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Research Article

Bregman Asymptotic Pointwise Nonexpansive Mappings in Banach Spaces

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We first introduce a new class of mappings called Bregman asymptotic pointwise nonexpansive mappings and investigate the existence and the approximation of fixed points of such mappings defined on a nonempty, bounded, closed, and convex subset C of a real Banach space E. Without using the original Opial property of a Banach space E, we prove weak convergence theorems for the sequences produced by generalized Mann and Ishikawa iteration processes for Bregman asymptotic pointwise nonexpansive mappings in a reflexive Banach space E. Our results are applicable in the function spaces L^p, where $1 < p < \infty$ is a real number.

1. Introduction

Throughout this paper, we denote the set of real numbers and the set of positive integers by $\mathbb{R}$ and $\mathbb{N}$, respectively. Let E be a Banach space with the norm $\|\cdot\|$ and the dual space E^*. For any $x \in E$, we denote the value of $x^* \in E^*$ at x by $\langle x, x^*\rangle$. Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in E; we denote the strong convergence of $\{x_n\}_{n\in\mathbb{N}}$ to $x \in E$ as $n \to \infty$ by $x_n \to x$ and the weak convergence by $x_n \rightharpoonup x$. The modulus δ of convexity of E is denoted by

$$\delta(\epsilon) = \inf\left\{1 - \frac{\|x + y\|}{2} : \|x\| \le 1, \|y\| \le 1, \|x - y\| \ge \epsilon\right\}, \tag{1}$$

for every ϵ with $0 \le \epsilon \le 2$. A Banach space E is said to be *uniformly convex* if $\delta(\epsilon) > 0$ for every $\epsilon > 0$. Let $S_E = \{x \in E : \|x\| = 1\}$. The norm of E is said to be *Gâteaux differentiable* if for each $x, y \in S_E$, the limit

$$\lim_{t\to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{2}$$

exists. In this case, E is called *smooth*. If the limit (2) is attained uniformly for all $x, y \in S_E$, then E is called *uniformly smooth*. The Banach space E is said to be *strictly convex* if $\|(x+y)/2\| < 1$ whenever $x, y \in S_E$ and $x \neq y$. It is well known that E is uniformly convex if and only if E^* is uniformly smooth. It is also known that if E is reflexive, then E is strictly convex if and only if E^* is smooth; for more details, see [1, 2].

Let C be a nonempty subset of E. Let $T : C \to E$ be a mapping. We denote the set of fixed points of T by $F(T)$; that is, $F(T) = \{x \in C : Tx = x\}$. A mapping $T : C \to E$ is said to be *nonexpansive* if $\|Tx - Ty\| \le \|x - y\|$ for all $x, y \in C$. A mapping $T : C \to E$ is said to be *quasi-nonexpansive* if $F(T) \neq \emptyset$ and $\|Tx - y\| \le \|x - y\|$ for all $x \in C$ and $y \in F(T)$. The nonexpansivity plays an important role in the study of *Mann iteration* [3] for finding fixed points of a mapping $T : C \to C$. Recall that the Mann iteration is given by the following formula:

$$x_{n+1} = \gamma_n T x_n + (1 - \gamma_n) x_n, \quad x_1 \in C. \tag{3}$$

Here, $\{\gamma_n\}_{n\in\mathbb{N}}$ is a sequence of real numbers in $[0, 1]$ satisfying some appropriate conditions. A more general iteration is the *Ishikawa iteration* [4], given by

$$y_n = \beta_n T x_n + (1 - \beta_n) x_n,$$

$$x_{n+1} = \gamma_n T y_n + (1 - \gamma_n) x_n, \tag{4}$$

where the sequences $\{\beta_n\}_{n\in\mathbb{N}}$ and $\{\gamma_n\}_{n\in\mathbb{N}}$ satisfy some appropriate conditions. When all $\beta_n = 0$, the Ishikawa iteration reduces to the classical Mann iteration. Construction of fixed points of nonexpansive mappings via Mann's and Ishikawa's algorithms [3] has been extensively investigated in the literature (see, e.g., [5] and the references therein). A powerful tool in deriving weak or strong convergence of iterative sequences is due to Opial [6]. A Banach space E is said to satisfy the *Opial property* [6] if for any weakly convergent sequence $\{x_n\}_{n\in\mathbb{N}}$ in E with weak limit x, we have

$$\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|, \tag{5}$$

for all y in E with $y \neq x$. It is well known that all Hilbert spaces, all finite dimensional Banach spaces, and the Banach spaces l^p $(1 \le p < \infty)$ satisfy the Opial property. However, not every Banach space satisfies the Opial property; see, for example, [7].

