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Research Article

A Note on Parabolic Homogenization with a Mismatch between the Spatial Scales

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We consider the homogenization of the linear parabolic problem $\rho(x/\varepsilon_2)\partial_t u^\varepsilon(x,t) - \nabla\cdot(a(x/\varepsilon_1, t/\varepsilon_1^2)\nabla u^\varepsilon(x,t)) = f(x,t)$ which exhibits a mismatch between the spatial scales in the sense that the coefficient $a(x/\varepsilon_1, t/\varepsilon_1^2)$ of the elliptic part has one frequency of fast spatial oscillations, whereas the coefficient $\rho(x/\varepsilon_2)$ of the time derivative contains a faster spatial scale. It is shown that the faster spatial microscale does not give rise to any corrector term and that there is only one local problem needed to characterize the homogenized problem. Hence, the problem is not of a reiterated type even though two rapid scales of spatial oscillation appear.

1. Introduction

The field of homogenization has its main source of inspiration in the problem of finding the macroscopic properties of strongly heterogeneous materials. Mathematically, the approach is to study a sequence of partial differential equations where a parameter ε associated with the length scales of the heterogeneities tends to zero. The sequence of solutions u^ε converges to the solution u to a so-called homogenized problem governed by a coefficient b, where b gives the effective property of the material and can be characterized by certain auxiliary problems called the local problems.

In this paper, we study the homogenization of the linear parabolic problem

$$\begin{gathered}\rho\left(\frac{x}{\varepsilon_2}\right)\partial_t u^\varepsilon(x,t) - \nabla\cdot\left(a\left(\frac{x}{\varepsilon_1},\frac{t}{\varepsilon_1^2}\right)\nabla u^\varepsilon(x,t)\right) \\ = f(x,t) \quad \text{in } \Omega_T, \\ u^\varepsilon(x,t) = 0 \quad \text{on } \partial\Omega\times(0,T), \\ u^\varepsilon(x,0) = u^0(x) \quad \text{in } \Omega,\end{gathered} \tag{1}$$

where $\Omega_T = \Omega\times(0,T)$, Ω is an open, bounded set in $\mathbb{R}^N$ with locally Lipschitz boundary, where both a and ρ possess unit periodicity in their respective arguments and the scales ε_1, ε_2, and ε_1^2 fulfill a certain separatedness assumption.

The problem exhibits rapid spatial oscillations in ρ and spatial as well as temporal oscillations in a. Furthermore, there is a "mismatch" between the spatial scales in the sense that the frequency of the spatial oscillations in $\rho(x/\varepsilon_2)$ is higher than that of $a(x/\varepsilon_1, t/\varepsilon_1^2)$. Since there are two spatial microscales represented in (1), one might expect two local problems with respect to one corrector each, see, for example, [1]. However, it is shown that no corrector corresponding to the scale emanating from $\rho(x/\varepsilon_2)$ appears in the local and homogenized problem and accordingly there is only one local problem appearing in the formulated theorem. Hence, the problem is not of a reiterated type. We prove by means of very weak multiscale convergence [2] that the corrector u_2 associated with the gradient for the second rapid spatial scale y_2 actually vanishes. Already, in [3, 4], it was observed that having more than one rapid temporal scale in parabolic problems does not generate a reiterated problem and in this paper we can see that nor does the addition of a spatial scale if it is contained in a coefficient that is multiplied with the time derivative of u^ε.

Thinking in terms of heat conduction, our result means that the heat capacity ρ may oscillate with completely different periodic patterns without making any difference for the homogenized coefficient as long as the arithmetic mean over one period is the same.

Parabolic homogenization problems for $\rho \equiv 1$ have been studied for different combinations of spatial and temporal scales in several papers by means of techniques of two-scale convergence type with approaches related to the one first introduced in [5], see, for example, [2, 3, 6–8], and in, for example, [9–11], techniques not of two-scale convergence type are applied. Concerning cases where, as in (1) above, we do not have $\rho \equiv 1$, Nandakumaran and Rajesh [12] studied a nonlinear parabolic problem with the same frequency of oscillation in time and space, respectively, in the elliptic part of the equation and an operator oscillating in space with the same frequency appearing in the temporal differentiation term. Recently, a number of papers have addressed various kinds of related problems where the temporal scale is not assumed to be identical with the spatial scale, see for example, [13, 14]. Up to the authors' knowledge, this is the first study of this type of problems where the oscillations of the coefficient of the term including the time derivative do not match the spatial oscillations of the elliptic part.

