, 9, 14].

Following the procedure of the previous example, this problem is solved using the convergence control parameter as $h = -1$.

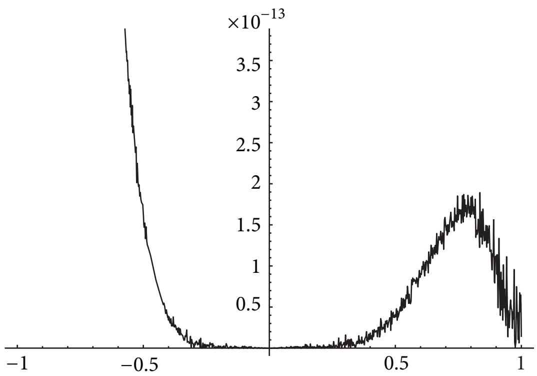

Figure 10: Absolute errors for problem (15).

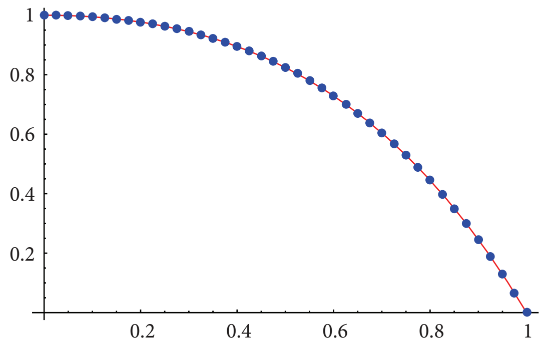

Figure 11: Comparison of the approximate solution with the exact solution for problem (16). Dotted line: approximate solution; solid line: the exact solution.

Table 5: Comparison of maximum absolute errors for Problem (15).

Present method	Akram and Rehman [9]	Lamnii et al. [14]	Siddiqi and Twizell [6] $x \in [x_4, x_{k-4}]$	Siddiqi and Akram [5]
1.89×10^{-13}	4.90×10^{-9}	5.01×10^{-9}	1.20×10^{-5}	1.02×10^{-8}

It is observed that the maximum absolute error values are better than those of Akram and Rehman [9], Lamnii et al. [14], Siddiqi and Akram [5], and Siddiqi and Twizell [6] as shown in Table 5. The comparison of exact solution with approximate solution is shown in Figure 9, and absolute errors are shown in Figure 10, respectively.

Example 6. Consider the following boundary value problem:

$$\begin{aligned} &u^{(8)}(x) = -8e^{x} + u(x), \quad 0 < x < 1, \\ &u(0) = 1, \quad u^{(4)}(0) = -3, \\ &u^{(1)}(0) = 0, \quad u^{(5)}(0) = -4, \\ &u^{(2)}(0) = -1, \quad u^{(1)}(1) = -e, \\ &u^{(3)}(0) = -2, \quad u^{(2)}(1) = -2e. \end{aligned} \tag{48}$$

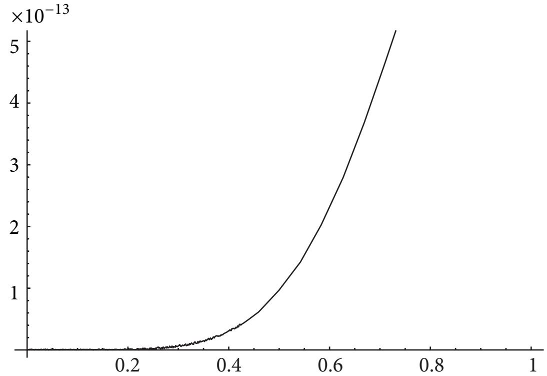

Figure 12: Absolute errors for problem (16).