Let E be a smooth, strictly convex, and reflexive Banach space and let J be the normalized duality mapping of E. Let C be a nonempty, closed, and convex subset of E. The generalized projection Π_C from E onto C [8] is defined and denoted by

$$\Pi_C(x) = \operatorname*{argmin}_{y\in C} \phi(y, x), \tag{6}$$

where $\phi(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2$. Let C be a nonempty, closed, and convex subset of a smooth Banach space E, and let T be a mapping from C into itself.

1.1. Some Facts about Gradients. For any convex function $g : E \to (-\infty, +\infty]$ we denote the domain of g by $\operatorname{dom} g = \{x \in E : g(x) < \infty\}$. For any $x \in \operatorname{int}\operatorname{dom} g$ and any $y \in E$, we denote by $g^o(x, y)$ the *right-hand derivative* of g at x in the direction y; that is,

$$g^o(x, y) = \lim_{t\downarrow 0} \frac{g(x + ty) - g(x)}{t}. \tag{7}$$

The function g is said to be *Gâteaux differentiable* at x if $\lim_{t\to 0}(g(x+ty)-g(x))/t$ exists for any y. In this case $g^o(x, y)$ coincides with $\nabla g(x)$, the value of the *gradient* ∇g of g at x. The function g is said to be *Gâteaux differentiable* if it is Gâteaux differentiable everywhere. The function g is said to be *Fréchet differentiable* at x if this limit is attained uniformly in $\|y\| = 1$. The function g is Fréchet differentiable at $x \in E$ (see, e.g., [9, page 13] or [10, page 508]) if for all $\epsilon > 0$, there exists $\delta > 0$ such that $\|y - x\| \le \delta$ implies that

$$|g(y) - g(x) - \langle y - x, \nabla g(x)\rangle| \le \epsilon \|y - x\|. \tag{8}$$

The function g is said to be *Fréchet differentiable* if it is Fréchet differentiable everywhere. It is well known that if a continuous convex function $g : E \to \mathbb{R}$ is Gâteaux differentiable, then ∇g is norm-to-weak* continuous (see, e.g., [9, Proposition 1.1.10]). Also, it is known that if g is Fréchet differentiable, then ∇g is norm-to-norm continuous (see, [10, page 508]). The mapping ∇g is said to be *weakly sequentially continuous* if $x_n \rightharpoonup x$ as $n \to \infty$ implies that $\nabla g(x_n) \rightharpoonup^* \nabla g(x)$ as $n \to \infty$ (for more details, see [9, Theorem 3.2.4] or [10, page 508]). The function g is said to be *strongly coercive* if

$$\lim_{\|x_n\|\to\infty} \frac{g(x_n)}{\|x_n\|} = \infty. \tag{9}$$

It is also said to be *bounded on bounded subsets of* E if $g(U)$ is bounded for each bounded subset U of E. Finally, g is said to be *uniformly Fréchet differentiable* on a subset X of E if the limit (7) is attained uniformly for all $x \in X$ and $\|y\| = 1$.

Let E be a reflexive Banach space. For any proper, lower semicontinuous, and convex function $g : E \to (-\infty, +\infty]$, the *conjugate function* g^* of g is defined by

$$g^*(x^*) = \sup_{x\in E} \{\langle x, x^*\rangle - g(x)\}, \tag{10}$$

for all $x^* \in E^*$. It is well known that $g(x) + g^*(x^*) \ge \langle x, x^*\rangle$ for all $(x, x^*) \in E \times E^*$. It is also known that $(x, x^*) \in \partial g$ is equivalent to

$$g(x) + g^*(x^*) = \langle x, x^*\rangle. \tag{11}$$

Here, ∂g is the subdifferential of g [11, 12]. We also know that if $g : E \to (-\infty, +\infty]$ is a proper, lower semicontinuous, and convex function, then $g^* : E^* \to (-\infty, +\infty]$ is a proper, weak* lower semicontinuous, and convex function; see [2] for more details on convex analysis.

