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Research Article

An Interior Inverse Problem for the Diffusion Operator

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An inverse problem for the diffusion operator on a finite interval with discontinuities conditions inside the interval is studied. We have shown that the potential function of the diffusion operator can be established uniquely by a set of values of eigenfunctions at the midpoint of the interval $(0, \pi)$ and one spectrum.

1. Introduction

In an inverse spectral problem, one seeks to determine coefficients in a differential operator from information about the spectrum of the operator, subject to specific side conditions. These kinds of problems arise in a remarkable variety of applications, for example, geophysics, seismology, seismic tomography, optics, and graph theory (see [1–7]).

We consider the boundary value problem $L(q_0(x), q_1(x), h, H)$ of the form

$$ly(x) := y''(x) + \left(\rho^2 - 2\rho q_1(x) - q_0(x)\right) y(x) = 0 \quad (1)$$

on the interval $0 < x < \pi$ with the boundary conditions

$$\begin{aligned} U(y) &:= y'(0) - hy(0) = 0, \\ V(y) &:= y'(\pi) + Hy(\pi) = 0 \end{aligned} \quad (2)$$

and with the jump conditions

$$\begin{aligned} y\left(\frac{\pi}{2} + 0\right) &= a_1 y\left(\frac{\pi}{2} - 0\right), \\ y'\left(\frac{\pi}{2} + 0\right) &= a_1^{-1} y'\left(\frac{\pi}{2} - 0\right) + a_2 y\left(\frac{\pi}{2} - 0\right), \end{aligned} \quad (3)$$

where ρ is the spectral parameter, $q_0(x)$ and $q_1(x)$ are real functions in $L^2[0, \pi]$, and the numbers h, H, a_1, and a_2 are real and $a_1 > 0$. Without loss of generality, we assume that

$$\int_0^\pi q_1(x)\, dx = 0. \quad (4)$$

Boundary value problems with discontinuities inside the interval are extensively studied [8, 9]. These kinds of problems are often appear in mathematics, mechanics, physics, and other branches of natural sciences. For example, discontinuous inverse problems appear in electronics for constructing parameters of heterogeneous electronic lines with desirable technical characteristics [10–12]. Also, boundary value problems with discontinuities in an interior point appear in geophysical models for oscillations of the Earth (see [13, 14]). Discontinuous inverse problems (in various formulations) have been considered in [15–17] and other works.

The inverse problem for interior spectral data of the differential operator consists in reconstruction of this operator from the known eigenvalues and some information on eigenfunctions at some internal point.

In the later years, interior inverse problems were studied by several authors [18–20]. In particular, research in [20] discussed the inverse problem for Sturm-Liouville operators with discontinuous boundary conditions and proved that the spectral data of parts of two spectra and some information on eigenfunctions at some interior point of the interval $(0, \pi)$ are sufficient to determine the potential.

The aim of this paper is to study the inverse problem of reconstructing the diffusion operator with discontinuous conditions on the basis of spectral data of a kind: one spectrum and some information on eigenfunctions at the midpoint of the interval $(0,\pi)$.

2. Auxiliary Assertions

Before giving the main results of this work, we will mention some results which will be needed later.

Let $\varphi(x,\rho)$, $C(x,\rho)$, and $S(x,\rho)$ be solutions of (1) under the initial conditions $\varphi(0,\rho) = C(0,\rho) = S'(0,\rho) = 1$, $\varphi'(0,\rho) = h$, $C'(0,\rho) = S(0,\rho) = 0$ and under the jump conditions (3). For each fixed $x \in [0,\pi]$, the functions $\varphi(x,\rho)$, $C(x,\rho)$, and $S(x,\rho)$ together with their derivatives with respect to x are entire in ρ.

Denote

$$\Delta(\rho) := -V(\varphi). \tag{5}$$

The function $\Delta(\rho)$ is called the characteristic function of L. The function $\Delta(\rho)$ is entire in ρ of order $1/2$, and its zeros coincide with the eigenvalues of L.

Denote

$$Q(x) := \int_0^x q_1(t)\,dt. \tag{6}$$

The functions

$$\cos(\rho x - Q(x)), \qquad \frac{\sin(\rho x - Q(x))}{\rho} \tag{7}$$

form a fundamental system of solutions for the differential equation

$$y'' + \frac{q_1'(x)}{\rho - q_1(x)} y' + (\rho - q_1(x))^2 y = 0. \tag{8}$$

We rewrite (1) in the form

$$\begin{aligned} y'' &+ \frac{q_1'(x)}{\rho - q_1(x)} y' + (\rho - q_1(x))^2 y \\ &= \left(q_0(x) + q_1^2(x)\right) y + \frac{q_1'(x)}{\rho - q_1(x)} y'. \end{aligned} \tag{9}$$

The function $y = C(x,\rho)$ is a solution of the Cauchy problem for (9) with the initial conditions $y(0) = 1$, $y'(0) = 0$.