Notation. We denote $Y_k = (0,1)^N$ for $k = 1,\ldots,n$, $Y^n = Y_1 \times \cdots \times Y_n$, $y^n = (y_1,\ldots,y_n)$, $dy^n = dy_1 \cdots dy_n$, $S_j = S = (0,1)$ for $j = 1,\ldots,m$, $S^m = S_1 \times \cdots \times S_m$, $s^m = (s_1,\ldots,s_m)$, and $ds^m = ds_1 \cdots ds_m$. Let $\varepsilon_k(\varepsilon)$, $k = 1,\ldots,n$, and $\varepsilon'_j(\varepsilon)$, $j = 1,\ldots,m$, be positive and go to zero when ε does. Furthermore, let $F_\sharp((0,1)^M)$ be the space of all functions in $F_{\text{loc}}(\mathbb{R}^M)$ that are $(0,1)^M$-periodic repetitions of some function in $F((0,1)^M)$.

2. Multiscale Convergence

A two-scale convergence was invented by Nguetseng [15] as a new approach for the homogenization of problems with fast oscillations in one scale in space. The method was further developed by Allaire [16] and generalized to multiple scales by Allaire and Briane [1]. To homogenize problem (1), we use the further generalization in the definition below, adapted to evolution settings, see, for example, [8].

Definition 1. A sequence $\{u^\varepsilon\}$ in $L^2(\Omega_T)$ is said to $(n+1, m+1)$-scale converge to $u_0 \in L^2(\Omega_T \times Y^n \times S^m)$ if

$$\begin{aligned}\int_{\Omega_T} u^\varepsilon(x,t)\, v\left(x,t,\frac{x}{\varepsilon_1},\ldots,\frac{x}{\varepsilon_n},\frac{t}{\varepsilon'_1},\ldots,\frac{t}{\varepsilon'_m}\right) dx\,dt \\ \longrightarrow \int_{\Omega_T}\int_{Y^n}\int_{S^m} u_0(x,t,y^n,s^m) \\ \times v(x,t,y^n,s^m)\, dy^n ds^m dx\,dt,\end{aligned} \tag{2}$$

for any $v \in L^2(\Omega_T; C_\sharp(Y^n \times S^m))$. We write

$$u^\varepsilon(x,t) \overset{n+1,m+1}{\rightharpoonup} u_0(x,t,y^n,s^m). \tag{3}$$

Usually, some assumptions are made about how the scales are related to each other. We say that the scales in a list $\{\varepsilon_1,\ldots,\varepsilon_n\}$ are separated if

$$\lim_{\varepsilon\to 0}\frac{\varepsilon_{k+1}}{\varepsilon_k} = 0, \tag{4}$$

for $k = 1,\ldots,n-1$ and that the scales are well-separated if there exists a positive integer l such that

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon_k}\left(\frac{\varepsilon_{k+1}}{\varepsilon_k}\right)^l = 0, \tag{5}$$

for $k = 1,\ldots,n-1$.

The concept in the following definition is used as an assumption in the proofs of the compactness results in Theorems 3 and 7. For a more technically formulated definition and some examples, see [17, Section 2.4].

Definition 2. Let $\{\varepsilon_1,\ldots,\varepsilon_n\}$ and $\{\varepsilon'_1,\ldots,\varepsilon'_m\}$ be lists of well-separated scales. Consider all elements from both lists. If from possible duplicates, where by duplicates we mean scales which tend to zero equally fast, one member of each pair is removed and the list in order of magnitude of all the remaining elements is well separated, the lists $\{\varepsilon_1,\ldots,\varepsilon_n\}$ and $\{\varepsilon'_1,\ldots,\varepsilon'_m\}$ are said to be jointly well separated.

In the theorem below, which will be used in the homogenization procedure in Section 3, $W_2^1(0,T;H_0^1(\Omega),L^2(\Omega))$ denotes all functions $u \in L^2(0,T;H_0^1(\Omega))$ such that $\partial_t u \in L^2(0,T;H^{-1}(\Omega))$, see, for example, [18, Chapter 23].

Theorem 3. *Let $\{u^\varepsilon\}$ be a bounded sequence in $W_2^1(0,T;H_0^1(\Omega),L^2(\Omega))$, and suppose that the lists $\{\varepsilon_1,\ldots,\varepsilon_n\}$ and $\{\varepsilon'_1,\ldots,\varepsilon'_m\}$ are jointly well separated. Then there exists a subsequence such that*

$$\begin{aligned} u^\varepsilon(x,t) &\longrightarrow u(x,t) \quad \text{in } L^2(\Omega_T), \\ u^\varepsilon(x,t) &\rightharpoonup u(x,t) \quad \text{in } L^2\left(0,T;H_0^1(\Omega)\right),\end{aligned} \tag{6}$$

$$\nabla u^\varepsilon(x,t) \overset{n+1,m+1}{\rightharpoonup} \nabla u(x,t) + \sum_{j=1}^{n}\nabla_{y_j} u_j\left(x,t,y^j,s^m\right), \tag{7}$$

where $u \in W_2^1(0,T;H_0^1(\Omega),L^2(\Omega))$, $u_1 \in L^2(\Omega_T \times S^m; H_\sharp^1(Y_1)/\mathbb{R})$, and $u_j \in L^2(\Omega_T \times Y^{j-1} \times S^m; H_\sharp^1(Y_j)/\mathbb{R})$ for $j = 2,\ldots,n$.