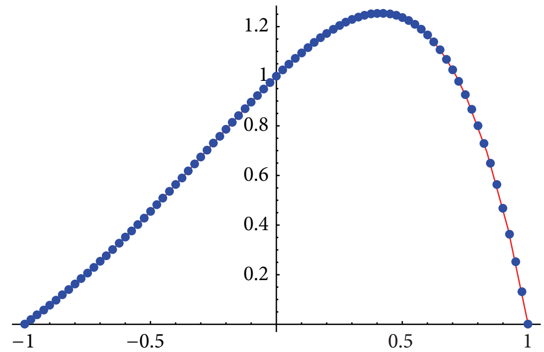

Figure 13: Comparison of the approximate solution with the exact solution for problem (17). Dotted line: approximate solution; solid line: the exact solution.

The exact solution of problem (16) is $u(x) = (1-x)e^{x}$ [8, 9, 33, 34].

Following the procedure of the previous example, this problem is solved using the convergence control parameter as $h = -1$.

It is observed that the errors in absolute values are better than those of Akram and Rehman [9], Golbabai and Javidi [8], He [33], and Torvattanabun and Koonprasert [34] as shown in Table 6. Figures 11 and 12 show the comparison of exact solution with approximate solution and absolute errors, respectively.

Example 7. Consider the following tenth-order boundary value problem:

$$\begin{aligned} &u^{(10)}(x) \\ &\quad = -e^{x}\left(89 + 21x + x^{2} - x^{3}\right) + xu(x), \quad -1 < x < 1, \\ &u(-1) = 0, \quad u(1) = 0, \\ &u^{(1)}(-1) = \frac{2}{e}, \quad u^{(1)}(1) = -2e, \\ &u^{(2)}(-1) = \frac{2}{e}, \quad u^{(2)}(1) = -6e, \end{aligned}$$

Table 6: Comparison of absolute errors for Problem (16).

x	Present method $\lvert u-u_1\rvert$	Akram and Rehman [9], $\lvert u-u_7\rvert$	Golbabai and Javidi [8] ($N=7$)	He [33]	Torvattanabun and Koonprasert [34]
0.25	$2.55351E-15$	$3.0291E-10$	$2.1630E-09$	$4.578E-09$	$3.8922E-10$
0.50	$9.65894E-14$	$7.7317E-09$	$1.1571E-07$	$9.840E-09$	$1.1571E-07$
0.75	$5.63438E-13$	$3.1222E-08$	$1.0479E-06$	$1.096E-05$	$1.0479E-06$
1.0	$9.23328E-13$	$4.3979E-08$	$4.2188E-06$	$1.861E-04$	$4.2188E-06$

Table 7: Comparison of maximum absolute errors for Problem (17).

Present method	Geng and Li [10]	Siddiqi et al. [11]	Siddiqi and Akram [12]	Siddiqi and Twizell [13] $x\in[x_5,x_{k-5}]$	Lamnii et al. [14]	Farajeyan and Maleki [19]
6.42×10^{-13}	9.08×10^{-12}	1.97×10^{-6}	3.28×10^{-6}	2.07×10^{-3}	1.86×10^{-8}	1.75×10^{-12}

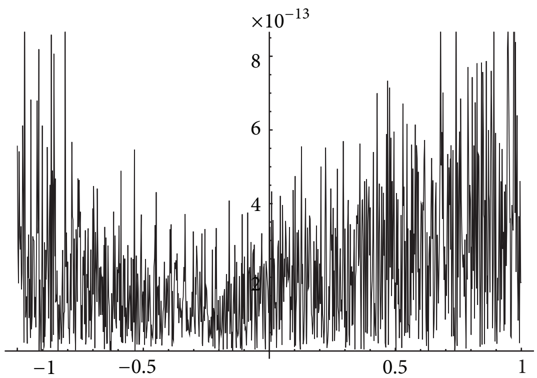

Figure 14: Absolute errors for problem (17).