By the method of variation of parameters, we deduce that

$$\begin{aligned} C(x,\rho) &= \cos(\rho x - Q(x)) \\ &+ \int_0^x \frac{\sin(\rho(x-t) - Q(x) + Q(t))}{\rho - q_1(t)} \\ &\quad \times \left(\left(q_0(t) + q_1^2(t)\right) C(t,\rho) + \frac{q_1'(t)}{\rho - q_1'(t)} C'(t,\rho) \right) dt. \end{aligned} \tag{10}$$

For $|\rho| \to \infty$,

$$C(x,\rho) = \cos(\rho x - Q(x)) + O\left(\frac{1}{\rho}\exp(|\operatorname{Im}\rho| x)\right). \tag{11}$$

Substituting this asymptotic into (10), we calculate

$$\begin{aligned} C(x,\rho) &= \cos(\rho x - Q(x)) \\ &+ \frac{\sin(\rho x - Q(x))}{2\rho} \int_0^x \left(q_0(t) + q_1^2(t)\right) dt \\ &+ \frac{1}{2\rho} \int_0^x \left(q_0(t) + q_1^2(t)\right) \\ &\quad \times \sin(\rho(x-2t) - Q(x) + 2Q(t))\, dt \\ &+ o\left(\frac{1}{\rho}\exp(|\operatorname{Im}\rho| x)\right). \end{aligned} \tag{12}$$

Differentiating (12) with respect to x, we get

$$\begin{aligned} C'(x,\rho) &= -\rho\sin(\rho x - Q(x)) \\ &+ \frac{\cos(\rho x - Q(x))}{2} \int_0^x \left(q_0(t) + q_1^2(t)\right) dt \\ &+ \frac{1}{2} \int_0^x \left(q_0(t) + q_1^2(t)\right) \\ &\quad \times \cos(\rho(x-2t) - Q(x) + 2Q(t))\, dt \\ &+ o(\exp(|\operatorname{Im}\rho| x)). \end{aligned} \tag{13}$$

Analogously, one can obtain for the function $S(x,\rho)$

$$\begin{aligned} S(x,\rho) &= \frac{\sin(\rho x - Q(x))}{\rho} - \frac{\cos(\rho x - Q(x))}{2\rho^2} \\ &\quad \times \int_0^x \left(q_0(t) + q_1^2(t)\right) dt \\ &\quad + \frac{1}{2\rho^2} \int_0^x \left(q_0(t) + q_1^2(t)\right) \\ &\quad \quad \times \cos(\rho(x-2t) - Q(x) + 2Q(t))\, dt \\ &\quad + o\left(\frac{1}{\rho^2}\exp(|\operatorname{Im}\rho| x)\right), \end{aligned} \tag{14}$$

$$S'(x,\rho) = \cos(\rho x - Q(x)) - \frac{\sin(\rho x - Q(x))}{2\rho}\int_0^x \left(q_0(t)+q_1^2(t)\right)dt - \frac{1}{2\rho}\int_0^x \left(q_0(t)+q_1^2(t)\right) \times \sin(\rho(x-2t) - Q(x) + 2Q(t))\,dt + o\left(\frac{1}{\rho}\exp(|\operatorname{Im}\rho|\,x)\right). \tag{15}$$

Since $\varphi(x,\rho) = C(x,\rho) + hS(x,\rho)$, by similar arguments in [8], one can calculate, for $|\rho| \to \infty$,

$$\varphi(x,\rho) = \cos(\rho x - Q(x)) + \left(h + \frac{1}{2}\int_0^x \left(q_0(t)+q_1^2(t)\right)dt\right) \times \frac{\sin(\rho x - Q(x))}{\rho} + o\left(\frac{1}{\rho}\exp(|\operatorname{Im}\rho|\,x)\right), \quad x < \frac{\pi}{2}, \tag{16}$$

$$\begin{aligned}\varphi(x,\rho) = {}& b_1\cos(\rho x - Q(x)) + b_2\cos\left(\rho(\pi - x) - 2Q\left(\frac{\pi}{2}\right) + Q(x)\right) \\ &+ M(x)\frac{\sin(\rho x - Q(x))}{2\rho} + N(x)\frac{\sin(\rho(\pi-x) - 2Q(\pi/2) + Q(x))}{2\rho} \\ &+ o\left(\frac{1}{\rho}\exp(|\operatorname{Im}\rho|\,x)\right), \quad x > \frac{\pi}{2},\end{aligned} \tag{17}$$