1.2. Some Facts about Bregman Distances. Let E be a Banach space and let E^* be the dual space of E. Let $g : E \to \mathbb{R}$ be a convex and Gâteaux differentiable function. Then the *Bregman distance* [13, 14] corresponding to g is the function $D_g : E \times E \to \mathbb{R}$ defined by

$$D_g(x, y) = g(x) - g(y) - \langle x - y, \nabla g(y)\rangle, \quad \forall x, y \in E. \tag{12}$$

It is clear that $D_g(x, y) \ge 0$ for all $x, y \in E$. In that case when E is a smooth Banach space, setting $g(x) = \|x\|^2$ for all $x \in E$, we obtain that $\nabla g(x) = 2Jx$ for all $x \in E$ and hence $D_g(x, y) = \phi(x, y)$ for all $x, y \in E$.

Let E be a Banach space and let C be a nonempty and convex subset of E. Let $g : E \to \mathbb{R}$ be a convex and Gâteaux differentiable function. Then, we know from [15] that for $x \in E$ and $x_0 \in C$, $D_g(x_0, x) = \min_{y\in C} D_g(y, x)$ if and only if

$$\langle y - x_0, \nabla g(x) - \nabla g(x_0)\rangle \le 0, \quad \forall y \in C. \tag{13}$$

Furthermore, if C is a nonempty, closed, and convex subset of a reflexive Banach space E and $g : E \to \mathbb{R}$ is a strongly coercive Bregman function, then for each $x \in E$, there exists a unique $x_0 \in C$ such that

$$D_g(x_0, x) = \min_{y\in C} D_g(y, x). \tag{14}$$

The *Bregman projection* proj_C^g from E onto C is defined by $\mathrm{proj}_C^g(x) = x_0$ for all $x \in E$. It is also well known that proj_C^g has the following property:

$$D_g\left(y, \mathrm{proj}_C^g x\right) + D_g\left(\mathrm{proj}_C^g x, x\right) \leq D_g\left(y, x\right), \quad (15)$$

for all $y \in C$ and $x \in E$ (see [9] for more details).

For any bounded subset A of a reflexive Banach space E, we denote the Bregman diameter of A by

$$B\operatorname{diam}(A) := \sup\left\{D_g\left(x, y\right) : x, y \in A\right\} < \infty. \quad (16)$$

1.3. Some Facts about Uniformly Convex Functions. Let E be a Banach space and let $B_s := \{z \in E : \|z\| \leq s\}$ for all $s > 0$. Then a function $g : E \to \mathbb{R}$ is said to be *uniformly convex on bounded subsets of* E ([16, Pages 203, 221]) if $\rho_s(t) > 0$ for all $s, t > 0$, where $\rho_s : [0, +\infty) \to [0, \infty]$ is defined by

$$\rho_s(t) = \inf_{x,y\in B_s, \|x-y\|=t, \alpha\in(0,1)} \left(\alpha g(x) + (1-\alpha) g(y) - g(\alpha x + (1-\alpha) y)\right) \times (\alpha(1-\alpha))^{-1}, \quad (17)$$

for all $t \geq 0$. The function ρ_s is called the gauge of uniform convexity of g. The function g is also said to be *uniformly smooth on bounded subsets of* E ([16, Pages 207, 221]) if $\lim_{t\downarrow 0}(\sigma_s(t)/t) = 0$ for all $s > 0$, where $\sigma_s : [0, +\infty) \to [0, \infty]$ is defined by

$$\sigma_s(t) = \sup_{x\in B_s, y\in S_E, \alpha\in(0,1)} \left(\alpha g(x + (1-\alpha) t y) + (1-\alpha) g(x - \alpha t y) - g(x)\right) \times (\alpha(1-\alpha))^{-1}, \quad (18)$$

for all $t \geq 0$. The function g is said to be *uniformly convex* if the function $\delta_g : [0, +\infty) \to [0, +\infty]$, defined by

$$\delta_g(t) := \sup\left\{\frac{1}{2} g(x) + \frac{1}{2} g(y) - g\left(\frac{x+y}{2}\right) : \|y - x\| = t\right\}, \quad (19)$$

satisfies that $\lim_{t\downarrow 0}(\sigma_s(t)/t) = 0$.

Remark 1. Let E be a Banach space, let $s > 0$ be a constant, and let $g : E \to \mathbb{R}$ be a convex function which is uniformly convex on bounded subsets. Then

$$g(\alpha x + (1-\alpha) y) \leq \alpha g(x) + (1-\alpha) g(y) - \alpha(1-\alpha)\rho_s(\|x - y\|), \quad (20)$$

for all $x, y \in B_s := \{z \in E : \|z\| \leq s\}$ and $\alpha \in (0, 1)$, where ρ_s is the gauge of uniform convexity of g.

