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Research Article

Representation Theorem for Generators of BSDEs Driven by G-Brownian Motion and Its Applications

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We obtain a representation theorem for the generators of BSDEs driven by G-Brownian motions and then we use the representation theorem to get a converse comparison theorem for G-BSDEs and some equivalent results for nonlinear expectations generated by G-BSDEs.

1. Introduction

Let $(\Omega, \mathscr{F}, P)$ be a probability space, and, for fixed $T \in [0, +\infty)$, let $(B_t)_{0\le t\le T}$ be a standard Brownian motion and let $\mathscr{F}_t$ be the augmentation of $\sigma\{B_s, 0 \le s \le t\}$. Then Pardoux and Peng [1] introduced the backward stochastic differential equations (BSDEs) and proved the existence and uniqueness result of the BSDEs. In 1997, Peng [2] promoted g-expectations based on BSDEs. One of the important properties of g-expectations is comparison theorem or monotonicity. Chen [3] first considers a converse result of BSDEs under equal case. After that, Briand et al. [4] obtained a converse comparison theorem for BSDEs under general case. They also derived a representation theorem for the generator g. Following this paper, Jiang [5] discussed a more general representation theorem then, in his another paper [6], showed a more general converse comparison theorem. Here the representation theorem is an important method in solving the converse comparison problem and other problems (see Jiang [7]).

Peng [8–13] defined the G-expectations and G-Brownian motions (G-BMs) and proved the representation theorem of G-expectation by a set of singular probabilities, which differs from nonlinear g-expectations because g-expectations are equivalent with a group of absolutely continuous probabilities with respect to the probability measure P. Soner et al. [14] obtained an existence and uniqueness result of 2 BSDEs. Recently, Hu et al. [15] proved another existence and uniqueness result on BSDEs driven by G-Brownian motions (G-BSDEs).

An important advantage of G-BSDEs is the easiness to define the nonlinear expectations. Hu et al. in [16] gave a comparison theorem for G-BSDEs and talked about the properties of corresponding nonlinear expectations. In this paper, we consider the representation theorem for generators of G-BSDEs and then consider the converse comparison theorem of G-BSDEs and some equivalent results for nonlinear expectations generated by G-BSDEs. In the following, in Section 2, we review some basic concepts and results about G-expectations. We give the representation theorem of G-BSDEs in Section 3. In Section 4, we consider the applications of representation theorem of G-BSDEs, which contain the converse comparison theorem and some equivalent results for nonlinear expectations generated by G-BSDEs.

2. Preliminaries

We review some basic notions and results of G-expectation, the related spaces of random variables, and the backward stochastic differential equations driven by a G-Brownian motion. The readers may refer to [10, 13, 15, 17–19] for more details.

Definition 1. Let Ω be a given set and let $\mathscr{H}$ be a vector lattice of real valued functions defined on Ω, namely, $c \in \mathscr{H}$ for each constant c and $|X| \in \mathscr{H}$ if $X \in \mathscr{H}$. $\mathscr{H}$ is considered as the space of random variables. A sublinear expectation $\widehat{\mathbb{E}}$ on $\mathscr{H}$ is a functional $\widehat{\mathbb{E}} : \mathscr{H} \rightarrow \mathbb{R}$ satisfying the following properties: for all $X, Y \in \mathscr{H}$, one has

(a) monotonicity: if $X \geq Y$, then $\widehat{\mathbb{E}}[X] \geq \widehat{\mathbb{E}}[Y]$;

(b) constant preservation: $\widehat{\mathbb{E}}[c] = c$;

(c) subadditivity: $\widehat{\mathbb{E}}[X+Y] \leq \widehat{\mathbb{E}}[X] + \widehat{\mathbb{E}}[Y]$;

(d) positive homogeneity: $\widehat{\mathbb{E}}[\lambda X] = \lambda \widehat{\mathbb{E}}[X]$ for each $\lambda \geq 0$. $(\Omega, \mathscr{H}, \widehat{\mathbb{E}})$ is called a sublinear expectation space.

Definition 2. Let X_1 and X_2 be two n-dimensional random vectors defined, respectively, in sublinear expectation spaces $(\Omega_1, \mathscr{H}_1, \widehat{\mathbb{E}}_1)$ and $(\Omega_2, \mathscr{H}_2, \widehat{\mathbb{E}}_2)$. They are called identically distributed, denoted by $X_1 \overset{d}{=} X_2$, if $\widehat{\mathbb{E}}_1[\varphi(X_1)] = \widehat{\mathbb{E}}_2[\varphi(X_2)]$, for all $\varphi \in C_{b\text{-Lip}}(\mathbb{R}^n)$, where $C_{b\text{-Lip}}(\mathbb{R}^n)$ denotes the space of bounded and Lipschitz functions on $\mathbb{R}^n$.

Definition 3. In a sublinear expectation space $(\Omega, \mathscr{H}, \widehat{\mathbb{E}})$, a random vector $Y = (Y_1, \ldots, Y_n)$, $Y_i \in \mathscr{H}$, is said to be independent of another random vector $X = (X_1, \ldots, X_m)$, $X_i \in \mathscr{H}$ under $\widehat{\mathbb{E}}[\cdot]$, denoted by $Y \perp X$, if for every test function $\varphi \in C_{b\text{-Lip}}(\mathbb{R}^m \times \mathbb{R}^n)$ one has $\widehat{\mathbb{E}}[\varphi(X, Y)] = \widehat{\mathbb{E}}[\widehat{\mathbb{E}}[\varphi(x, Y)]_{x=X}]$.

Definition 4 (G-normal distribution). A d-dimensional random vector $X = (X_1, \ldots, X_d)$ in a sublinear expectation space $(\Omega, \mathscr{H}, \widehat{\mathbb{E}})$ is called G-normally distributed if for each $a, b \geq 0$ one has

$$aX + b\overline{X} \overset{d}{=} \sqrt{a^2 + b^2} X, \tag{1}$$

where $\overline{X}$ is an independent copy of X; that is, $\overline{X} \overset{d}{=} X$ and $\overline{X} \perp X$. Here, the letter G denotes the function

$$G(A) := \frac{1}{2} \widehat{\mathbb{E}}[\langle AX, X\rangle] : \mathbb{S}_d \longrightarrow \mathbb{R}, \tag{2}$$

where $\mathbb{S}_d$ denotes the collection of $d \times d$ symmetric matrices.

Peng [13] showed that $X = (X_1, \ldots, X_d)$ is G-normally distributed if and only if for each $\varphi \in C_{b\text{-Lip}}(\mathbb{R}^d)$, $u(t,x) := \widehat{\mathbb{E}}[\varphi(x + \sqrt{t}X)]$, $(t,x) \in [0,\infty) \times \mathbb{R}^d$, is the solution of the following G-heat equation:

$$\partial_t u - G(D_x^2 u) = 0, \qquad u(0,x) = \varphi(x). \tag{3}$$

The function $G(\cdot) : \mathbb{S}_d \rightarrow \mathbb{R}$ is a monotonic, sublinear mapping on $\mathbb{S}_d$ and $G(A) = (1/2)\widehat{\mathbb{E}}[\langle AX, X\rangle] \leq (1/2)|A|\widehat{\mathbb{E}}[|X|^2]$ implies that there exists a bounded, convex, and closed subset $\Gamma \subset \mathbb{S}_d^+$ such that

$$G(A) = \frac{1}{2} \sup_{\gamma \in \Gamma} \operatorname{tr}[\gamma A], \tag{4}$$

where $\mathbb{S}_d^+$ denotes the collection of nonnegative elements in $\mathbb{S}_d$.

In this paper, we only consider nondegenerate G-normal distribution; that is, there exists some $\underline{\sigma}^2 > 0$ such that $G(A) - G(B) \geq \underline{\sigma}^2 \operatorname{tr}[A - B]$ for any $A \geq B$.

Definition 5. (i) Let $\Omega = C_0^d(\mathbb{R}^+)$ denote the space of $\mathbb{R}^d$-valued continuous functions on $[0,\infty)$ with $\omega_0 = 0$ and let $B_t(\omega) = \omega_t$ be the canonical process. Set

$$L_{ip}(\Omega) := \{\varphi(B_{t_1}, \ldots, B_{t_n}) : n \geq 1, t_1, \ldots, t_n \in [0,\infty), \varphi \in C_{b\text{-Lip}}(\mathbb{R}^{d\times n})\}. \tag{5}$$

Let $G : \mathbb{S}_d \rightarrow \mathbb{R}$ be a given monotonic and sublinear function. G-expectation is a sublinear expectation defined by

$$\widehat{\mathbb{E}}[X] = \widetilde{\mathbb{E}}[\varphi(\sqrt{t_1 - t_0}\xi_1, \ldots, \sqrt{t_m - t_{m-1}}\xi_m)], \tag{6}$$

for all $X = \varphi(B_{t_1} - B_{t_0}, B_{t_2} - B_{t_1}, \ldots, B_{t_m} - B_{t_{m-1}})$, where $\xi_1, \ldots, \xi_n$ are identically distributed d-dimensional G-normally distributed random vectors in a sublinear expectation space $(\widetilde{\Omega}, \widetilde{\mathscr{H}}, \widetilde{\mathbb{E}})$ such that ξ_{i+1} is independent of $(\xi_1, \ldots, \xi_i)$ for every $i = 1, \ldots, m-1$. The corresponding canonical process $B_t = (B_t^i)_{i=1}^d$ is called a G-Brownian motion.