Proof. See [17, Theorem 2.74]. □

To treat evolution problems with fast time oscillations, such as (1), we also need the concept of very weak multiscale convergence, see, for example, [2, 5].

Definition 4. A sequence $\{g^\varepsilon\}$ in $L^1(\Omega_T)$ is said to $(n+1,m+1)$-scale converge very weakly to $g_0 \in L^1(\Omega_T \times Y^n \times S^m)$ if

$$\begin{aligned}\int_{\Omega_T} & g^\varepsilon(x,t)\, v\left(x, \frac{x}{\varepsilon_1}, \ldots, \frac{x}{\varepsilon_{n-1}}\right) \\ &\times c\left(t, \frac{t}{\varepsilon_1'}, \ldots, \frac{t}{\varepsilon_m'}\right) \varphi\left(\frac{x}{\varepsilon_n}\right) dx\, dt \\ \longrightarrow & \int_{\Omega_T}\int_{Y^n}\int_{S^m} g_0(x,t,y^n,s^m)\, v(x,y^{n-1}) \\ &\times c(t,s^m)\,\varphi(y_n)\, dy^n ds^m dx\, dt,\end{aligned} \tag{8}$$

for any $v \in D(\Omega, C_\sharp^\infty(Y^{n-1}))$, $c \in D(0,T; C_\sharp^\infty(S^m))$, and $\varphi \in C_\sharp^\infty(Y_n)/\mathbb{R}$, where

$$\int_{Y_n} g_0(x,t,y^n,s^m)\, dy_n = 0. \tag{9}$$

We write

$$g^\varepsilon(x,t) \overset{n+1,m+1}{\underset{vw}{\rightharpoonup}} g_0(x,t,y^n,s^m). \tag{10}$$

Remark 5. The requirement (9) is imposed in order to ensure the uniqueness of the limit. For details, see [17, Proposition 2.26].

Remark 6. The convergence in Definition 1 may take place only if $\{u^\varepsilon\}$ is bounded in $L^2(\Omega_T)$ and hence also is a weakly convergent in $L^2(\Omega_T)$, at least up to suitable subsequences. For very weak multiscale convergence, this is not so. The main intention with the concept is to study sequences of the type $\{u^\varepsilon/\varepsilon_n\}$, which are in general not bounded in $L^2(\Omega_T)$. This requires a more restrictive class of test functions.

The theorem below is a key result for the homogenization procedure in Section 3.

Theorem 7. *Let $\{u^\varepsilon\}$ be a bounded sequence in $W_2^1(0,T; H_0^1(\Omega), L^2(\Omega))$, and assume that the lists $\{\varepsilon_1, \ldots, \varepsilon_n\}$ and $\{\varepsilon_1', \ldots, \varepsilon_m'\}$ are jointly well separated. Then there exists a subsequence such that*

$$\frac{u^\varepsilon(x,t)}{\varepsilon_n} \overset{n+1,m+1}{\underset{vw}{\rightharpoonup}} u_n(x,t,y^n,s^m), \tag{11}$$

where, for $n = 1$, $u_1 \in L^2(\Omega_T \times S^m; H_\sharp^1(Y_1)/\mathbb{R})$ and, for $n = 2, 3, \ldots$, $u_n \in L^2(\Omega_T \times Y^{n-1} \times S^m; H_\sharp^1(Y_n)/\mathbb{R})$ are the same as those in Theorem 3.

Proof. See [17, Theorem 2.54]. □

Remark 8. For a sequence of solutions $\{u^\varepsilon\}$ to (1), we may replace the requirement that $\{\partial_t u^\varepsilon\}$ should be bounded in $L^2(0,T; H^{-1}(\Omega))$ by the assumption that $\{u^\varepsilon\}$ is bounded in $L^\infty(\Omega_T)$ and still obtain (6), see [12, Lemmas 3.3 and (4.1)] and thereby also (7) and (11). The only difference is that u will belong to $L^2(0,T; H_0^1(\Omega))$ instead of the space $W_2^1(0,T; H_0^1(\Omega), L^2(\Omega))$. See also [13].