Definition 2. Let E be a reflexive Banach space and let $g : E \to \mathbb{R}$ be a convex, continuous, strongly coercive, and Gâteaux differentiable function which is bounded on bounded subsets and uniformly convex on bounded subsets of E. Let C be a nonempty, bounded, closed, and convex subset of E. A mapping $T : C \to E$ is said to be *Bregman asymptotic pointwise nonexpansive* if there exists a sequence of mappings $\theta_n : C \to [0, \infty)$ such that

$$D_g\left(T^n x, T^n y\right) \leq \theta_n(x) D_g(x, y), \quad \forall x, y \in C,$$
$$\limsup_{n\to\infty} \theta_n(x) \leq 1, \quad \forall x \in C. \quad (21)$$

Denoting $a_n(x) = \max\{\theta_n(x), 1\}$, we note that without loss of generality we can assume that T is Bregman asymptotic pointwise nonexpansive if

$$D_g\left(T^n x, T^n y\right) \leq a_n(x) D_g(x, y), \quad \forall x, y \in C,\ n \in \mathbb{N}, \quad (22)$$

$$\lim_{n\to\infty} a_n(x) = 1, \quad a_n(x) \geq 1, \quad \forall x \in C,\ n \in \mathbb{N}. \quad (23)$$

Define $b_n(x) = a_n(x) - 1$. In view of (23), we obtain

$$\lim_{n\to\infty} b_n(x) = 0. \quad (24)$$

Next, we denote by $\mathscr{BT}(C)$ the class of all Bregman asymptotic pointwise nonexpansive mappings $T : C \to C$.

Imposing some restrictions on the behavior of a_n and b_n, we can define the following subclass of Bregman asymptotic pointwise nonexpansive mappings.

Definition 3. Let C and $\mathscr{BT}(C)$ be as in Definition 2. We define $\mathscr{BT}_r(C)$ as a class of all $T \in \mathscr{BT}(C)$ such that

$$\sum_{n=1}^{\infty} b_n(x) < \infty, \quad \forall x \in C,\ a_n \text{ is a bounded function for every } n \text{ in } \mathbb{N}. \quad (25)$$

Kirk and Xu [17] studied the existence of fixed points of asymptotic pointwise nonexpansive mappings with respect to the norm of a Banach space E. Recently, Kozlowski [18] proved weak and strong convergence theorems for asymptotic pointwise nonexpansive mappings in a Banach space. To see some other related works, we refer the reader to [19, 20].

In this paper, we first investigate the approximation of fixed points of a new class of Bregman asymptotic pointwise nonexpansive mappings defined on a nonempty, bounded, closed, and convex subset C of a real Banach space E. Without using the Opial property of a Banach space E, we prove weak convergence theorems for the sequences produced by generalized Mann and Ishikawa iteration processes. Our results improve and generalize many known results in the current literature; see, for example, [18, 21].

2. Preliminaries

In this section, we begin by recalling some preliminaries and lemmas which will be used in the sequel.

Definition 4 (see [10]). Let E be a Banach space. The function $g : E \to \mathbb{R}$ is said to be a Bregman function if the following conditions are satisfied:

(1) g is continuous, strictly convex, and Gâteaux differentiable;

(2) the set $\{y \in E : D_g(x, y) \leq r\}$ is bounded for all $x \in E$ and $r > 0$.

Lemma 5 (see [9, 16]). *Let E be a reflexive Banach space and $g : E \to \mathbb{R}$ a strongly coercive Bregman function. Then*

(1) *$\nabla g : E \to E^*$ is one-to-one, onto, and norm-to-weak* continuous;*

(2) *$\langle x - y, \nabla g(x) - \nabla g(y)\rangle = 0$ if and only if $x = y$;*

(3) *$\{x \in E : D_g(x, y) \leq r\}$ is bounded for all $y \in E$ and $r > 0$;*

(4) *$\operatorname{dom} g^* = E^*$, g^* is Gâteaux differentiable and $\nabla g^* = (\nabla g)^{-1}$.*

We know the following two results; see [16, Proposition 3.6.4].