(ii) For each fixed $t \in [0,\infty)$, the conditional G-expectation $\widehat{\mathbb{E}}_t$ for $\xi = \varphi(B_{t_1} - B_{t_0}, B_{t_2} - B_{t_1}, \ldots, B_{t_m} - B_{t_{m-1}}) \in L_{ip}(\Omega)$, where without loss of generality we suppose $t_i = t$, is defined by

$$\widehat{\mathbb{E}}_t[\varphi(B_{t_1} - B_{t_0}, B_{t_2} - B_{t_1}, \ldots, B_{t_m} - B_{t_{m-1}})] = \psi(B_{t_1} - B_{t_0}, B_{t_2} - B_{t_1}, \ldots, B_{t_i} - B_{t_{i-1}}), \tag{7}$$

where

$$\psi(x_1, \ldots, x_i) = \widehat{\mathbb{E}}[\varphi(x_1, \ldots, x_i, B_{t_{i+1}} - B_{t_i}, \ldots, B_{t_m} - B_{t_{m-1}})]. \tag{8}$$

For each fixed $T > 0$, we set

$$L_{ip}(\Omega_T) := \{\varphi(B_{t_1}, \ldots, B_{t_n}) : n \geq 1, t_1, \ldots, t_n \in [0,T], \varphi \in C_{b\text{-Lip}}(\mathbb{R}^{d\times n})\}. \tag{9}$$

For each $p \geq 1$, we denote by $L_G^p(\Omega)$ (resp., $L_G^p(\Omega_T)$) the completion of $L_{ip}(\Omega)$ (resp., $L_{ip}(\Omega_T)$) under the norm $\|\xi\|_{p,G} = (\widehat{\mathbb{E}}[|\xi|^p])^{1/p}$. It is easy to check that $L_G^q(\Omega) \subset L_G^p(\Omega)$ for $1 \leq p \leq q$ and $\widehat{\mathbb{E}}_t[\cdot]$ can be extended continuously to $L_G^1(\Omega)$.

For each fixed $\mathbf{a} \in \mathbb{R}^d$, $B_t^{\mathbf{a}} = \langle \mathbf{a}, B_t\rangle$ is a 1-dimensional $G_{\mathbf{a}}$-Brownian motion, where $G_{\mathbf{a}}(\alpha) = (1/2)(\sigma_{\mathbf{a}\mathbf{a}^T}^2 \alpha^+ - \sigma_{-\mathbf{a}\mathbf{a}^T}^2 \alpha^-)$, $\sigma_{\mathbf{a}\mathbf{a}^T}^2 = 2G(\mathbf{a}\mathbf{a}^T)$ and $\sigma_{-\mathbf{a}\mathbf{a}^T}^2 = -2G(-\mathbf{a}\mathbf{a}^T)$. Let $\pi_t^N = \{t_0^N, \ldots, t_N^N\}$, $N = 1, 2, \ldots$, be a sequence of partitions of $[0,t]$ such that $\mu(\pi_t^N) = \max\{|t_{i+1}^N - t_i^N| : i = 0, \ldots, N-1\} \rightarrow 0$; the quadratic variation process of $B^{\mathbf{a}}$ is defined by

$$\langle B^{\mathbf{a}}\rangle_t = \lim_{\mu(\pi_t^N) \rightarrow 0} \sum_{j=0}^{N-1} \left(B_{t_{j+1}^N}^{\mathbf{a}} - B_{t_j^N}^{\mathbf{a}}\right)^2. \tag{10}$$

For each fixed $\mathbf{a}, \bar{\mathbf{a}} \in \mathbb{R}^d$, the mutual variation process of $B^{\mathbf{a}}$ and $B^{\bar{\mathbf{a}}}$ is defined by

$$\left\langle B^{\mathbf{a}}, B^{\bar{\mathbf{a}}}\right\rangle_t = \frac{1}{4}\left[\left\langle B^{\mathbf{a}+\bar{\mathbf{a}}}\right\rangle_t - \left\langle B^{\mathbf{a}-\bar{\mathbf{a}}}\right\rangle_t\right]. \quad (11)$$

Definition 6. For fixed $T > 0$, let $M_G^0(0,T)$ be the collection of processes in the following form: for a given partition $\{t_0, \ldots, t_N\} = \pi_T$ of $[0,T]$,

$$\eta_t(\omega) = \sum_{j=0}^{N-1} \xi_j I_{[t_j, t_{j+1})}(t), \quad (12)$$

where $\xi_j \in L_{ip}(\Omega_{t_j})$, $j = 0, 1, 2, \ldots, N-1$. For $p \geq 1$, one denotes by $H_G^p(0,T)$, $M_G^p(0,T)$ the completion of $M_G^0(0,T)$ under the norms $\|\eta\|_{H_G^p} = \{\hat{\mathbb{E}}[(\int_0^T |\eta_s|^2 ds)^{p/2}]\}^{1/p}$, $\|\eta\|_{M_G^p} = \{\hat{\mathbb{E}}[\int_0^T |\eta_s|^p ds]\}^{1/p}$, respectively.

For each $\eta \in M_G^1(0,T)$, we can define the integrals $\int_0^T \eta_t dt$ and $\int_0^T \eta_t d\langle B^{\mathbf{a}}, B^{\bar{\mathbf{a}}}\rangle_t$ for each $\mathbf{a}, \bar{\mathbf{a}} \in \mathbb{R}^d$. For each $\eta \in H_G^p(0,T;\mathbb{R}^d)$ with $p \geq 1$, we can define Itô's integral $\int_0^T \eta_t dB_t$.

Let $S_G^0(0,T) = \{h(t, B_{t_1 \wedge t}, \ldots, B_{t_n \wedge t}) : t_1, \ldots, t_n \in [0,T], h \in C_{b,\mathrm{Lip}}(\mathbb{R}^{n+1})\}$. For $p \geq 1$ and $\eta \in S_G^0(0,T)$, set $\|\eta\|_{S_G^p} = \{\hat{\mathbb{E}}[\sup_{t\in[0,T]} |\eta_t|^p]\}^{1/p}$. Denote by $S_G^p(0,T)$ the completion of $S_G^0(0,T)$ under the norm $\|\cdot\|_{S_G^p}$.

We consider the following type of G-BSDEs (in this paper, we always use Einstein convention):

$$\begin{aligned} Y_t = \xi &+ \int_t^T f(s, Y_s, Z_s)\, ds + \int_t^T g_{ij}(s, Y_s, Z_s)\, d\left\langle B^i, B^j\right\rangle_s \\ &- \int_t^T Z_s dB_s - (K_T - K_t), \end{aligned} \quad (13)$$

where

$$f(t,\omega,y,z), g_{ij}(t,\omega,y,z) : [0,T] \times \Omega_T \times \mathbb{R} \times \mathbb{R}^d \longrightarrow \mathbb{R}, \quad (14)$$

satisfy the following properties.

(H1) There exists some $\beta > 1$ such that for any y, z, $f(\cdot,\cdot,y,z), g_{ij}(\cdot,\cdot,y,z) \in M_G^\beta(0,T)$.

(H2) There exists some $L > 0$ such that

$$\begin{aligned} &\left|f(t,y,z) - f(t,y',z')\right| + \sum_{i,j=1}^d \left|g_{ij}(t,y,z) - g_{ij}(t,y',z')\right| \\ &\quad \leq L\left(\left|y-y'\right| + \left|z-z'\right|\right). \end{aligned} \quad (15)$$

For simplicity, we denote by $\mathfrak{S}_G^\alpha(0,T)$ the collection of processes (Y,Z,K) such that $Y \in S_G^\alpha(0,T)$, $Z \in H_G^\alpha(0,T;\mathbb{R}^d)$, K is a decreasing G-martingale with $K_0 = 0$ and $K_T \in L_G^\alpha(\Omega_T)$.