$$\varphi'(x,\rho) = -\rho\sin(\rho x - Q(x)) + \left(h + \frac{1}{2}\int_0^x \left(q_0(t)+q_1^2(t)\right)dt\right) \times \cos(\rho x - Q(x)) + o(\exp(|\operatorname{Im}\rho|\,x)), \quad x < \frac{\pi}{2}, \tag{18}$$

$$\begin{aligned}\varphi'(x,\rho) = {}& \rho\left(-b_1\sin(\rho x - Q(x)) + b_2\sin\left(\rho(\pi-x) - 2Q\left(\frac{\pi}{2}\right) + Q(x)\right)\right) \\ &+ M(x)\frac{\cos(\rho x - Q(x))}{2} - N(x)\frac{\cos(\rho(\pi-x) - 2Q(\pi/2) + Q(x))}{2} \\ &+ o(\exp(|\operatorname{Im}\rho|\,x)), \quad x > \frac{\pi}{2},\end{aligned} \tag{19}$$

where

$$\begin{aligned}M(x) &= b_1\left(2h + \int_0^x \left(q_0(t)+q_1^2(t)\right)dt\right) + a_2, \\ N(x) &= b_2\left(2h - \int_0^x \left(q_0(t)+q_1^2(t)\right)dt + 2\int_0^{\pi/2}\left(q_0(t)+q_1^2(t)\right)dt\right) - a_2.\end{aligned} \tag{20}$$

It follows from (6), (17), and (19) that

$$\Delta(\rho) = b_1\left[\rho\left(\sin\rho\pi + \frac{b_2}{b_1}\sin 2Q\left(\frac{\pi}{2}\right)\right) - \omega\frac{\cos\rho\pi}{2} + \frac{\omega_1}{2}\right] + o(\exp(|\operatorname{Im}\rho|\,x)), \tag{21}$$

where

$$\begin{aligned}\omega &= 2H + 2h + \int_0^\pi \left(q_0(t)+q_1^2(t)\right)dt + \frac{a_2}{b_1}, \\ \omega &= -\left[\frac{b_2}{b_1}\left(2H - 2h + \int_0^\pi \left(q_0(t)+q_1^2(t)\right)dt - 2\int_0^{\pi/2}\left(q_0(t)+q_1^2(t)\right)dt\right) + \frac{a_2}{b_1}\right]\cos 2Q\left(\frac{\pi}{2}\right).\end{aligned} \tag{22}$$

Using (21) by the well-known method (see, e.g., [3]), one has that, for $n \to \infty$,

$$\rho_n = n + \omega_0 + \frac{1}{2n\pi}\left(\omega + (-1)^{n-1}\omega_1\right) + o\left(\frac{1}{n}\right), \tag{23}$$

where

$$\omega_0 = (-1)^{n-1}\frac{b_2}{b_1\pi}\sin 2Q\left(\frac{\pi}{2}\right). \tag{24}$$

3. Main Result

In this section, we will give a uniqueness theorem. It says that the potential function $q_0(x)$ for a diffusion operator is uniquely determined by one spectrum and some information on eigenfunctions at the midpoint of the interval $(0,\pi)$. The technique we used is similar to those used in [6, 9].

Together with L, we consider a boundary value problem $\widetilde{L} = L(\widetilde{q_0}, q_1, h, H)$ of the same form but with a different coefficient $\widetilde{q_0}$. We agree that, if a certain symbol α denotes an object related to L, then $\widetilde{\alpha}$ will denote an analogous object related to $\widetilde{L}$.

Consider the problems

$$\varphi'' + \left(\rho^2 - 2\rho q_1(x) - q_0(x)\right)\varphi = 0, \tag{25}$$

with the initial conditions $\varphi(0) = 1$, $\varphi'(0) = h$ and

$$\widetilde{\varphi}'' + \left(\rho^2 - 2\rho q_1(x) - \widetilde{q}_0(x)\right)\widetilde{\varphi} = 0, \tag{26}$$

with the initial conditions $\widetilde{\varphi}(0) = 1$, $\widetilde{\varphi}'(0) = h$.