Table 8: Comparison of numerical results for Example 8.

x	Exact solution	Absolute error present method	Absolute error Kasi Viswanadham and Raju [35]
0.1	−0.0473684	$3.95413E-11$	$1.322478E-06$
0.2	−0.0888889	$7.33317E-10$	$4.231930E-06$
0.3	−0.123529	$7.33317E-09$	$1.676381E-05$
0.4	−0.15	$6.06524E-09$	$4.245341E-05$
0.5	−0.166667	$7.74775E-09$	$6.663799E-05$
0.6	−0.171429	$6.56402E-09$	$6.940961E-05$
0.7	−0.161538	$3.48667E-09$	$4.750490E-05$
0.8	−0.133333	$9.23198E-10$	$1.643598E-05$
0.9	−0.08181824	$5.33521E-11$	$2.607703E-07$

$$u^{(3)}(-1)=0,\quad u^{(3)}(1)=-12e,$$
$$u^{(4)}(-1)=\frac{-4}{e},\quad u^{(3)}(1)=-20e. \tag{49}$$

The exact solution of problem (17) is $u(x)=(1-x^2)e^x$ [10–14, 19].

Following the procedure of the previous example, this problem is solved using the convergence control parameter as $h=-1$.

It is observed that the errors in absolute values of the present method are better than those of Farajeyan and Maleki [19], Geng and Li [10], Lamnii et al. [14], Siddiqi and Akram [12], Siddiqi et al. [11], and Siddiqi and Twizell [13] as shown in Table 7. Figures 13 and 14 show the comparison of exact solution with approximate solution and absolute errors for Example 7, respectively.

Example 8. Consider the following tenth-order nonlinear boundary value problem:

$$\begin{gathered}
u^{(10)}(x)=\frac{14175}{4}(x+u(x)+1)^{11},\quad 0<x<1,\\
u(0)=0,\quad u(1)=0,\\
u^{(1)}(0)=\frac{-1}{2},\quad u^{(1)}(1)=1,\\
u^{(2)}(0)=\frac{1}{2},\quad u^{(2)}(1)=4,\\
u^{(3)}(0)=\frac{3}{4},\quad u^{(3)}(1)=12,\\
u^{(4)}(0)=\frac{3}{2},\quad u^{(3)}(1)=48.
\end{gathered} \tag{50}$$

The exact solution of problem (18) is $u(x)=(2/(2-x))-x-1$ [35].

Following the procedure of the previous example, this problem is solved using the convergence control parameter as $h=-1$.

It is observed that the errors in absolute values are better than those of Viswanadham and Raju [35] as shown in Table 8. In Figure 15, the comparison of exact solution with approximate solution is shown, and in Figure 16, absolute errors for Example 8 are shown, respectively.

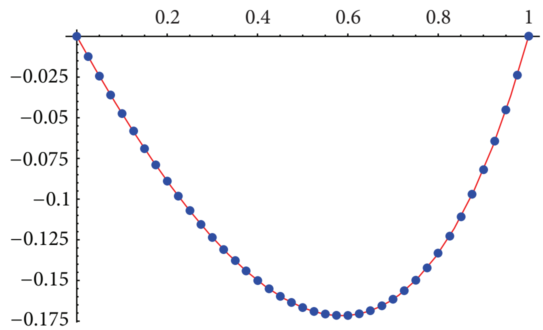

FIGURE 15: Comparison of the approximate solution with the exact solution for problem (18). Dotted line: approximate solution; solid line: the exact solution.

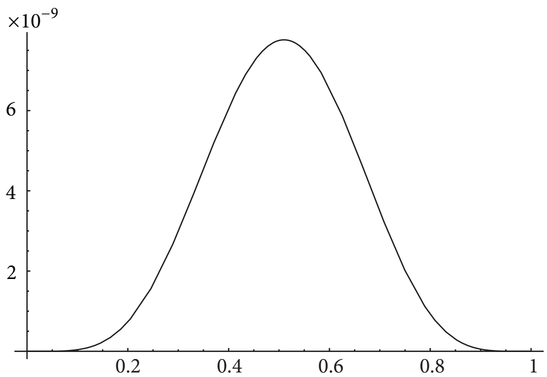

FIGURE 16: Absolute errors for problem (18).

4. Conclusion

In this paper, the homotopy analysis method (HAM) has been applied to obtain the numerical solutions of seventh-eighth-, and tenth-order boundary value problems. All computational work was carried out using Mathematica software. The numerical results show that only a few number of approximations can be used for numerical purpose with a high degree of accuracy. It is observed that the absolute errors are better than the methods in [3, 5–14, 19, 32–35]. It is also observed that our proposed method is well suited for the solution of higher order boundary value problems and reduces the computational work. HAM converges to exact solutions more rapidly as compared to the other method. Therefore, the present method is an accurate and reliable analytical technique for boundary value problems.