Definition 7. Let $\xi \in L_G^\beta(\Omega_T)$ and f and g_{ij} satisfy (H1) and (H2) for some $\beta > 1$. A triplet of processes (Y,Z,K) is called a solution of (13) if for some $1 < \alpha \leq \beta$ the following properties hold:

(a) $(Y,Z,K) \in \mathfrak{S}_G^\alpha(0,T)$;

(b) $Y_t = \xi + \int_t^T f(s,Y_s,Z_s)ds + \int_t^T g_{ij}(s,Y_s,Z_s)d\langle B^i, B^j\rangle_s - \int_t^T Z_s dB_s - (K_T - K_t)$.

Theorem 8 (see [15]). *Assume that $\xi \in L_G^\beta(\Omega_T)$ and f and g_{ij} satisfy (H1) and (H2) for some $\beta > 1$. Then, (13) has a unique solution (Y,Z,K). Moreover, for any $1 < \alpha < \beta$, one has $Y \in S_G^\alpha(0,T)$, $Z \in H_G^\alpha(0,T;\mathbb{R}^d)$, and $K_T \in L_G^\alpha(\Omega_T)$.*

We have the following estimates.

Proposition 9 (see [15]). *Let $\xi \in L_G^\beta(\Omega_T)$ and f, g_{ij} satisfy (H1) and (H2) for some $\beta > 1$. Assume that $(Y,Z,K) \in \mathfrak{S}_G^\alpha(0,T)$ for some $1 < \alpha < \beta$ is a solution of (13). Then, there exists a constant $C_\alpha > 0$ depending on α, T, G, L such that*

$$\begin{aligned} &\left|Y_t\right|^\alpha \leq C_\alpha \hat{\mathbb{E}}_t\left[|\xi|^\alpha + \left(\int_t^T \left|h_s^0\right| ds\right)^\alpha\right], \\ &\hat{\mathbb{E}}\left[\left(\int_0^T \left|Z_s\right|^2 ds\right)^{\alpha/2}\right] \\ &\quad \leq C_\alpha \left\{\hat{\mathbb{E}}\left[\sup_{t\in[0,T]} \left|Y_t\right|^\alpha\right] + \left(\hat{\mathbb{E}}\left[\sup_{t\in[0,T]} \left|Y_t\right|^\alpha\right]\right)^{1/2} \right.\\ &\qquad \left. \times \left(\hat{\mathbb{E}}\left[\left(\int_0^T h_s^0 ds\right)^\alpha\right]\right)^{1/2}\right\}, \\ &\hat{\mathbb{E}}\left[\left|K_t\right|^\alpha\right] \leq C_\alpha \left\{\hat{\mathbb{E}}\left[\sup_{t\in[0,T]} \left|Y_t\right|^\alpha\right] + \hat{\mathbb{E}}\left[\left(\int_0^T h_s^0 ds\right)^\alpha\right]\right\}, \end{aligned} \quad (16)$$

where $h_s^0 = |f(s,0,0)| + \sum_{i,j=1}^d |g_{ij}(s,0,0)|$.

Proposition 10 (see [15, 20]). *Let $\alpha \geq 1$ and $\delta > 0$ be fixed. Then, there exists a constant C depending on α and δ such that*

$$\begin{aligned} &\hat{\mathbb{E}}\left[\sup_{t\in[0,T]} \hat{\mathbb{E}}_t\left[|\xi|^\alpha\right]\right] \\ &\quad \leq C\left\{\left(\hat{\mathbb{E}}\left[|\xi|^{\alpha+\delta}\right]\right)^{\alpha/(\alpha+\delta)} + \hat{\mathbb{E}}\left[|\xi|^{\alpha+\delta}\right]\right\}, \\ &\qquad \forall \xi \in L_G^{\alpha+\delta}(\Omega_T). \end{aligned} \quad (17)$$

Theorem 11 (see [16]). *Let (Y^l, Z^l, K^l), $l = 1, 2$, be the solutions of the following G-BSDEs:*

$$\begin{aligned} Y_t^l = \xi + \int_t^T f\left(s, Y_s^l, Z_s^l\right) ds + \int_t^T g_{ij}\left(s, Y_s^l, Z_s^l\right) d\left\langle B^i, B^j\right\rangle_s \\ + V_T^l - V_t^l - \int_t^T Z_s^l dB_s - \left(K_T^l - K_t^l\right), \end{aligned} \tag{18}$$

where $\xi \in L_G^\beta(\Omega_T)$, f and g_{ij} satisfy (H1) and (H2) for some $\beta > 1$ and $(V_t^l)_{t\le T}$ are RCLL processes in $M_G^\beta(0,T)$ such that $\hat{\mathbb{E}}[\sup_{t\in[0,T]}|V_t^l|^\beta] < \infty$. If $V_t^1 - V_t^2$ is an increasing process, then $Y_t^1 \ge Y_t^2$ for $t \in [0,T]$.

In this paper, we also need the following assumptions for G-BSDE (13).

(H3) For each fixed $(\omega, y, z) \in \Omega_T \times \mathbb{R} \times \mathbb{R}^d$, $t \to f(t,\omega,y,z)$ and $t \to g_{ij}(t,\omega,y,z)$ are continuous.

(H4) For each fixed $(t, y, z) \in [0,T) \times \mathbb{R} \times \mathbb{R}^d$, $f(t,y,z)$, $g_{ij}(t,y,z) \in L_G^\beta(\Omega_t)$, and

$$\lim_{\varepsilon\to 0+}\frac{1}{\varepsilon}\hat{\mathbb{E}}\left[\int_t^{t+\varepsilon}\left(\left|f(u,y,z)-f(t,y,z)\right|^\beta + \sum_{i,j=1}^d\left|g_{ij}(u,y,z)-g_{ij}(t,y,z)\right|^\beta\right)du\right] = 0. \tag{19}$$

(H5) For each $(t,\omega,y) \in [0,T] \times \Omega_T \times \mathbb{R}$, $f(t,\omega,y,0) = g_{ij}(t,\omega,y,0) = 0$.

Assume that $\xi \in L_G^\beta(\Omega_T)$; f and g_{ij} satisfy (H1), (H2), and (H5) for some $\beta > 1$. Let $(Y^{T,\xi}, Z^{T,\xi}, K^{T,\xi})$ be the solution of G-BSDE (13) corresponding to ξ, f, and g_{ij} on $[0,T]$. It is easy to check that $Y^{T,\xi} = Y^{T',\xi}$ on $[0,T]$ for $T' > T$. Following [16], we can define consistent nonlinear expectation

$$\tilde{\mathbb{E}}_t[\xi] = Y_t^{T,\xi} \quad \text{for } t \in [0,T] \tag{20}$$

and set $\tilde{\mathbb{E}}[\xi] = \tilde{\mathbb{E}}_0[\xi] = Y_0^{T,\xi}$.

3. Representation Theorem of Generators of G-BSDEs

We consider the following type of G-FBSDEs:

$$\begin{aligned} X_s^{t,x} &= x + \int_t^s b\left(X_u^{t,x}\right) du \\ &\quad + \int_t^s h_{ij}\left(X_u^{t,x}\right) d\left\langle B^i, B^j\right\rangle_u + \int_t^s \sigma\left(X_u^{t,x}\right) dB_u, \end{aligned} \tag{21}$$

$$\begin{aligned} {}^{\varepsilon}Y_s^{t,x,y,p} &= y + \left\langle p, X_{t+\varepsilon}^{t,x} - x\right\rangle \\ &\quad + \int_s^{t+\varepsilon} f\left(u, {}^{\varepsilon}Y_u^{t,x,y,p}, {}^{\varepsilon}Z_u^{t,x,y,p}\right) du \\ &\quad + \int_s^{t+\varepsilon} g_{ij}\left(u, {}^{\varepsilon}Y_u^{t,x,y,p}, {}^{\varepsilon}Z_u^{t,x,y,p}\right) d\left\langle B^i, B^j\right\rangle_u \\ &\quad - \int_s^{t+\varepsilon} {}^{\varepsilon}Z_u^{t,x,y,p} dB_u - \left({}^{\varepsilon}K_{t+\varepsilon}^{t,x,y,p} - {}^{\varepsilon}K_s^{t,x,y,p}\right), \end{aligned} \tag{22}$$

where $h_{ij} = h_{ji}$ and $g_{ij} = g_{ji}$, $1 \le i, j \le d$.

We now give the main result in this section.