3. Homogenization

Let us now investigate the heat conduction problem

$$\begin{aligned}\rho\left(\frac{x}{\varepsilon_2}\right)\partial_t u^\varepsilon(x,t) - \nabla\cdot\left(a\left(\frac{x}{\varepsilon_1}, \frac{t}{\varepsilon_1^2}\right)\nabla u^\varepsilon(x,t)\right) \\ = f(x,t) \quad \text{in } \Omega_T, \\ u^\varepsilon(x,t) = 0 \quad \text{on } \partial\Omega\times(0,T), \\ u^\varepsilon(x,0) = u^0(x) \quad \text{in } \Omega,\end{aligned} \tag{12}$$

which takes into consideration heat capacity oscillations. We assume that $\rho \in C_\sharp^\infty(Y_2)$, is positive, $f \in L^2(\Omega_T)$, $u^0 \in L^2(\Omega)$, and

$$a(y_1,s)\,\xi\cdot\xi \geq \alpha|\xi|^2 \tag{13}$$

for some $\alpha > 0$, all $(y_1,s) \in \mathbb{R}^{N+1}$, and all $\xi \in \mathbb{R}^N$, where $a \in C_\sharp(Y_1\times S)^{N\times N}$. Moreover, we assume that $\{u^\varepsilon\}$ is bounded in $L^\infty(\Omega_T)$, see Remark 8, and that the lists $\{\varepsilon_1, \varepsilon_2\}$ and $\{\varepsilon_1^2\}$ are jointly well separated. Note that this separatedness assumption implies, for example, that ε_2 tends to zero faster than ε_1, which means that we have a mismatch between the spatial scales in (12).

We give a homogenization result for this problem in the theorem below. In the proof, it is shown that the local problem associated with the slower spatial microscale is enough to characterize the homogenized problem; that is, the fastest spatial scale does not give rise to any corrector involved in the homogenization. We also prove that the second corrector u_2 actually vanishes.

Theorem 9. *Let $\{u^\varepsilon\}$ be a sequence of solutions to (12). Then*

$$u^\varepsilon(x,t) \rightharpoonup u(x,t) \quad \text{in } L^2\left(0,T; H_0^1(\Omega)\right), \tag{14}$$

$$\begin{aligned}\nabla u^\varepsilon(x,t) \overset{3,2}{\rightharpoonup} \nabla u(x,t) + \nabla_{y_1} u_1(x,t,y_1,s) \\ + \nabla_{y_2} u_2\left(x,t,y^2,s\right),\end{aligned} \tag{15}$$

where u is the unique solution to

$$\begin{aligned}\left(\int_{Y_2}\rho(y_2)\,dy_2\right)\partial_t u(x,t) - \nabla\cdot(b\nabla u(x,t)) \\ = f(x,t) \quad \text{in } \Omega_T, \\ u(x,t) = 0 \quad \text{on } \partial\Omega\times(0,T), \\ u(x,0) = u^0(x) \quad \text{in } \Omega,\end{aligned} \tag{16}$$

with

$$\begin{aligned}b\nabla u(x,t) \\ = \int_S\int_{Y_1} a(y_1,s)\left(\nabla u(x,t) + \nabla_{y_1} u_1(x,t,y_1,s)\right) dy_1 ds.\end{aligned} \tag{17}$$

Here, $u_1 \in L^2(\Omega_T \times S; H^1_\sharp(Y_1)/\mathbb{R})$ *uniquely solves*

$$\left(\int_{Y_2} \rho(y_2)\, dy_2\right) \partial_s u_1(x,t,y_1,s) - \nabla_{y_1} \cdot \left(a(y_1,s)\left(\nabla u(x,t) + \nabla_{y_1} u_1(x,t,y_1,s)\right)\right) = 0. \tag{18}$$

Furthermore, the corrector u_2 *vanishes.*

Remark 10. After a separation of variables, we can write the local problem as

$$\left(\int_{Y_2} \rho(y_2)\, dy_2\right) \partial_s z_k(y_1,s) - \sum_{i,j=1}^{N} \partial_{y_i}\left(a_{ij}(y_1,s)\left(\delta_{jk} + \partial_{y_j} z_k(y_1,s)\right)\right) = 0 \tag{19}$$

and the homogenized coefficient as

$$b_{ik} = \int_S \int_{Y_1} \sum_{j=1}^{N} \left(a_{ij}(y_1,s)\left(\delta_{jk} + \partial_{y_j} z_k(y_1,s)\right)\right) dy_1\, ds, \tag{20}$$

where $k = 1, \dots, N$ and

$$u_1(x,t,y_1,s) = \sum_{k=1}^{N} \partial_{x_k} u(x,t) \cdot z_k(y_1,s). \tag{21}$$

Remark 11. Periodic homogenization problems of, for example, elliptic or parabolic type may be seen as special cases of the more general concepts of G-convergence, which gives a characterization of the limit problem but no suggestion of how to compute the homogenized matrix. Essential features of G-convergence for parabolic problems are that boundary conditions, and initial conditions are preserved in the limit. G-convergence for linear parabolic problems were studied already in [19] by Spagnolo and extended to the monotone case by Svanstedt in [20]. A treatment of this problem in a quite general setting is found in the recent work [21] by Paronetto.