References

[1] G. Richards and P. R. R. Sarma, "Reduced order models for induction motors with two rotor circuits," *IEEE Transactions on Energy Conversion*, vol. 9, no. 4, pp. 673–678, 1994.

[2] S. Chandrasekhar, *Hydrodynamic and Hydromagnetic Stability*, The International Series of Monographs on Physics, Clarendon Press, Oxford, UK, 1961.

[3] S. S. Siddiqi and M. Iftikhar, "Solution of seventh order boundary value problems by variation of parameters method," *Research Journal of Applied Sciences, Engineering and Technology*, vol. 5, no. 1, pp. 176–179, 2013.

[4] G. R. Liu and T. Y. Wu, "Differential quadrature solutions of eighth-order boundary-value differential equations," *Journal of Computational and Applied Mathematics*, vol. 145, no. 1, pp. 223–235, 2002.

[5] S. S. Siddiqi and G. Akram, "Solution of eight-order boundary value problems using the non-polynomial spline technique," *International Journal of Computer Mathematics*, vol. 84, no. 3, pp. 347–368, 2007.

[6] S. S. Siddiqi and E. H. Twizell, "Spline solutions of linear eighth-order boundary-value problems," *Computer Methods in Applied Mechanics and Engineering*, vol. 131, no. 3-4, pp. 309–325, 1996.

[7] M. Inc and D. J. Evans, "An efficient approach to approximate solutions of eighth-order boundary-value problems," *International Journal of Computer Mathematics*, vol. 81, no. 6, pp. 685–692, 2004.

[8] A. Golbabai and M. Javidi, "Application of homotopy perturbation method for solving eighth-order boundary value problems," *Applied Mathematics and Computation*, vol. 191, no. 2, pp. 334–346, 2007.

[9] G. Akram and H. U. Rehman, "Numerical solution of eighth order boundary value problems in reproducing Kernel space," *Numerical Algorithms*, vol. 62, no. 3, pp. 527–540, 2013.

[10] F. Geng and X. Li, "Variational iteration method for solving tenth-order boundary value problems," *Journal of Mathematical Sciences*, vol. 3, no. 2, pp. 161–172, 2009.

[11] S. S. Siddiqi, G. Akram, and S. Zaheer, "Solution of tenth order boundary value problems using variational iteration technique," *European Journal of Scientific Research*, vol. 30, no. 3, pp. 326–347, 2009.

[12] S. S. Siddiqi and G. Akram, "Solutions of tenth-order boundary value problems using eleventh degree spline," *Applied Mathematics and Computation*, vol. 185, no. 1, pp. 115–127, 2007.

[13] S. S. Siddiqi and E. H. Twizell, "Spline solutions of linear tenth-order boundary-value problems," *International Journal of Computer Mathematics*, vol. 68, no. 3-4, pp. 345–362, 1998.

[14] A. Lamnii, H. Mraoui, D. Sbibih, A. Tijini, and A. Zidna, "Spline solution of some linear boundary value problems," *Applied Mathematics E-Notes*, vol. 8, pp. 171–178, 2008.

[15] G. Domairry and N. Nadim, "Assessment of homotopy analysis method and homotopy perturbation method in non-linear heat transfer equation," *International Communications in Heat and Mass Transfer*, vol. 35, no. 1, pp. 93–102, 2008.

[16] Z. Ziabakhsh and G. Domairry, "Solution of the laminar viscous flow in a semi-porous channel in the presence of a uniform magnetic field by using the homotopy analysis method," *Communications in Nonlinear Science and Numerical Simulation*, vol. 14, no. 4, pp. 1284–1294, 2009.

[17] G. Domairry and M. Fazeli, "Application of homotopy analysis method to solve mhd jef-feryhamel flows in non-parallel walls," *Communications in Nonlinear Science and Numerical Simulation*, vol. 14, pp. 489–499, 2009.