Theorem 12. *Let $b : \mathbb{R}^n \to \mathbb{R}^n$, $h_{ij} : \mathbb{R}^n \to \mathbb{R}^n$, and $\sigma : \mathbb{R}^n \to \mathbb{R}^{n\times d}$ be Lipschitz functions and let f and g_{ij} satisfy (H1), (H2), (H3), and (H4) for some $\beta > 1$. Then, for each $(t,x,y,p) \in [0,T) \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n$ and $\alpha \in (1,\beta)$, one has*

$$\begin{aligned} L_G^\alpha - \lim_{\varepsilon\to 0+}\frac{1}{\varepsilon}\left\{{}^{\varepsilon}Y_t^{t,x,y,p} - y\right\} &= f\left(t, y, \sigma^T(x)p\right) + \left\langle p, b(x)\right\rangle \\ &\quad + 2G\left(\left(g_{ij}\left(t, y, \sigma^T(x)p\right) + \left\langle p, h_{ij}(x)\right\rangle\right)_{i,j=1}^d\right). \end{aligned} \tag{23}$$

Proof. For each fixed $(t,x,y,p) \in [0,T) \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n$, we write $(Y^\varepsilon, Z^\varepsilon, K^\varepsilon)$ instead of $({}^{\varepsilon}Y^{t,x,y,p}, {}^{\varepsilon}Z^{t,x,y,p}, {}^{\varepsilon}K^{t,x,y,p})$ for simplicity. We have $\hat{\mathbb{E}}[|X_{t+\varepsilon}^{t,x}|^\gamma] < \infty$ for each $\gamma \ge 1$ (see [16, 19]). Thus, by Theorem 8, G-BSDE (22) has a unique solution $(Y^\varepsilon, Z^\varepsilon, K^\varepsilon)$ and $Y_t^\varepsilon \in L_G^\alpha(\Omega_t)$. We set, for $s \in [t, t+\varepsilon]$,

$$\begin{aligned} \tilde{Y}_s^\varepsilon &= Y_s^\varepsilon - \left(y + \left\langle p, X_s^{t,x} - x\right\rangle\right), \\ \tilde{Z}_s^\varepsilon &= Z_s^\varepsilon - \sigma^T\left(X_s^{t,x}\right)p, \qquad \tilde{K}_s^\varepsilon = K_s^\varepsilon. \end{aligned} \tag{24}$$

Applying Itô's formula to $\tilde{Y}_s^\varepsilon$ on $[t, t+\varepsilon]$, it is easy to verify that $(\tilde{Y}^\varepsilon, \tilde{Z}^\varepsilon, \tilde{K}^\varepsilon)$ solves the following G-BSDE:

$$\begin{aligned} \tilde{Y}_s^\varepsilon &= \int_s^{t+\varepsilon} f\left(u, \tilde{Y}_u^\varepsilon + y + \left\langle p, X_u^{t,x} - x\right\rangle, \tilde{Z}_u^\varepsilon + \sigma^T\left(X_u^{t,x}\right)p\right) du \\ &\quad + \int_s^{t+\varepsilon} \left\langle p, b\left(X_u^{t,x}\right)\right\rangle du \\ &\quad + \int_s^{t+\varepsilon} g_{ij}\left(u, \tilde{Y}_u^\varepsilon + y + \left\langle p, X_u^{t,x} - x\right\rangle, \tilde{Z}_u^\varepsilon + \sigma^T\left(X_u^{t,x}\right)p\right) d\left\langle B^i, B^j\right\rangle_u \\ &\quad + \int_s^{t+\varepsilon} \left\langle p, h_{ij}\left(X_u^{t,x}\right)\right\rangle d\left\langle B^i, B^j\right\rangle_u \\ &\quad - \int_s^{t+\varepsilon} \tilde{Z}_u^\varepsilon dB_u - \left(\tilde{K}_{t+\varepsilon}^\varepsilon - \tilde{K}_s^\varepsilon\right). \end{aligned} \tag{25}$$

From Proposition 9,

$$\begin{aligned}\left|\widetilde{Y}_s^\varepsilon\right|^\alpha \le C_\alpha \widehat{\mathbb{E}}_s\Bigg[\Bigg(\int_s^{t+\varepsilon}\Big(&\left|f\left(u, y+\left\langle p, X_u^{t,x}-x\right\rangle, \sigma^T\left(X_u^{t,x}\right)p\right)\right| \\ &+\left|\langle p, b(X_u^{t,x})\rangle\right| \\ &+\sum_{i,j=1}^d\left|g_{ij}\left(u, y+\left\langle p, X_u^{t,x}-x\right\rangle, \sigma^T\left(X_u^{t,x}\right)p\right)\right| \\ &+\left|\left\langle p, h_{ij}\left(X_u^{t,x}\right)\right\rangle\right|\Big)du\Bigg)^\alpha\Bigg],\end{aligned}$$

$$\begin{aligned}&\widehat{\mathbb{E}}\left[\left(\int_t^{t+\varepsilon}\left|\widetilde{Z}_u^\varepsilon\right|^2du\right)^{\alpha/2}\right] \\ &\le C_\alpha\Bigg\{\widehat{\mathbb{E}}\Bigg[\Bigg(\int_t^{t+\varepsilon}\Big(\left|f\left(u, y+\left\langle p, X_u^{t,x}-x\right\rangle, \sigma^T\left(X_u^{t,x}\right)p\right)\right| \\ &\qquad+\left|\langle p, b(X_u^{t,x})\rangle\right|+\left|\left\langle p, h_{ij}\left(X_u^{t,x}\right)\right\rangle\right| \\ &\qquad+\sum_{i,j=1}^d\left|g_{ij}\left(u, y+\left\langle p, X_u^{t,x}-x\right\rangle,\right.\right. \\ &\qquad\qquad\left.\left.\sigma^T\left(X_u^{t,x}\right)p\right)\right|\Big)du\Bigg)^\alpha\Bigg] \\ &\qquad+\widehat{\mathbb{E}}\left[\sup_{s\in[t,t+\varepsilon]}\left|\widetilde{Y}_s^\varepsilon\right|^\alpha\right]\Bigg\},\end{aligned} \tag{26}$$

hold for some constant $C_\alpha > 0$, only depending on α, T, G, and L. By Proposition 10 and the Lipschitz assumption, we obtain

$$\begin{aligned}&\widehat{\mathbb{E}}\left[\sup_{s\in[t,t+\varepsilon]}\left|\widetilde{Y}_s^\varepsilon\right|^\alpha+\left(\int_t^{t+\varepsilon}\left|\widetilde{Z}_u^\varepsilon\right|^2du\right)^{\alpha/2}\right] \\ &\le C_1\varepsilon^\alpha\widehat{\mathbb{E}}\Bigg[1+\Bigg(\frac{1}{\varepsilon}\int_t^{t+\varepsilon}\Bigg(\left|f(u,0,0)\right|^\beta \\ &\qquad+\sum_{i,j=1}^d\left|g_{ij}(u,0,0)\right|^\beta\Bigg)du\Bigg)^{\alpha/\beta} \\ &\qquad+\sup_{s\in[t,t+\varepsilon]}\left|X_s^{t,x}\right|^\beta\Bigg],\end{aligned} \tag{27}$$

where C_1 is a constant depending on x, y, p, α, β, T, G, and L. Noting that $\widehat{\mathbb{E}}[\sup_{s\in[t,t+\varepsilon]}|X_s^{t,x}|^\beta] \le C_2(1+|x|^\beta)$ (see [16, 19]), where C_2 depends on T and L, and the following inequality holds:

$$\begin{aligned}&\int_t^{t+\varepsilon}\left(\left|f(u,0,0)\right|^\beta+\sum_{i,j=1}^d\left|g_{ij}(u,0,0)\right|^\beta\right)du \\ &\le 2^{\beta-1}\Bigg\{\varepsilon\left(\left|f(t,0,0)\right|^\beta+\sum_{i,j=1}^d\left|g_{ij}(t,0,0)\right|^\beta\right) \\ &\qquad+\int_t^{t+\varepsilon}\Bigg(\left|f(u,0,0)-f(t,0,0)\right|^\beta \\ &\qquad+\sum_{i,j=1}^d\left|g_{ij}(u,0,0)-g_{ij}(t,0,0)\right|^\beta\Bigg)du\Bigg\}.\end{aligned} \tag{28}$$