For $x \le \pi/2$, the following representation holds (see [21]):

$$\begin{aligned}\varphi(x,\rho) = {}& \cos(\rho x - \alpha(x)) + \int_0^x A(x,t)\cos\rho t\,dt \\ & + \int_0^x B(x,t)\sin\rho t\,dt,\end{aligned} \tag{27}$$

$$\begin{aligned}\widetilde{\varphi}(x,\rho) = {}& \cos(\rho x - \alpha(x)) + \int_0^x \widetilde{A}(x,t)\cos\rho t\,dt \\ & + \int_0^x \widetilde{B}(x,t)\sin\rho t\,dt,\end{aligned} \tag{28}$$

where

$$\begin{aligned}&\alpha(x) \\ &\quad = xp(0) + 2\int_0^x \left[A(\zeta,\zeta)\sin\alpha(\zeta) - B(\zeta,\zeta)\cos\alpha(\zeta)\right]d\zeta.\end{aligned} \tag{29}$$

The kernels $A(x,t)$ and $B(x,t)$ are the solution of the problem

$$\begin{gathered}\frac{\partial^2 A(x,t)}{\partial x^2} - 2p(x)\frac{\partial B(x,t)}{\partial t} - q(x)A(x,t) = \frac{\partial^2 A(x,t)}{\partial t^2}, \\ \frac{\partial^2 B(x,t)}{\partial x^2} + 2p(x)\frac{\partial A(x,t)}{\partial t} - q(x)B(x,t) = \frac{\partial^2 B(x,t)}{\partial t^2}, \\ q(x) = -p^2(x) \\ + 2\frac{d}{dx}\left[A(x,x)\cos\alpha(x) + B(x,x)\sin\alpha(x)\right], \\ A(0,0) = h, \qquad B(x,0) = 0, \\ \left.\frac{\partial A(x,t)}{\partial t}\right|_{t=o} = 0, \qquad \alpha(x) = \int_0^x p(t)\,dt.\end{gathered} \tag{30}$$

Hence,

$$\begin{aligned}\varphi\widetilde{\varphi} = \frac{1}{2}\Big\{ & 1 + \cos 2\left[\rho x - \alpha(x)\right] \\ & + \int_0^x H(x,\tau)\cos 2\left[\rho\tau - \alpha(\tau)\right]\Big\}\,d\tau,\end{aligned} \tag{31}$$

where

$$\begin{aligned}H(x,t) = 2\Big[& A(x,x-2\tau) + \widetilde{A}(x,x-2\tau) \\ & + B(x,x-2\tau) + \widetilde{B}(x,x-2\tau) \\ & + \int_{-x+2\tau}^{x} A(x,s)\widetilde{A}(x,s-2\tau)\,ds \\ & + \int_{-x+2\tau}^{x} B(x,s)\widetilde{B}(x,s-2\tau)\,ds \\ & + \int_{-x}^{x-2\tau} A(x,s)\widetilde{A}(x,s+2\tau)\,ds \\ & + \int_{-x}^{x-2\tau} B(x,s)\widetilde{B}(x,s+2\tau)\,ds\Big].\end{aligned} \tag{32}$$

The eigenvalues and the corresponding eigenfunctions of the problem L are denoted by ρ_n and $\varphi_n(x)$, $n \in N$, respectively.

Theorem 1. *If for any $n \in N$,*

$$\rho_n = \widetilde{\rho}_n, \qquad \left[\varphi_n, \widetilde{\varphi}_n\right]\left(\frac{\pi}{2} - 0\right) = 0, \tag{33}$$

then $q_0 = \widetilde{q}_0$ almost everywhere on $[0,\pi]$.

Proof. If we multiply (25) by $\widetilde{\varphi}$ and (26) by φ, and then subtract, after integrating on $[0,(\pi/2)-0]$, we obtain

$$\begin{aligned}&D(\rho) \\ &\quad := \int_0^{(\pi/2)-0} \left[\widetilde{q}(x) - q(x)\right]\varphi(x,\rho)\widetilde{\varphi}(x,\rho)\,dx \\ &\quad = \left[\widetilde{\varphi}'(x,\rho)\varphi(x,\rho) - \widetilde{\varphi}(x,\rho)\varphi'(x,\rho)\right]\Big|_0^{(\pi/2)-0}.\end{aligned} \tag{34}$$

By using the properties of φ and $\widetilde{\varphi}$, we conclude that the function $D(\rho)$ is an entire function. From condition of the theorem, together with the initial-value condition at 0, it follows that $D(\rho_n) = 0$, $n \in N$.