Proof of Theorem 9. Following the procedure in Section 23.9 in [18], we obtain that $\{u^\varepsilon\}$ is bounded in $L^2(0,T; H^1_0(\Omega))$, see also [22]. Hence, (14) holds up to a subsequence. We proceed by studying the weak form of (12); that is,

$$\begin{aligned}\int_{\Omega_T} -\rho\left(\frac{x}{\varepsilon_2}\right) u^\varepsilon(x,t)\, v(x)\, \partial_t c(t) &+ a\left(\frac{x}{\varepsilon_1}, \frac{t}{\varepsilon_1^2}\right) \nabla u^\varepsilon(x,t) \nabla v(x)\, c(t)\, dx\, dt \\ &= \int_{\Omega_T} f(x,t)\, v(x)\, c(t)\, dx\, dt,\end{aligned} \tag{22}$$

for all $v \in H^1_0(\Omega)$ and $c \in D(0,T)$. We pass to the limit by applying (6), taking into consideration Remark 8, and (7) with $n = 1$ and $m = 1$ and arrive at the homogenized problem

$$\begin{aligned}\int_{\Omega_T} \int_S \int_{Y_1} &-\left(\int_{Y_2} \rho(y_2)\, dy_2\right) u(x,t)\, v(x)\, \partial_t c(t) \\ &+ a(y_1,s)\left(\nabla u(x,t) + \nabla_{y_1} u_1(x,t,y_1,s)\right) \\ &\times \nabla v(x)\, c(t)\, dy_1 ds\, dx\, dt \\ = \int_{\Omega_T} & f(x,t)\, v(x)\, c(t)\, dx\, dt.\end{aligned} \tag{23}$$

To find the local problem associated with u_1, let us again consider (22) in which we choose

$$v(x) = \varepsilon_1 v_1(x)\, v_2\left(\frac{x}{\varepsilon_1}\right); \quad v_1 \in D(\Omega), \ v_2 \in \frac{C^\infty_\sharp(Y_1)}{\mathbb{R}}, \tag{24}$$

$$c(t) = c_1(t)\, c_2\left(\frac{t}{\varepsilon_1^2}\right); \quad c_1 \in D(0,T); \ c_2 \in C^\infty_\sharp(S); \tag{25}$$

that is, we study

$$\begin{aligned}\int_{\Omega_T} &-\rho\left(\frac{x}{\varepsilon_2}\right) u^\varepsilon(x,t)\, v_1(x)\, v_2\left(\frac{x}{\varepsilon_1}\right) \\ &\times \left(\varepsilon_1 \partial_t c_1(t)\, c_2\left(\frac{t}{\varepsilon_1^2}\right) + \varepsilon_1^{-1} c_1(t)\, \partial_s c_2\left(\frac{t}{\varepsilon_1^2}\right)\right) \\ &+ a\left(\frac{x}{\varepsilon_1}, \frac{t}{\varepsilon_1^2}\right) \nabla u^\varepsilon(x,t) \\ &\cdot \left(\varepsilon_1 \nabla v_1(x)\, v_2\left(\frac{x}{\varepsilon_1}\right) + v_1(x)\, \nabla_{y_1} v_2\left(\frac{x}{\varepsilon_1}\right)\right) \\ &\times c_1(t)\, c_2\left(\frac{t}{\varepsilon_1^2}\right) dx\, dt \\ = \int_{\Omega_T} & f(x,t)\, \varepsilon_1 v_1(x)\, v_2\left(\frac{x}{\varepsilon_1}\right) c_1(t)\, c_2\left(\frac{t}{\varepsilon_1^2}\right) dx\, dt.\end{aligned} \tag{26}$$