Together with assumption (H4), we get

$$\widehat{\mathbb{E}}\left[\sup_{s\in[t,t+\varepsilon]}\left|\widetilde{Y}_s^\varepsilon\right|^\alpha+\left(\int_t^{t+\varepsilon}\left|\widetilde{Z}_u^\varepsilon\right|^2du\right)^{\alpha/2}\right] \le C_3\varepsilon^\alpha, \tag{29}$$

where C_3 depends on x, y, p, α, β, T, G, and L. Now, we prove (23). Let us consider

$$\begin{aligned}\frac{1}{\varepsilon}\{Y_t^\varepsilon-y\} &= \frac{1}{\varepsilon}\widetilde{Y}_t^\varepsilon=\frac{1}{\varepsilon}\widehat{\mathbb{E}}_t\left[\widetilde{Y}_t^\varepsilon+\widetilde{K}_{t+\varepsilon}^\varepsilon-\widetilde{K}_t^\varepsilon\right] \\ &= \frac{1}{\varepsilon}\widehat{\mathbb{E}}_t\Bigg[\int_t^{t+\varepsilon}f\left(u, y+\left\langle p, X_u^{t,x}-x\right\rangle, \sigma^T\left(X_u^{t,x}\right)p\right)du \\ &\qquad+\int_t^{t+\varepsilon}\left\langle p, b\left(X_u^{t,x}\right)\right\rangle du \\ &\qquad+\int_t^{t+\varepsilon}g_{ij}\left(u, y+\left\langle p, X_u^{t,x}-x\right\rangle, \sigma^T\left(X_u^{t,x}\right)p\right) \\ &\qquad\times d\left\langle B^i, B^j\right\rangle_u \\ &\qquad+\int_t^{t+\varepsilon}\left\langle p, h_{ij}\left(X_u^{t,x}\right)\right\rangle d\left\langle B^i, B^j\right\rangle_u\Bigg]+L_\varepsilon,\end{aligned} \tag{30}$$

where

$$\begin{aligned}L_\varepsilon=\frac{1}{\varepsilon}\Bigg\{&\widehat{\mathbb{E}}_t\Bigg[\int_t^{t+\varepsilon}f\Big(u, \widetilde{Y}_u^\varepsilon+y+\left\langle p, X_u^{t,x}-x\right\rangle, \\ &\qquad\widetilde{Z}_u^\varepsilon+\sigma^T\left(X_u^{t,x}\right)p\Big)du \\ &\quad+\int_t^{t+\varepsilon}\left\langle p, b\left(X_u^{t,x}\right)\right\rangle du \\ &\quad+\int_t^{t+\varepsilon}g_{ij}\Big(u, \widetilde{Y}_u^\varepsilon+y+\left\langle p, X_u^{t,x}-x\right\rangle, \\ &\qquad\widetilde{Z}_u^\varepsilon+\sigma^T\left(X_u^{t,x}\right)p\Big)d\left\langle B^i, B^j\right\rangle_u \\ &\quad+\int_t^{t+\varepsilon}\left\langle p, h_{ij}\left(X_u^{t,x}\right)\right\rangle d\left\langle B^i, B^j\right\rangle_u\Bigg] \\ &-\widehat{\mathbb{E}}_t\Bigg[\int_t^{t+\varepsilon}f\left(u, y+\left\langle p, X_u^{t,x}-x\right\rangle, \sigma^T\left(X_u^{t,x}\right)p\right)du \\ &\quad+\int_t^{t+\varepsilon}\left\langle p, b\left(X_u^{t,x}\right)\right\rangle du \\ &\quad+\int_t^{t+\varepsilon}g_{ij}\left(u, y+\left\langle p, X_u^{t,x}-x\right\rangle, \sigma^T\left(X_u^{t,x}\right)p\right) \\ &\qquad\times d\left\langle B^i, B^j\right\rangle_u \\ &\quad+\int_t^{t+\varepsilon}\left\langle p, h_{ij}\left(X_u^{t,x}\right)\right\rangle d\langle B^i, B^j\rangle_u\Bigg]\Bigg\}.\end{aligned} \tag{31}$$

It is easy to check that $|L_\varepsilon| \le (C_4/\varepsilon)\widehat{\mathbb{E}}_t[\int_t^{t+\varepsilon}(|\widetilde{Y}_u^\varepsilon| + |\widetilde{Z}_u^\varepsilon|)du]$, where C_4 depends on G, L, and T. Thus, by (29), we get

$$\begin{aligned}
&\widehat{\mathbb{E}}\left[|L_\varepsilon|^\alpha\right] \\
&\le \frac{C_4^\alpha}{\varepsilon^\alpha}\widehat{\mathbb{E}}\left[\left(\int_t^{t+\varepsilon}\left(\left|\widetilde{Y}_u^\varepsilon\right| + \left|\widetilde{Z}_u^\varepsilon\right|\right)du\right)^\alpha\right] \\
&\le \frac{2^{\alpha-1}C_4^\alpha}{\varepsilon^\alpha}\widehat{\mathbb{E}}\left[\left(\int_t^{t+\varepsilon}\left|\widetilde{Y}_u^\varepsilon\right|du\right)^\alpha + \left(\int_t^{t+\varepsilon}\left|\widetilde{Z}_u^\varepsilon\right|du\right)^\alpha\right] \\
&\le 2^{\alpha-1}C_4^\alpha\left\{\widehat{\mathbb{E}}\left[\sup_{s\in[t,t+\varepsilon]}\left|\widetilde{Y}_s^\varepsilon\right|^\alpha\right] + \varepsilon^{-\alpha/2}\widehat{\mathbb{E}}\left[\left(\int_t^{t+\varepsilon}\left|\widetilde{Z}_u^\varepsilon\right|^2du\right)^{\alpha/2}\right]\right\} \\
&\le 2^{\alpha-1}C_4^\alpha C_3\left(\varepsilon^\alpha + \varepsilon^{\alpha/2}\right),
\end{aligned} \tag{32}$$

which implies $L_G^\alpha - \lim_{\varepsilon\to 0+}L_\varepsilon = 0$. We set

$$\begin{aligned}
M_\varepsilon = \frac{1}{\varepsilon}\Bigg\{&\widehat{\mathbb{E}}_t\Bigg[\int_t^{t+\varepsilon} f\left(u, y + \left\langle p, X_u^{t,x} - x\right\rangle, \sigma^T\left(X_u^{t,x}\right)p\right)du \\
&+ \int_t^{t+\varepsilon}\left\langle p, b\left(X_u^{t,x}\right)\right\rangle du \\
&+ \int_t^{t+\varepsilon} g_{ij}\left(u, y + \left\langle p, X_u^{t,x} - x\right\rangle, \sigma^T\left(X_u^{t,x}\right)p\right) \\
&\times d\left\langle B^i, B^j\right\rangle_u \\
&+ \int_t^{t+\varepsilon}\left\langle p, h_{ij}\left(X_u^{t,x}\right)\right\rangle d\left\langle B^i, B^j\right\rangle_u\Bigg] \\
&- \widehat{\mathbb{E}}_t\Bigg[\int_t^{t+\varepsilon} f\left(u, y, \sigma^T(x)p\right)du + \langle p, b(x)\rangle\varepsilon \\
&+ \int_t^{t+\varepsilon} g_{ij}\left(u, y, \sigma^T(x)p\right)d\left\langle B^i, B^j\right\rangle_u \\
&+ \int_t^{t+\varepsilon}\left\langle p, h_{ij}(x)\right\rangle d\left\langle B^i, B^j\right\rangle_u\Bigg]\Bigg\}.
\end{aligned} \tag{33}$$

By the Lipschitz condition, we can get $|M_\varepsilon| \le (C_5/\varepsilon)\widehat{\mathbb{E}}_t[\int_t^{t+\varepsilon}|X_u^{t,x} - x|du]$, where C_5 depends on p, G, L, and T. Noting that $\widehat{\mathbb{E}}[\sup_{s\in[t,t+\varepsilon]}|X_s^{t,x} - x|^\alpha] \le C_6(1+|x|^\alpha)\varepsilon^{\alpha/2}$ (see [16, 19]), where C_6 depends on L, G, and α, we obtain

$$\begin{aligned}
\widehat{\mathbb{E}}\left[|M_\varepsilon|^\alpha\right] &\le C_5^\alpha\widehat{\mathbb{E}}\left[\sup_{s\in[t,t+\varepsilon]}\left|X_s^{t,x} - x\right|^\alpha\right] \\
&\le C_5^\alpha C_6\left(1 + |x|^\alpha\right)\varepsilon^{\alpha/2},
\end{aligned} \tag{34}$$

which implies $L_G^\alpha - \lim_{\varepsilon\to 0+}M_\varepsilon = 0$. Now, we set

$$\begin{aligned}
N_\varepsilon = \frac{1}{\varepsilon}\Bigg\{&\widehat{\mathbb{E}}_t\Bigg[\int_t^{t+\varepsilon} f\left(u, y, \sigma^T(x)p\right)du + \langle p, b(x)\rangle\varepsilon \\
&+ \int_t^{t+\varepsilon} g_{ij}\left(u, y, \sigma^T(x)p\right)d\left\langle B^i, B^j\right\rangle_u \\
&+ \int_t^{t+\varepsilon}\left\langle p, h_{ij}(x)\right\rangle d\left\langle B^i, B^j\right\rangle_u\Bigg] \\
&- \widehat{\mathbb{E}}_t\Bigg[\int_t^{t+\varepsilon} f\left(t, y, \sigma^T(x)p\right)du + \langle p, b(x)\rangle\varepsilon \\
&+ \int_t^{t+\varepsilon} g_{ij}\left(t, y, \sigma^T(x)p\right)d\left\langle B^i, B^j\right\rangle_u \\
&+ \int_t^{t+\varepsilon}\left\langle p, h_{ij}(x)\right\rangle d\left\langle B^i, B^j\right\rangle_u\Bigg]\Bigg\}.
\end{aligned} \tag{35}$$