Theorem 6. *Let E be a reflexive Banach space and $g : E \to \mathbb{R}$ a convex function which is bounded on bounded subsets of E. Then the following assertions are equivalent:*

(1) *g is strongly coercive and uniformly convex on bounded subsets of E;*

(2) *$\operatorname{dom} g^* = E^*$, g^* is bounded on bounded subsets and uniformly smooth on bounded subsets of E^*;*

(3) *$\operatorname{dom} g^* = E^*$, g^* is Fréchet differentiable and ∇g^* is uniformly norm-to-norm continuous on bounded subsets of E^*.*

Theorem 7. *Let E be a reflexive Banach space and $g : E \to \mathbb{R}$ a continuous convex function which is strongly coercive. Then the following assertions are equivalent:*

(1) *g is bounded on bounded subsets and uniformly smooth on bounded subsets of E;*

(2) *g^* is Fréchet differentiable and ∇g^* is uniformly norm-to-norm continuous on bounded subsets of E^*;*

(3) *$\operatorname{dom} g^* = E^*$, g^* is strongly coercive and uniformly convex on bounded subsets of E^*.*

Let E be a Banach space and let $g : E \to \mathbb{R}$ be a convex and Gâteaux differentiable function. Then the Bregman distance [13, 14] does not satisfy the well known properties of a metric, but it does have the following important property, which is called the *three point identity* [22]:

$$D_g(x, z) = D_g(x, y) + D_g(y, z) + \langle x - y, \nabla g(y) - \nabla g(z)\rangle, \quad \forall x, y, z \in E. \tag{26}$$

In particular, it can easily be seen that

$$D_g(x, y) = -D_g(y, x) + \langle y - x, \nabla g(y) - \nabla g(x)\rangle, \quad \forall x, y \in E. \tag{27}$$

Indeed, by letting $z = x$ in (26) and taking into account that $D_g(x, x) = 0$, we get the desired result.

Lemma 8 (see [23]). *Let E be a Banach space and $g : E \to \mathbb{R}$ a Gâteaux differentiable function which is uniformly convex on bounded subsets of E. Let $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$ be bounded sequences in E. Then the following assertions are equivalent:*

(1) $\lim_{n\to\infty} D_g(x_n, y_n) = 0$;

(2) $\lim_{n\to\infty} \|x_n - y_n\| = 0$.

Lemma 9 (see [10, 24]). *Let E be a reflexive Banach space, $g : E \to \mathbb{R}$ a strongly coercive Bregman function, and V the function defined by*

$$V(x, x^*) = g(x) - \langle x, x^*\rangle + g^*(x^*), \quad x \in E, \ x^* \in E^*. \tag{28}$$

Then the following assertions hold:

(1) *$D_g(x, \nabla g^*(x^*)) = V(x, x^*)$ for all $x \in E$ and $x^* \in E^*$;*

(2) *$V(x, x^*) + \langle \nabla g^*(x^*) - x, y^*\rangle \leq V(x, x^* + y^*)$ for all $x \in E$ and $x^*, y^* \in E^*$.*

Let C and D be nonempty subsets of a real Banach space E with $D \subset C$. A mapping $R_D : C \to D$ is said to be *sunny* if

$$R_D(R_D x + t(x - R_D x)) = R_D x, \tag{29}$$

for each $x \in E$ and $t \geq 0$. A mapping $R_D : C \to D$ is said to be a *retraction* if $R_D x = x$ for each $x \in C$.

Lemma 10 (see [25]). *Suppose $\{r_k\}_{k\in\mathbb{N}}$ is a bounded sequence of real numbers and $\{d_{k,n}\}_{k,n\in\mathbb{N}}$ is a doubly index sequence of real numbers which satisfy*

$$\limsup_{k\to\infty} \limsup_{n\to\infty} d_{k,n} \leq 0, \qquad r_{k,n} \leq r_k + d_{k,n}, \tag{30}$$

for each $k, n \geq 1$. Then $\{r_k\}_{k\in\mathbb{N}}$ converges to an $r \in \mathbb{R}$.