In addition, by (27), (28), and (34), for $0 < x < \pi$, we find

$$\left|D(\rho)\right| \le M, \tag{35}$$

where $M > 0$ is constant. Now, we define an entire function

$$\Psi(\rho) = \frac{D(\rho)}{\Delta(\rho)}. \tag{36}$$

From (21) and (35), it follows that

$$\left|\Psi(\rho)\right| = O\left(\frac{1}{\rho}\right), \tag{37}$$

for large $|\rho|$. So, for all ρ, from the Liouville theorem, we get

$$\Psi(\rho) = 0 \quad \text{or} \quad D(\rho) = 0. \tag{38}$$

Define $r(x) = \widetilde{q}(x) - q(x)$. Further substituting (31) into (34) and (38), we obtain

$$\frac{1}{2}\int_0^{(\pi/2)-0} r(x)\left\{[1+\cos 2(\rho x - \alpha(x))] + \int_0^x H(x,\tau)\cos 2(\rho\tau - \alpha(\tau))\,d\tau\right\}dx = 0, \tag{39}$$

which can be rewritten as

$$\int_0^{(\pi/2)-0} r(x)\,dx + \int_0^{(\pi/2)-0}\cos 2(\rho\tau - \alpha(\tau)) \times\left[r(\tau) + \int_\tau^{(\pi/2)-0} r(x)H(x,\tau)\,dx\right]d\tau = 0. \tag{40}$$

Letting $\rho \to \infty$ for real ρ, we conclude from Riemann-Lebesgue lemma that

$$\int_0^{(\pi/2)-0} r(x)\,dx = 0, \tag{41}$$

$$\int_0^{(\pi/2)-0}\cos 2(\rho\tau - \alpha(\tau)) \times\left[r(\tau) + \int_\tau^{(\pi/2)-0} r(x)H(x,\tau)\,dx\right]d\tau = 0. \tag{42}$$

Then, by using the trigonometric expansion of cos function and the completeness of the functions cos and sin, we obtain

$$r(\tau) + \int_\tau^{(\pi/2)-0} r(x)H(x,\tau)\,dx = 0, \quad 0 < \tau < \frac{\pi}{2}. \tag{43}$$

Since (43) is a Volterra integral equation, it has only trivial solution. Hence, we have obtained our result $r(x) = 0$ on $0 < x < \pi/2$; that is, $q_0(x) = \widetilde{q}_0(x)$ almost everywhere on $[0, \pi/2]$.

To prove that $q_0(x) = \widetilde{q}_0(x)$ almost everywhere on $[\pi/2, \pi]$, we will consider the supplementary problem $\widehat{L}$:

$$\begin{gathered}\varphi''(x) + \left(\rho^2 - 2\rho q_{11}(x) - q_{01}(x)\right)\varphi(x) = 0, \\ 0 < x < \pi, \\ U(\varphi) := \varphi'(0) - H\varphi(0) = 0, \\ V(\varphi) := \varphi'(\pi) + h\varphi(\pi) = 0, \\ \varphi\left(\frac{\pi}{2}+0\right) = a_1^{-1}\varphi\left(\frac{\pi}{2}-0\right), \\ \varphi'\left(\frac{\pi}{2}+0\right) = a_1\varphi'\left(\frac{\pi}{2}-0\right) + a_1a_2\varphi\left(\frac{\pi}{2}-0\right),\end{gathered} \tag{44}$$

where

$$q_{11}(x) = q_1(\pi - x), \qquad q_{01}(x) = q_0(\pi - x). \tag{45}$$

Note that, if $y(x)$ and $z(x)$ satisfy the matching conditions (3), then a direct calculation yields

$$[y,z]\left(\frac{\pi}{2}+0\right) = [y,z]\left(\frac{\pi}{2}-0\right). \tag{46}$$

The assumption of Theorem 1 and (46) imply that

$$[\varphi_n, \widetilde{\varphi}_n]\left(\frac{\pi}{2}+0\right) = 0. \tag{47}$$

A direct calculation implies that $\widehat{\varphi}_n(x) := \varphi_n(\pi - x)$ is the solution to the supplementary problem $\widehat{L}$ and $\widehat{\varphi}_n((\pi/2)-0) := \varphi_n((\pi/2)+0)$. Thus, for the supplementary problem $\widehat{L}$, the assumption conditions in the theorem are still satisfied.

If we repeat the previous arguments, then this yields $r(\pi - x) = 0$ on $0 < x < \pi/2$; that is, $q_0(x) = \widetilde{q}_0(x)$ almost everywhere on $[\pi/2, \pi]$. The proof of the theorem is finished. □

We suggest to extend this work for fractional differential equations and local fractional differential equations [22–24] when the order of $y(x)$ is noninteger in (1).

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