It is easy to deduce that $|N_\varepsilon| \le (C_7/\varepsilon)\widehat{\mathbb{E}}_t[\int_t^{t+\varepsilon}(|f(u, y, \sigma^T(x)p) - f(t, y, \sigma^T(x)p)| + \sum_{i,j=1}^d |g_{ij}(u, y, \sigma^T(x)p) - g_{ij}(t, y, \sigma^T(x)p)|)du]$, where C_7 depends on G. Then,

$$\begin{aligned}
&\widehat{\mathbb{E}}\left[|N_\varepsilon|^\alpha\right] \\
&\le C_7^\alpha\frac{1}{\varepsilon}\widehat{\mathbb{E}}\Bigg[\int_t^{t+\varepsilon}\Bigg(\left|f\left(u, y, \sigma^T(x)p\right) - f\left(t, y, \sigma^T(x)p\right)\right| \\
&\qquad + \sum_{i,j=1}^d\Big|g_{ij}\left(u, y, \sigma^T(x)p\right) \\
&\qquad - g_{ij}\left(t, y, \sigma^T(x)p\right)\Big|\Bigg)^\alpha du\Bigg] \\
&\le C_7^\alpha\Bigg(\frac{1}{\varepsilon}\widehat{\mathbb{E}}\Bigg[\int_t^{t+\varepsilon}\Bigg(\Big|f\left(u, y, \sigma^T(x)p\right) \\
&\qquad - f\left(t, y, \sigma^T(x)p\right)\Big| \\
&\qquad + \sum_{i,j=1}^d\Big|g_{ij}\left(u, y, \sigma^T(x)p\right) \\
&\qquad - g_{ij}\left(t, y, \sigma^T(x)p\right)\Big|\Bigg)^\beta du\Bigg]\Bigg)^{\alpha/\beta}.
\end{aligned} \tag{36}$$

Take limit on both sides of the above inequality and use assumption (H4); then, we have

$$L_G^\alpha - \lim_{\varepsilon\to 0+}N_\varepsilon = 0. \tag{37}$$

On the other hand,

$$
\begin{aligned}
&\widehat{\mathbb{E}}_t\left[\int_t^{t+\varepsilon} f\left(t, y, \sigma^T(x) p\right) du+\langle p, b(x)\rangle \varepsilon\right. \\
&\quad+\int_t^{t+\varepsilon} g_{ij}\left(t, y, \sigma^T(x) p\right) d\left\langle B^i, B^j\right\rangle_u \\
&\quad\left.+\int_t^{t+\varepsilon}\left\langle p, h_{ij}(x)\right\rangle d\left\langle B^i, B^j\right\rangle_u\right] \\
&=f\left(t, y, \sigma^T(x) p\right) \varepsilon+\langle p, b(x)\rangle \varepsilon \\
&\quad+\widehat{\mathbb{E}}_t\left[\left(g_{ij}\left(t, y, \sigma^T(x) p\right)+\left\langle p, h_{ij}(x)\right\rangle\right)\right. \\
&\qquad\left.\times\left(\left\langle B^i, B^j\right\rangle_{t+\varepsilon}-\left\langle B^i, B^j\right\rangle_t\right)\right] \\
&=\left(f\left(t, y, \sigma^T(x) p\right)+\langle p, b(x)\rangle\right. \\
&\quad\left.+2G\left(\left(g_{ij}\left(t, y, \sigma^T(x) p\right)+\left\langle p, h_{ij}(x)\right\rangle\right)_{i,j=1}^d\right)\right) \varepsilon.
\end{aligned} \tag{38}
$$

Then, we have

$$
\begin{aligned}
&L_G^\alpha-\lim_{\varepsilon \rightarrow 0+} \frac{1}{\varepsilon}\left\{Y_t^{\varepsilon}-y\right\} \\
&\quad=f\left(t, y, \sigma^T(x) p\right)+\langle p, b(x)\rangle \\
&\quad\quad+2G\left(\left(g_{ij}\left(t, y, \sigma^T(x) p\right)+\left\langle p, h_{ij}(x)\right\rangle\right)_{i,j=1}^d\right).
\end{aligned} \tag{39}
$$

The proof is complete. □

4. Some Applications

4.1. Converse Comparison Theorem for G-BSDEs. We consider the following G-BSDEs:

$$
\begin{aligned}
Y_t^{l,\xi}=\xi &+\int_t^T f^l\left(s, Y_s^{l,\xi}, Z_s^{l,\xi}\right) ds \\
&+\int_t^T g_{ij}^l\left(s, Y_s^{l,\xi}, Z_s^{l,\xi}\right) d\left\langle B^i, B^j\right\rangle_s \\
&-\int_t^T Z_s^{l,\xi} dB_s-\left(K_T^{l,\xi}-K_t^{l,\xi}\right), \quad l=1,2,
\end{aligned} \tag{40}
$$

where $g_{ij}^l=g_{ji}^l$.

We first generalized the comparison theorem in [16].

Proposition 13. *Let f^l and g_{ij}^l satisfy (H1) and (H2) for some $\beta>1$, $l=1,2$. If $f^2-f^1+2G((g_{ij}^2-g_{ij}^1)_{i,j=1}^d) \leq 0$, then, for each $\xi \in L_G^\beta(\Omega_T)$, one has $Y_t^{1,\xi} \geq Y_t^{2,\xi}$ for $t \in[0,T]$.*

Proof. From the above G-BSDEs, we have

$$
\begin{aligned}
Y_t^{2,\xi}&=\xi+\int_t^T f^2\left(s, Y_s^{2,\xi}, Z_s^{2,\xi}\right) ds \\
&\quad+\int_t^T g_{ij}^2\left(s, Y_s^{2,\xi}, Z_s^{2,\xi}\right) d\left\langle B^i, B^j\right\rangle_s \\
&\quad-\int_t^T Z_s^{2,\xi} dB_s-\left(K_T^{2,\xi}-K_t^{2,\xi}\right) \\
&=\xi+\int_t^T f^1\left(s, Y_s^{2,\xi}, Z_s^{2,\xi}\right) ds \\
&\quad+\int_t^T g_{ij}^1\left(s, Y_s^{2,\xi}, Z_s^{2,\xi}\right) d\left\langle B^i, B^j\right\rangle_s \\
&\quad+V_T-V_t-\int_t^T Z_s^{2,\xi} dB_s-\left(K_T^{2,\xi}-K_t^{2,\xi}\right),
\end{aligned} \tag{41}
$$

where

$$
\begin{aligned}
V_t=&\int_0^t\left(f^2-f^1\right)\left(s, Y_s^{2,\xi}, Z_s^{2,\xi}\right) ds \\
&+\int_0^t\left(g_{ij}^2-g_{ij}^1\right)\left(s, Y_s^{2,\xi}, Z_s^{2,\xi}\right) d\left\langle B^i, B^j\right\rangle_s \\
=&\int_0^t\left(f^2-f^1+2G\left(\left(g_{ij}^2-g_{ij}^1\right)_{i,j=1}^d\right)\right)\left(s, Y_s^{2,\xi}, Z_s^{2,\xi}\right) ds \\
&+\int_0^t\left(g_{ij}^2-g_{ij}^1\right)\left(s, Y_s^{2,\xi}, Z_s^{2,\xi}\right) d\left\langle B^i, B^j\right\rangle_s \\
&-\int_0^t 2G\left(\left(g_{ij}^2-g_{ij}^1\right)_{i,j=1}^d\right)\left(s, Y_s^{2,\xi}, Z_s^{2,\xi}\right) ds.
\end{aligned} \tag{42}
$$

By the assumption, it is easy to check that $(V_t)_{t \leq T}$ is a decreasing process. Thus, using Theorem 11, we obtain $Y_t^{1,\xi} \geq Y_t^{2,\xi}$ for $t \in[0,T]$. □

Remark 14. Suppose $d=1$ and let $f^1=10|z|$, $f^2=|z|$, $g^1=|z|$, and $g^2=2|z|$. It is easy to check that $f^2-f^1+2G(g^2-g^1) \leq 0$. Thus, $f^2-f^1+2G((g_{ij}^2-g_{ij}^1)_{i,j=1}^d) \leq 0$ does not imply $f^2 \leq f^1$ and $(g_{ij}^2)_{i,j=1}^d \leq (g_{ij}^1)_{i,j=1}^d$.

Now, we give the converse comparison theorem.