We first investigate the second term of the part of the expression containing time derivatives. We have

$$\int_{\Omega_T} -\rho\left(\frac{x}{\varepsilon_2}\right) u^\varepsilon(x,t)\, v_1(x)\, v_2 \times \left(\frac{x}{\varepsilon_1}\right) \varepsilon_1^{-1} c_1(t)\, \partial_s c_2\left(\frac{t}{\varepsilon_1^2}\right) dx\, dt$$

$$
\begin{aligned}
&= \int_{\Omega_T} -\varepsilon_1^{-1} u^\varepsilon(x,t)\, v_1(x)\, v_2\left(\frac{x}{\varepsilon_1}\right)\left(\rho\left(\frac{x}{\varepsilon_2}\right) - \int_{Y_2}\rho(y_2)\,dy_2\right) c_1(t)\,\partial_s c_2\left(\frac{t}{\varepsilon_1^2}\right) dx\,dt \\
&\quad + \int_{\Omega_T} -\varepsilon_1^{-1} u^\varepsilon(x,t)\, v_1(x)\, v_2\left(\frac{x}{\varepsilon_1}\right) \times \left(\int_{Y_2}\rho(y_2)\,dy_2\right) c_1(t)\,\partial_s c_2\left(\frac{t}{\varepsilon_1^2}\right) dx\,dt \\
&\longrightarrow \int_{\Omega_T}\int_S\int_{Y_1} -\left(\int_{Y_2}\rho(y_2)\,dy_2\right) u_1(x,t,y_1,s) \times v_1(x)\, v_2(y_1)\, c_1(t) \times \partial_s c_2(s)\,dy_1\,ds\,dx\,dt,
\end{aligned}
\tag{27}
$$

where we have applied (11) with $n = 2$ and $m = 1$ and with $n = 1$ and $m = 1$, respectively, in the last step. The passage to the limit in the remaining part of (26) is a simple application of (7) with $n = 1$ and $m = 1$. This provides us with the weak form,

$$
\begin{aligned}
&\int_{\Omega_T}\int_S\int_{Y_1} -\left(\int_{Y_2}\rho(y_2)\,dy_2\right) u_1(x,t,y_1,s) \times v_1(x)\, v_2(y_1)\, c_1(t)\,\partial_s c_2(s) \\
&\quad + a(y_1,s)\left(\nabla u(x,t) + \nabla_{y_1} u_1(x,t,y_1,s)\right) \cdot v_1(x)\,\nabla_{y_1} v_2(y_1)\, c_1(t)\, c_2(s)\,dy_1\,ds\,dx\,dt = 0,
\end{aligned}
\tag{28}
$$

of the local problem (18). This means that u_1, and thus also u, is uniquely determined and hence the entire sequence $\{u^\varepsilon\}$ converges and not just the extracted subsequence.

This far, we have only used test functions oscillating with a period ε_1, and hence we have not given the coefficient $\rho(x/\varepsilon_2)$ a fair chance to produce a second corrector u_2. In order to do so, we use a slightly different set of test functions in (22). Again, we let c be as in (25), whereas v is chosen according to

$$
\begin{aligned}
&v(x) = \varepsilon_2 v_1(x)\, v_2\left(\frac{x}{\varepsilon_1}\right) \tilde{v}\left(\frac{x}{\varepsilon_2}\right); \\
&v_1 \in D(\Omega), \quad v_2 \in C_\sharp^\infty(Y_1),
\end{aligned}
\tag{29}
$$

where

$$
\tilde{v}(y_2) = v_3(y_2) - \frac{K}{\rho(y_2)}; \quad v_3 \in C_\sharp^\infty(Y_2), \tag{30}
$$

with

$$
K = \int_{Y_2}\rho(y_2)\, v_3(y_2)\,dy_2. \tag{31}
$$

Note that

$$
\int_{Y_2}\rho(y_2)\,\tilde{v}(y_2)\,dy_2 = 0, \tag{32}
$$

which means that $\rho\tilde{v} \in C_\sharp^\infty(Y_2)/\mathbb{R}$. We get

$$
\begin{aligned}
&\int_{\Omega_T} -\rho\left(\frac{x}{\varepsilon_2}\right) u^\varepsilon(x,t)\, v_1(x)\, v_2\left(\frac{x}{\varepsilon}\right)\tilde{v}\left(\frac{x}{\varepsilon_2}\right) \times \left(\varepsilon_2 \partial_t c_1(t)\, c_2\left(\frac{t}{\varepsilon_1^2}\right) + \frac{\varepsilon_2}{\varepsilon_1^2} c_1(t)\,\partial_s c_2\left(\frac{t}{\varepsilon_1^2}\right)\right) \\
&\quad + a\left(\frac{x}{\varepsilon_1}, \frac{t}{\varepsilon_1^2}\right)\nabla u^\varepsilon(x,t) \cdot \left(\varepsilon_2 \nabla v_1(x)\, v_2\left(\frac{x}{\varepsilon_1}\right)\tilde{v}\left(\frac{x}{\varepsilon_2}\right) + \frac{\varepsilon_2}{\varepsilon_1} v_1(x)\,\nabla_{y_1} v_2\left(\frac{x}{\varepsilon_1}\right)\tilde{v}\left(\frac{x}{\varepsilon_2}\right) + v_1(x)\, v_2\left(\frac{x}{\varepsilon_1}\right)\nabla_{y_2}\tilde{v}\left(\frac{x}{\varepsilon_2}\right)\right) \\
&\quad \times c_1(t)\, c_2\left(\frac{t}{\varepsilon_1^2}\right) dx\,dt \\
&= \int_{\Omega_T} f(x,t)\,\varepsilon_2 v_1(x)\, v_2\left(\frac{x}{\varepsilon_1}\right) \times \tilde{v}\left(\frac{x}{\varepsilon_2}\right) c_1(t)\, c_2\left(\frac{t}{\varepsilon_1^2}\right) dx\,dt,
\end{aligned}
\tag{33}
$$