Let E be a reflexive Banach space and let $g : E \to (-\infty, +\infty]$ be an admissible function, that is, a proper, lower-semicontinuous, convex, and Gâteaux differentiable function. Let C be a nonempty, closed, and convex subset of E and let $\{x_n\}_{n\in\mathbb{N}}$ be a bounded sequence in E. For any x in E, we set

$$\operatorname{Br}(\{x_n\}, x) = \limsup_{n\to\infty} D_g(x_n, x). \tag{31}$$

The *Bregman asymptotic radius* of $\{x_n\}_{n\in\mathbb{N}}$ relative to C is defined by

$$\operatorname{Br}(\{x_n\}, C) = \inf\{\operatorname{Br}(\{x_n\}, x) : x \in C\}. \tag{32}$$

The *Bregman asymptotic center* of $\{x_n\}_{n\in\mathbb{N}}$ relative to C is the set

$$\operatorname{BA}(\{x_n\}, C) = \{x \in C : \operatorname{Br}(\{x_n\}, x) = \operatorname{Br}(\{x_n\}, C)\}. \tag{33}$$

The following Bregman Opial-like inequality has been proved in [26]. It is worth mentioning that the Bregman Opial-like inequality is different from the ordinary Opial inequality [6] and can be applied in uniformly convex Banach spaces.

Lemma 11 (see [26]). *Let E be a Banach space and let $g : E \to (-\infty, +\infty]$ be a proper strictly convex function so that it is Gâteaux differentiable on* int dom g. *Suppose $\{x_n\}_{n\in\mathbb{N}}$ is a sequence in* dom g *such that $x_n \rightharpoonup v$ for some $v \in$* int dom g. *Then*

$$\limsup_{n\to\infty} D_g(x_n, v) < \limsup_{n\to\infty} D_g(x_n, y), \quad \forall y \in \text{ int dom } g \text{ with } y \neq v. \tag{34}$$

Theorem 12 (see [16]). *Let $g : E \to (-\infty, +\infty]$ be a function. Then the following assertions are equivalent:*

(1) *g is convex and lower semicontinuous;*

(2) *g is convex and weakly lower semicontinuous;*

(3) epi (g) *is convex and closed;*

(4) epi (g) *is convex and weakly closed,*

where epi $(g) = \{(x,t) \in E \times \mathbb{R} : g(x) \leq t\}$ *denotes the epigraph of g.*

3. Fixed Point Theorems and Demiclosedness Principle

Proposition 13. *Let E be a reflexive Banach space and let $g : E \to \mathbb{R}$ be a strongly admissible function which is bounded on bounded subsets and uniformly convex on bounded subsets of E. Let C be a nonempty, bounded, closed, and convex subset of E and let $T \in \mathcal{BT}_r(C)$. Then T has a fixed point. Moreover, $F(T)$ is closed and convex.*

Proof. We first show that $F(T)$ is nonempty. Let x in C be fixed. We define a function $f : C \to [0, \infty)$ by

$$f(y) = \limsup_{n\to\infty} D_g(y, T^n x), \quad y \in C. \tag{35}$$

In view of Remark 1, it is easy to see that f is convex. Since g is continuous, by Theorem 12 we conclude that the Bregman distance D_g is weakly lower-semicontinuous in the first argument. Since E is a uniformly convex Banach space and C is weakly compact, in view of [1] there exists a unique point $z \in C$ such that

$$f(z) = \min_{y\in C} f(y). \tag{36}$$

We show that $\{T^n z\}_{n\in\mathbb{N}}$ is convergent in norm. To this end, let $s_1 = \sup\{\|T^n z\|, \|z\| : n \in \mathbb{N}\}$ and $\rho_{s_1} : E \to \mathbb{R}$ be the gauge of uniform convexity of g. For any $k, m \in \mathbb{N}$, put $u_{k,m} = (1/2)T^k z + (1/2)T^m z$. Then we have $u_{k,m} \in C$. In view of Remark 1, we obtain