Theorem 15. *Let f^l and g_{ij}^l satisfy (H1), (H2), (H3), (H4), and (H5) for some $\beta>1$, $l=1,2$. If $Y_t^{1,\xi} \geq Y_t^{2,\xi}$ for each $t \in[0,T]$ and $\xi \in L_G^\beta(\Omega_T)$, then $f^2-f^1+2G((g_{ij}^2-g_{ij}^1)_{i,j=1}^d) \leq 0$ q.s..*

Proof. For simplicity, we take the notation $\widetilde{\mathbb{E}}_t^l[\xi]=Y_t^{l,\xi}$, $l=1,2$. For each fixed $(t,y,z) \in[0,T) \times \mathbb{R} \times \mathbb{R}^d$, let us consider

$$
\eta_\varepsilon=y+\left\langle z, h_{ij}\right\rangle\left(\left\langle B^i, B^j\right\rangle_{t+\varepsilon}-\left\langle B^i, B^j\right\rangle_t\right)+\left\langle z, B_{t+\varepsilon}-B_t\right\rangle, \tag{43}
$$

where $h_{ij} = h_{ji} \in \mathbb{R}^d$. By Theorem 12, we have, for each $\alpha \in (1, \beta)$,

$$L_G^\alpha - \lim_{\varepsilon \to 0+} \frac{1}{\varepsilon} \left(\widetilde{\mathbb{E}}_t^l [\eta_\varepsilon] - y \right) = f^l(t, y, z) + 2G\left(\left(g_{ij}^l(t, y, z) + \langle z, h_{ij} \rangle \right)_{i,j=1}^d \right). \tag{44}$$

Since $\widetilde{\mathbb{E}}_t^1[\eta_\varepsilon] \geq \widetilde{\mathbb{E}}_t^2[\eta_\varepsilon]$,

$$f^1(t, y, z) + 2G\left(\left(g_{ij}^1(t, y, z) + \langle z, h_{ij} \rangle \right)_{i,j=1}^d \right) \geq f^2(t, y, z) + 2G\left(\left(g_{ij}^2(t, y, z) + \langle z, h_{ij} \rangle \right)_{i,j=1}^d \right) \text{ q.s.} \tag{45}$$

Take a h_{ij} such that $\langle z, h_{ij} \rangle = -g_{ij}^1(t, y, z)$. Therefore, $\{f^2 - f^1 + 2G((g_{ij}^2 - g_{ij}^1)_{i,j=1}^d)\}(t, y, z) \leq 0$ q.s. By the assumptions (H2) and (H3), it is easy to deduce that $f^2 - f^1 + 2G((g_{ij}^2 - g_{ij}^1)_{i,j=1}^d) \leq 0$ q.s. □

In the following, we use the notation $\widetilde{\mathbb{E}}_t^l[\xi] = Y_t^{l,\xi}$, $l = 1, 2$.

Corollary 16. *Let f^l and g_{ij}^l be deterministic functions and satisfy (H1), (H2), (H3), and (H5) for some $\beta > 1$, $l = 1, 2$. If $\widetilde{\mathbb{E}}^1[\xi] \geq \widetilde{\mathbb{E}}^2[\xi]$ for each $\xi \in L_G^\beta(\Omega_T)$, then $f^2 - f^1 + 2G((g_{ij}^2 - g_{ij}^1)_{i,j=1}^d) \leq 0$.*

Proof. Taking η_ε as in Theorem 15, since f^l and g_{ij}^l are deterministic, we could get $\widetilde{\mathbb{E}}_t^l[\eta_\varepsilon] = \widetilde{\mathbb{E}}^l[\eta_\varepsilon]$, for $l = 1, 2$. And the proof in Theorem 15 still holds true. □

4.2. Some Equivalent Relations. We consider the following G-BSDE:

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)\, ds + \int_t^T g_{ij}(s, Y_s, Z_s)\, d\langle B^i, B^j \rangle_s - \int_t^T Z_s dB_s - (K_T - K_t), \tag{46}$$

where $g_{ij} = g_{ji}$. We use the notation $\widetilde{\mathbb{E}}_t[\xi] = Y_t$.

Proposition 17. *Let f and g_{ij} satisfy (H1), (H2), (H3), (H4), and (H5) for some $\beta > 1$ and fix $\alpha \in (1, \beta)$. Then, one has*

(1) *$\widetilde{\mathbb{E}}_t[\xi + \eta] = \widetilde{\mathbb{E}}_t[\xi] + \eta$ for $t \in [0, T]$, $\xi \in L_G^\alpha(\Omega_T)$, and $\eta \in L_G^\alpha(\Omega_t)$ if and only if for each $t \in [0, T]$, y, $y' \in \mathbb{R}$, $z \in \mathbb{R}^d$,*

$$f(t, y, z) - f(t, y', z) + 2G\left(\left(g_{ij}(t, y, z) - g_{ij}(t, y', z) \right)_{i,j=1}^d \right) = 0; \tag{47}$$

(2) *$\widetilde{\mathbb{E}}_t[\xi + \eta] \leq \widetilde{\mathbb{E}}_t[\xi] + \widetilde{\mathbb{E}}_t[\eta]$ for $t \in [0, T]$, $\xi \in L_G^\alpha(\Omega_T)$, and $\eta \in L_G^\alpha(\Omega_T)$ if and only if for each $t \in [0, T]$, y, $y' \in \mathbb{R}$, z, $z' \in \mathbb{R}^d$,*

$$\begin{aligned} 0 \geq{}& f(t, y + y', z + z') - f(t, y, z) - f(t, y', z') \\ &+ 2G\Big(\big(g_{ij}(t, y + y', z + z') - g_{ij}(t, y, z) \\ &\quad - g_{ij}(t, y', z') \big)_{i,j=1}^d \Big); \end{aligned} \tag{48}$$

(3) *$\widetilde{\mathbb{E}}_t[\lambda\xi + (1-\lambda)\eta] \leq \lambda\widetilde{\mathbb{E}}_t[\xi] + (1-\lambda)\widetilde{\mathbb{E}}_t[\eta]$ for $t \in [0, T]$, $\lambda \in [0, 1]$, $\xi \in L_G^\alpha(\Omega_T)$, and $\eta \in L_G^\alpha(\Omega_T)$ if and only if for each $t \in [0, T]$, y, $y' \in \mathbb{R}$, z, $z' \in \mathbb{R}^d$, $\lambda \in [0, 1]$,*

$$\begin{aligned} 0 \geq{}& f(t, \lambda y + (1-\lambda) y', \lambda z + (1-\lambda) z') \\ &- \lambda f(t, y, z) - (1-\lambda) f(t, y', z') \\ &+ 2G\Big(\big(g_{ij}(t, \lambda y + (1-\lambda) y', \lambda z + (1-\lambda) z') \\ &\quad - \lambda g_{ij}(t, y, z) - (1-\lambda) g_{ij}(t, y', z') \big)_{i,j=1}^d \Big); \end{aligned} \tag{49}$$

(4) *$\widetilde{\mathbb{E}}_t[\lambda\xi] = \lambda\widetilde{\mathbb{E}}_t[\xi]$ for $t \in [0, T]$, $\lambda \geq 0$, and $\xi \in L_G^\alpha(\Omega_T)$ if and only if for each $t \in [0, T]$, $y \in \mathbb{R}$, $z \in \mathbb{R}^d$, $\lambda \geq 0$,*

$$\begin{aligned} f(t, \lambda y, \lambda z) - \lambda f(t, y, z) &= 2G\left(\left(\lambda g_{ij}(t, y, z) - g_{ij}(t, \lambda y, \lambda z) \right)_{i,j=1}^d \right) \\ &= -2G\left(\left(g_{ij}(t, \lambda y, \lambda z) - \lambda g_{ij}(t, y, z) \right)_{i,j=1}^d \right). \end{aligned} \tag{50}$$

Proof. (1) "⇒" part. For each fixed $t \in [0, T)$, y, $y' \in \mathbb{R}$, $z \in \mathbb{R}^d$, we take

$$\xi_\varepsilon = y + \langle z, h_{ij} \rangle \left(\langle B^i, B^j \rangle_{t+\varepsilon} - \langle B^i, B^j \rangle_t \right) + \langle z, B_{t+\varepsilon} - B_t \rangle, \qquad \eta = y' - y, \tag{51}$$

where $h_{ij} = h_{ji} \in \mathbb{R}^d$. Then, by Theorem 12 and $\widetilde{\mathbb{E}}_t[\xi_\varepsilon + \eta] = \widetilde{\mathbb{E}}_t[\xi_\varepsilon] + \eta$, we can obtain

$$f(t, y', z) + 2G\left(\left(g_{ij}(t, y', z) + \langle z, h_{ij} \rangle \right)_{i,j=1}^d \right) = f(t, y, z) + 2G\left(\left(g_{ij}(t, y, z) + \langle z, h_{ij} \rangle \right)_{i,j=1}^d \right). \tag{52}$$

We choose h_{ij} such that $g_{ij}(t, y', z) + \langle z, h_{ij} \rangle = 0$, which implies (47).