and applying (11) with $n = 2$ and $m = 1$ together with (15), that is, (7) with $n = 2$ and $m = 1$, we achieve

$$
\begin{aligned}
&\int_{\Omega_T}\int_S\int_{Y^2} a(y_1,s) \times \left(\nabla u(x,t) + \nabla_{y_1} u_1(x,t,y_1,s) + \nabla_{y_2} u_2(x,t,y^2,s)\right) \\
&\quad \cdot v_1(x)\, v_2(y_1)\,\nabla_{y_2}\tilde{v}(y_2)\, c_1(t)\, c_2(s)\,dy^2\,ds\,dx\,dt = 0.
\end{aligned}
\tag{34}
$$

Noting that a, u, and u_1 are all independent of y_2, (34) reduces to

$$
\int_{\Omega_T}\int_S\int_{Y^2} a(y_1,s)\,\nabla_{y_2} u_2(x,t,y^2,s) \cdot v_1(x)\, v_2(y_1)\,\nabla_{y_2}\tilde{v}(y_2)\, c_1(t)\, c_2(s)\,dy^2\,ds\,dx\,dt = 0. \tag{35}
$$

Recalling (30), we have

$$
\begin{aligned}
&\int_{\Omega_T}\int_S\int_{Y^2} a(y_1,s)\,\nabla_{y_2} u_2(x,t,y^2,s) \cdot v_1(x)\, v_2(y_1)\,\nabla_{y_2}\left(v_3(y_2) - \frac{K}{\rho(y_2)}\right) \\
&\quad \times c_1(t)\, c_2(s)\,dy^2\,ds\,dx\,dt = 0,
\end{aligned}
\tag{36}
$$

which after rearranging can be written as

$$\begin{aligned}\int_{\Omega_T}\int_S\int_{Y^2} & a\left(y_1,s\right)\nabla_{y_2}u_2\left(x,t,y^2,s\right)\\ &\cdot v_1\left(x\right)v_2\left(y_1\right)\nabla_{y_2}v_3\left(y_2\right)c_1\left(t\right)c_2\left(s\right)dy^2 ds\,dx\,dt\\ = K\int_{\Omega_T}\int_S\int_{Y_1} & \left(\int_{Y_2}a\left(y_1,s\right)\nabla_{y_2}u_2\left(x,t,y^2,s\right)\right.\\ &\left.\cdot\nabla_{y_2}\left(\frac{1}{\rho\left(y_2\right)}\right)dy_2\right)\\ &\times v_1\left(x\right)v_2\left(y_1\right)c_1\left(t\right)c_2\left(s\right)dy_1 ds\,dx\,dt.\end{aligned} \tag{37}$$

If we replace $\widetilde{v}$ with $1/\rho$ in (33), let $\varepsilon \to 0$, and use (6) and (7) with $n = 2$ and $m = 1$, we find that

$$\begin{aligned}\int_{\Omega_T}\int_S\int_{Y^2} & a\left(y_1,s\right)\nabla_{y_2}u_2\left(x,t,y^2,s\right)\\ &\cdot v_1\left(x\right)v_2\left(y_1\right)\nabla_{y_2}\left(\frac{1}{\rho\left(y_2\right)}\right)c_1\left(t\right)\\ &\times c_2\left(s\right)dy^2 ds\,dx\,dt = 0.\end{aligned} \tag{38}$$

This means that the right-hand side in (37) is zero. Applying several times the variational lemma on the remaining part, we obtain

$$\int_{Y_2}a\left(y_1,s\right)\nabla_{y_2}u_2\left(x,t,y^2,s\right)\cdot\nabla_{y_2}v_3\left(y_2\right)dy_2 = 0, \tag{39}$$

and hence the corrector u_2 is zero. □

Remark 12. That u_2 vanishes means that $u^\varepsilon/\varepsilon_2$ tends to zero in the sense of very weak $(3,2)$-scale convergence. However, there might still be oscillations originating from the oscillations of $\rho(x/\varepsilon_2)$ that have an impact on u^ε. The possibility is that their amplitude is so small that the magnification by $1/\varepsilon_2$ is not enough for the oscillations to be recognized in the limit. In this sense, the concept of very weak multiscale convergence gives us a more precise idea of what a corrector equals zero means.