$$\begin{aligned}
&D_g(u_{k,m}, T^{k+m+n}x) \\
&\quad = g(u_{k,m}) - g(T^{k+m+n}x) \\
&\qquad - \langle u_{k,m} - T^{k+m+n}x, \nabla g(T^{k+m+n}x)\rangle \\
&\quad = g\left(\frac{1}{2}T^k z + \frac{1}{2}T^m z\right) - g(T^{k+m+n}x) \\
&\qquad - \left\langle \frac{1}{2}T^k z + \frac{1}{2}T^m z - T^{k+m+n}x, \nabla g(T^{k+m+n}x)\right\rangle \\
&\quad \leq \frac{1}{2}g(T^k z) + \frac{1}{2}g(T^m z) \\
&\qquad - \frac{1}{4}\rho_{s_1}(\|T^k z - T^m z\|) - g(T^{k+m+n}x) \\
&\qquad - \frac{1}{2}\langle T^k z - T^{k+m+n}x, \nabla g(T^{k+m+n}x)\rangle \\
&\qquad - \frac{1}{2}\langle T^m z - T^{k+m+n}x, \nabla g(T^{k+m+n}x)\rangle \\
&\quad = \frac{1}{2}\left[g(T^k z) - g(T^{k+m+n}x) - \langle T^k z - T^{k+m+n}x, \nabla g(T^{k+m+n}x)\rangle\right] \\
&\qquad + \frac{1}{2}\left[g(T^m z) - g(T^{k+m+n}x) - \langle T^m z - T^{k+m+n}x, \nabla g(T^{k+m+n}x)\rangle\right] \\
&\qquad - \frac{1}{4}\rho_{s_1}(\|T^k z - T^m z\|) \\
&\quad = \frac{1}{2}D_g(T^k z, T^{k+m+n}x) + \frac{1}{2}D_g(T^m z, T^{k+m+n}x) \\
&\qquad - \frac{1}{4}\rho_{s_1}(\|T^k z - T^m z\|) \\
&\quad \leq \frac{1}{2}a_k(z) D_g(z, T^{m+n}x) + \frac{1}{2}a_k(z) D_g(z, T^{k+n}x) \\
&\qquad - \frac{1}{4}\rho_{s_1}(\|T^k z - T^m z\|).
\end{aligned} \tag{37}$$

Applying to both sides of the above inequalities $\limsup_{n\to\infty}$ we obtain

$$f(u_{k,m}) \leq \frac{1}{2}[a_k(z) + a_m(z)] f(z) - \frac{1}{4}\rho_{s_1}(\|T^k z - T^m z\|). \tag{38}$$

This, together with $f(z) \leq f(u_{k,m})$, implies that

$$\begin{aligned}
&\rho_{s_1}(\|T^k z - T^m z\|) \\
&\quad \leq 4\left[\frac{1}{2}[a_k(z) + a_m(z)] f(z) - f(u_{k,m})\right] \\
&\quad \leq 4\left[\frac{1}{2}[a_k(z) + a_m(z)] f(z) - f(z)\right].
\end{aligned} \tag{39}$$

Letting $k, m \to \infty$ in (39) we conclude that

$$\lim_{k,m\to\infty} \rho_{s_1}(\|T^k z - T^m z\|) = 0. \tag{40}$$

From the properties of ρ_{s_1}, we deduce that $\lim_{k,m\to\infty}\|T^k z - T^m z\| = 0$. Thus, $\{T^k z\}_{k\in\mathbb{N}}$ is a norm-Cauchy sequence and hence convergent. Let

$$v = \lim_{n\to\infty} T^n z. \quad (41)$$

Since T is a Bregman asymptotic pointwise nonexpansive mapping, we have, for all $n \in \mathbb{N}$,

$$D_g\left(T^{n+1}z, Tv\right) \le a_1\left(T^n z\right) D_g\left(T^n z, v\right). \quad (42)$$

Letting $n \to \infty$ in (42), we conclude that $Tv = v$. This shows that $F(T) \neq \emptyset$.

Now, we show that $F(T)$ is closed. Let $\{p_n\}_{n\in\mathbb{N}}$ be a sequence in $F(T)$ such that $p_n \to p$ as $n \to \infty$. Then we have that $\{p_n\}_{n\in\mathbb{N}}$ is a bounded sequence in E. We claim that $p \in F(T)$. Since g is continuous, we conclude that $g(p_n) \to g(p)$ as $n \to \infty$. This implies that

$$\begin{aligned} D_g\left(p_n, p\right) &= g\left(p_n\right) - g\left(p\right) - \left\langle p_n - p, \nabla g\left(p\right)\right\rangle \\ &\le \left|g\left(p_n\right) - g\left(p\right)\right| + \left\|p_n - p\right\| \left\|\nabla g\left(p\right)\right\| \longrightarrow 0 \\ &\qquad (n \longrightarrow \infty). \end{aligned} \quad (43)$$

In view of the definition of T, we obtain

$$D_g\left(Tp, p_n\right) \le a_n\left(p\right) D_g\left(p, p_n\right) \longrightarrow 0 \quad (n \longrightarrow \infty). \quad (44)$$

This implies that

$$\begin{aligned} 0 &= \lim_{n\to\infty} D_g\left(p_n, Tp\right) \\ &= \lim_{n\to\infty} \left[g\left(p_n\right) - g\left(Tp\right) - \left\langle p_n - Tp, \nabla g\left(Tp\right)\right\rangle\right] \\ &= D_g\left(p, Tp\right). \end{aligned} \quad (45)$$

It follows from Lemma 8 that $Tp = p$. Thus we have $p \in F(T)$.