[18] S. M. Moghimi, G. Domairry, S. Soleimani, E. Ghasemi, and H. Bararnia, "Application of homotopy analysis method to solve MHD Jeffery-Hamel flows in non-parallel walls," *Advances in Engineering Software*, vol. 42, no. 3, pp. 108–113, 2011.

[19] K. Farajeyan and N. R. Maleki, "Numerical solution of tenth-order boundary value problems in off step points," *Journal of*

Basic and Applied Scientific Research, vol. 2, no. 6, pp. 6235–6244, 2012.

[20] M. Khan and M. Hussain, "Application of Laplace decomposition method on semi-infinite domain," *Numerical Algorithms*, vol. 56, no. 2, pp. 211–218, 2011.

[21] M. Khan and M. A. Gondal, "A reliable treatment of Abel's second kind singular integral equations," *Applied Mathematics Letters*, vol. 25, no. 11, pp. 1666–1670, 2012.

[22] M. Khan, M. A. Gondal, and S. Kumar, "A new analytical solution procedure for nonlinear integral equations," *Mathematical and Computer Modelling*, vol. 55, no. 7-8, pp. 1892–1897, 2012.

[23] M. Khan, M. A. Gondal, and S. Karimi Vanani, "On the coupling of homotopy perturbation and Laplace transformation for system of partial differential equations," *Applied Mathematical Sciences*, vol. 6, no. 9–12, pp. 467–478, 2012.

[24] S. Nadeem, A. Hussain, and M. Khan, "HAM solutions for boundary layer flow in the region of the stagnation point towards a stretching sheet," *Communications in Nonlinear Science and Numerical Simulation*, vol. 15, no. 3, pp. 475–481, 2010.

[25] M. Shaban, S. Kazem, and J. A. Rad, "A modification of the homotopy analysis method based on chebyshev operational matrices," *Mathematical and Computer Modelling*, vol. 57, pp. 1227–1239, 2013.

[26] O. A. Arqub and A. El-Ajou, "Solution of the fractional epidemic model by homotopy analysis method," *Journal of King Saud University*, vol. 25, pp. 73–781, 2013.

[27] M. Russo and R. A. Van Gorder, "Control of error in the homotopy analysis of nonlinear Klein–Gordon initial value problems," *Applied Mathematics and Computation*, vol. 219, no. 12, pp. 6494–6509, 2013.

[28] S.-J. Liao, "A kind of approximate solution technique which does not depend upon small parameters. II. An application in fluid mechanics," *International Journal of Non-Linear Mechanics*, vol. 32, no. 5, pp. 815–822, 1997.

[29] S.-J. Liao, *Beyond Perturbation: Introduction to the Homotopy Analysis Method*, vol. 2, Chapman & Hall/CRC, Boca Raton, Fla, USA, 2004.

[30] S.-J. Liao, "On the homotopy analysis method for nonlinear problems," *Applied Mathematics and Computation*, vol. 147, no. 2, pp. 499–513, 2004.

[31] S. Liao and Y. Tan, "A general approach to obtain series solutions of nonlinear differential equations," *Studies in Applied Mathematics*, vol. 119, no. 4, pp. 297–354, 2007.

[32] S. S. Siddiqi, G. Akram, and M. Iftikhar, "Solution of seventh order boundary value problem by differential transformation method," *World Applied Sciences Journal*, vol. 16, no. 11, pp. 1521–1526, 2012.

[33] J.-H. He, "The variational iteration method for eighth-order initial-boundary value problems," *Physica Scripta*, vol. 76, no. 6, pp. 680–682, 2007.

[34] M. Torvattanabun and S. Koonprasert, "Variational iteration method for solving eighth-order boundary value problems," *Thai Journal of Mathematics*, vol. 8, no. 4, pp. 121–129, 2010.

[35] K. N. S. Kasi Viswanadham and Y. S. Raju, "Quintic B-spline collocation method for tenth order boundary value problems," *International Journal of Computer Applications*, vol. 51, no. 15, pp. 7–13, 2012.