References

[1] G. Allaire and M. Briane, "Multiscale convergence and reiterated homogenisation," *Proceedings of the Royal Society of Edinburgh A*, vol. 126, no. 2, pp. 297–342, 1996.

[2] L. Flodén, A. Holmbom, M. Olsson, and J. Persson, "Very weak multiscale convergence," *Applied Mathematics Letters*, vol. 23, no. 10, pp. 1170–1173, 2010.

[3] J. L. Woukeng, "Periodic homogenization of nonlinear nonmonotone parabolic operators with three time scales," *Annali di Matematica Pura ed Applicata*, vol. 189, no. 3, pp. 357–379, 2010.

[4] L. Flodén and M. Olsson, "Homogenization of some parabolic operators with several time scales," *Applications of Mathematics*, vol. 52, no. 5, pp. 431–446, 2007.

[5] A. Holmbom, "Homogenization of parabolic equations: an alternative approach and some corrector-type results," *Applications of Mathematics*, vol. 42, no. 5, pp. 321–343, 1997.

[6] G. Nguetseng and J. L. Woukeng, "Σ-convergence of nonlinear parabolic operators," *Nonlinear Analysis. Theory, Methods & Applications*, vol. 66, no. 4, pp. 968–1004, 2007.

[7] J. L. Woukeng, "Σ-convergence and reiterated homogenization of nonlinear parabolic operators," *Communications on Pure and Applied Analysis*, vol. 9, no. 6, pp. 1753–1789, 2010.

[8] L. Flodén, A. Holmbom, M. O. Lindberg, and J. Persson, "Detection of scales of heterogeneity and parabolic homogenization applying very weak multiscale convergence," *Annals of Functional Analysis*, vol. 2, no. 1, pp. 84–99, 2011.

[9] A. Bensoussan, J.-L. Lions, and G. Papanicolaou, *Asymptotic analysis for periodic structures*, vol. 5 of *Studies in Mathematics and its Applications*, North-Holland Publishing Co., Amsterdam, The Netherlands, 1978.

[10] A. Piatnitski, "A parabolic equation with rapidly oscillating coefficients," *Moscow University Mathematics Bulletin*, vol. 35, no. 3, pp. 35–42, 1980.

[11] J. Garnier, "Homogenization in a periodic and time-dependent potential," *SIAM Journal on Applied Mathematics*, vol. 57, no. 1, pp. 95–111, 1997.

[12] A. K. Nandakumaran and M. Rajesh, "Homogenization of a nonlinear degenerate parabolic differential equation," *Electronic Journal of Differential Equations*, vol. 2001, no. 17, pp. 1–19, 2001.

[13] N. Svanstedt and J. L. Woukeng, "Periodic homogenization of strongly nonlinear reaction-diffusion equations with large reaction terms," *Applicable Analysis*, vol. 92, no. 7, pp. 1–22, 2012.

[14] L. Flodén, A. Holmbom, and M. Olsson Lindberg, "A strange term in the homogenization of parabolic equations with two spatial and two temporal scales," *Journal of Function Spaces and Applications*, vol. 2012, Article ID 643458, 9 pages, 2012.

[15] G. Nguetseng, "A general convergence result for a functional related to the theory of homogenization," *SIAM Journal on Mathematical Analysis*, vol. 20, no. 3, pp. 608–623, 1989.

[16] G. Allaire, "Homogenization and two-scale convergence," *SIAM Journal on Mathematical Analysis*, vol. 23, no. 6, pp. 1482–1518, 1992.

[17] J. Persson, *Selected topics in homogenization [Doctoral thesis]*, Mid Sweden University, Östersund, Sweden, 2012.

[18] E. Zeidler, *Nonlinear Functional Analysis and Its Applications*, Springer, New York, NY, USA, 1990.

[19] S. Spagnolo, "Convergence of parabolic equations," *Bollettino della Unione Matematica Italiana. Serie VIII. Sezione B*, vol. 14, no. 2, pp. 547–568, 1977.

[20] N. Svanstedt, *G-convergence and homogenization of sequences of linear and nonlinear partial differential operators [Doctoral thesis]*, Department of Mathematics, Luleå University of Technology, Luleå, Sweden, 1992.

[21] F. Paronetto, "G-convergence of mixed type evolution operators," *Journal de Mathématiques Pures et Appliquées*, vol. 93, no. 4, pp. 361–407, 2010.

[22] L. E. Persson, L. Persson, N. Svanstedt, and J. Wyller, *The homogenization Method. An Introduction*, Studentlitteratur, Lund, Sweden, Chartwell-Bratt, Bromley, UK, 1993.