"⇐" part. Let (Y, Z, K) be the solution of G-BSDE (46) corresponding to terminal condition ξ. We claim that $(Y_s + \eta, Z_s, K_s)_{s \in [t,T]}$ is the solution of G-BSDE (46) corresponding to terminal condition $\xi+\eta$ on $[t,T]$. For this, we only need to check that, for $s\in[t,T]$,

$$\begin{aligned}\int_s^T f(u,Y_u,Z_u)\,du+\int_s^T g_{ij}(u,Y_u,Z_u)\,d\langle B^i,B^j\rangle_u &= \int_s^T f(u,Y_u+\eta,Z_u)\,du \\ &\quad+\int_s^T g_{ij}(u,Y_u+\eta,Z_u)\,d\langle B^i,B^j\rangle_u.\end{aligned}\tag{53}$$

By (47) we can get

$$\begin{aligned}&\int_s^T \left(g_{ij}(u,Y_u,Z_u)-g_{ij}(u,Y_u+\eta,Z_u)\right)d\langle B^i,B^j\rangle_u \\ &\qquad -2\int_s^T G\left(\left(g_{ij}(u,Y_u,Z_u)-g_{ij}(u,Y_u+\eta,Z_u)\right)_{i,j=1}^d\right)du \\ &\quad=\int_s^T \left(g_{ij}(u,Y_u,Z_u)-g_{ij}(u,Y_u+\eta,Z_u)\right)d\langle B^i,B^j\rangle_u \\ &\qquad+\int_s^T \left(f(u,Y_u,Z_u)-f(u,Y_u+\eta,Z_u)\right)du\le 0, \\ &\int_s^T \left(g_{ij}(u,Y_u+\eta,Z_u)-g_{ij}(u,Y_u,Z_u)\right)d\langle B^i,B^j\rangle_u \\ &\qquad -2\int_s^T G\left(\left(g_{ij}(u,Y_u+\eta,Z_u)-g_{ij}(u,Y_u,Z_u)\right)_{i,j=1}^d\right)du \\ &\quad=\int_s^T \left(g_{ij}(u,Y_u+\eta,Z_u)-g_{ij}(u,Y_u,Z_u)\right)d\langle B^i,B^j\rangle_u \\ &\qquad+\int_s^T \left(f(u,Y_u+\eta,Z_u)-f(u,Y_u,Z_u)\right)du\le 0,\end{aligned}\tag{54}$$

which implies (53). The proof of (1) is complete.

(2) "$\Rightarrow$" part. For each fixed $t\in[0,T)$, $y,y'\in\mathbb{R}$, $z,z'\in\mathbb{R}^d$, we consider $\xi_\varepsilon=y+\langle z,h_{ij}\rangle(\langle B^i,B^j\rangle_{t+\varepsilon}-\langle B^i,B^j\rangle_t)+\langle z,B_{t+\varepsilon}-B_t\rangle$ and $\eta_\varepsilon=y'+\langle z',h'_{ij}\rangle(\langle B^i,B^j\rangle_{t+\varepsilon}-\langle B^i,B^j\rangle_t)+\langle z',B_{t+\varepsilon}-B_t\rangle$, where $h_{ij}=h_{ji}\in\mathbb{R}^d$ and $h'_{ij}=h'_{ji}\in\mathbb{R}^d$. Then, by Theorem 12 and $\widetilde{\mathbb{E}}_t[\xi_\varepsilon+\eta_\varepsilon]=\widetilde{\mathbb{E}}_t[\xi_\varepsilon]+\widetilde{\mathbb{E}}_t[\eta_\varepsilon]$, we obtain

$$\begin{aligned}&f(t,y+y',z+z') \\ &\quad+2G\Big(\big(g_{ij}(t,y+y',z+z') \\ &\qquad+\langle z,h_{ij}\rangle+\langle z',h'_{ij}\rangle\big)_{i,j=1}^d\Big) \\ &\le f(t,y,z)+f(t,y',z') \\ &\quad+2G\left(\left(g_{ij}(t,y,z)+\langle z,h_{ij}\rangle\right)_{i,j=1}^d\right) \\ &\quad+2G\left(\left(g_{ij}(t,y',z')+\langle z',h'_{ij}\rangle\right)_{i,j=1}^d\right).\end{aligned}\tag{55}$$

We choose h_{ij}, h'_{ij} such that $g_{ij}(t,y,z)+\langle z,h_{ij}\rangle=0$ and $g_{ij}(t,y',z')+\langle z',h'_{ij}\rangle=0$, which implies (48).

"$\Leftarrow$" part. Let (Y,Z,K) and (Y',Z',K') be the solutions of G-BSDE (46) corresponding to terminal condition ξ and η, respectively. Then, $(Y+Y',Z+Z',K)$ solves the following G-BSDE:

$$\begin{aligned}Y_t+Y'_t&=\xi+\eta+\int_t^T f(s,Y_s+Y'_s,Z_s+Z'_s)\,ds \\ &\quad+\int_t^T g_{ij}(s,Y_s+Y'_s,Z_s+Z'_s)\,d\langle B^i,B^j\rangle_s \\ &\quad+V_T-V_t-\int_t^T (Z_s+Z'_s)\,dB_s-(K_T-K_t),\end{aligned}\tag{56}$$

where

$$\begin{aligned}V_t&=-K'_t-\int_0^t\big(f(s,Y_s+Y'_s,Z_s+Z'_s) \\ &\qquad -f(s,Y_s,Z_s)-f(s,Y'_s,Z'_s)\big)\,ds \\ &\quad-\int_0^t\big(g_{ij}(s,Y_s+Y'_s,Z_s+Z'_s)-g_{ij}(s,Y_s,Z_s) \\ &\qquad -g_{ij}(s,Y'_s,Z'_s)\big)\,d\langle B^i,B^j\rangle_s \\ &=-K'_t-\Big\{\int_0^t\big(g_{ij}(s,Y_s+Y'_s,Z_s+Z'_s) \\ &\qquad -g_{ij}(s,Y_s,Z_s)-g_{ij}(s,Y'_s,Z'_s)\big)\,d\langle B^i,B^j\rangle_s \\ &\qquad -2\int_0^t G\Big(\big(g_{ij}(s,Y_s+Y'_s,Z_s+Z'_s) \\ &\qquad\quad -g_{ij}(s,Y_s,Z_s) \\ &\qquad\quad -g_{ij}(s,Y'_s,Z'_s)\big)_{i,j=1}^d\Big)\,ds\Big\} \\ &\quad-\int_0^t\Big\{f(s,Y_s+Y'_s,Z_s+Z'_s) \\ &\qquad -f(s,Y_s,Z_s)-f(s,Y'_s,Z'_s) \\ &\qquad +2G\Big(\big(g_{ij}(s,Y_s+Y'_s,Z_s+Z'_s)-g_{ij}(s,Y_s,Z_s) \\ &\qquad\quad -g_{ij}(s,Y'_s,Z'_s)\big)_{i,j=1}^d\Big)\Big\}\,ds.\end{aligned}\tag{57}$$

By (48), it is easy to check that V_t is an increasing process. Then, by Theorem 11, we can get $\widetilde{\mathbb{E}}_t[\xi+\eta]\le\widetilde{\mathbb{E}}_t[\xi]+\widetilde{\mathbb{E}}_t[\eta]$. The proof of (2) is complete.

Finally, we could prove (3) as in (2) and (4) as in (1). □

Proposition 18. *One has the following.*

(1) *If $G(A)+G(-A)>0$ for any $A\in\mathbb{S}_d$ and $A\neq 0$, then (47) holds if and only if f and g_{ij} are independent of y.*

(2) *If there exists an* $A \in \mathbb{S}_d$ *with* $A \neq 0$ *such that* $G(A) + G(-A) = 0$ *and* $G(A) \neq 0$*, then, for any fixed* $g(t, y, z)$ *satisfying (H1)–(H5), one has* $f(t, y, z) = -2G(A)g(t, y, z)$ *and* $(g_{ij}(t, y, z))_{i,j=1}^{d} = g(t, y, z)A$ *satisfying* (47).

Proof. It is easy to verify (2), and we only need to prove (1). If (47) holds, it is easy to check that $G((g_{ij}(t, y, z) - g_{ij}(t, 0, z))_{i,j=1}^{d}) + G((g_{ij}(t, 0, z) - g_{ij}(t, y, z))_{i,j=1}^{d}) = 0$ holds. Then, from the assumption, we get $g_{ij}(t, y, z) = g_{ij}(t, 0, z)$. Therefore, by (47), we have $f(t, y, z) = f(t, 0, z)$, which implies that f and g_{ij} are independent of y. The converse part is obvious. □

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