Let us show that $F(T)$ is convex. For any $p, q \in F(T)$, $t \in (0,1)$, and $n \in \mathbb{N}$, we set $x = tp + (1-t)q$ and $e_n(p,q) = \max\{a_n(p), a_n(q)\}$. We prove that $x \in F(T)$. By the definition of Bregman distance (see (12)), we get

$$\begin{aligned} D_g\left(x, T^n x\right) &= g\left(x\right) - g\left(T^n x\right) \\ &\quad - \left\langle x - T^n x, \nabla g\left(T^n x\right)\right\rangle \\ &= g\left(x\right) - g\left(T^n x\right) \\ &\quad - \left\langle tp + \left(1-t\right)q - T^n x, \nabla g\left(T^n x\right)\right\rangle \\ &= g\left(x\right) - g\left(T^n x\right) - t\left\langle p - T^n x, \nabla g\left(T^n x\right)\right\rangle \\ &\quad - \left(1-t\right)\left\langle q - T^n x, \nabla g\left(T^n x\right)\right\rangle \\ &\quad + tg\left(p\right) + \left(1-t\right)g\left(q\right) \\ &\quad - \left[tg\left(p\right) + \left(1-t\right)g\left(q\right)\right] \\ &= g\left(x\right) + t\left[g\left(p\right) - g\left(T^n x\right)\right. \\ &\qquad \left. - \left\langle p - T^n x, \nabla g\left(T^n x\right)\right\rangle\right] \\ &\quad + \left(1-t\right)\left[g\left(q\right) - g\left(T^n x\right)\right. \\ &\qquad \left. - \left\langle q - T^n x, \nabla g\left(T^n x\right)\right\rangle\right] \\ &= g\left(x\right) + tD_g\left(p, T^n x\right) \\ &\quad + \left(1-t\right)D_g\left(q, T^n x\right) \\ &\quad - \left[tg\left(p\right) + \left(1-t\right)g\left(q\right)\right] \\ &\le g\left(x\right) + ta_n\left(p\right)D_g\left(p, x\right) \\ &\quad + \left(1-t\right)a_n\left(q\right)D_g\left(q, x\right) \\ &\quad - tg\left(p\right) - \left(1-t\right)g\left(q\right) \\ &\le g\left(x\right) + te_n\left(p,q\right)D_g\left(p, x\right) \\ &\quad + \left(1-t\right)e_n\left(p,q\right)D_g\left(q, x\right) \\ &\quad - tg\left(p\right) - \left(1-t\right)g\left(q\right) \\ &= g\left(x\right) + e_n\left(p,q\right) \\ &\quad \times \left[t\left(g\left(p\right) - g\left(x\right) - \left\langle p - x, \nabla g\left(x\right)\right\rangle\right)\right. \\ &\qquad \left. + \left(1-t\right)\left(g\left(q\right) - g\left(x\right) - \left\langle q - x, \nabla g\left(x\right)\right\rangle\right)\right] \\ &\quad - tg\left(p\right) - \left(1-t\right)g\left(q\right) \\ &= g\left(x\right) + e_n\left(p,q\right) \\ &\quad \times \left[-g\left(x\right) - \left\langle t\left(p - x\right), \nabla g\left(x\right)\right\rangle\right. \\ &\qquad \left. - \left\langle \left(1-t\right)\left(q - x\right), \nabla g\left(x\right)\right\rangle\right] \\ &\quad + e_n\left(p,q\right)\left[tg\left(p\right) + \left(1-t\right)g\left(q\right)\right] \\ &\quad - tg\left(p\right) - \left(1-t\right)g\left(q\right) \\ &= g\left(x\right) - e_n\left(p,q\right)g\left(x\right) - e_n\left(p,q\right) \\ &\quad \times \left[\left\langle tp + \left(1-t\right)q - x, \nabla g\left(x\right)\right\rangle\right] \\ &\quad + e_n\left(p,q\right)\left[tg\left(p\right) + \left(1-t\right)g\left(q\right)\right] \\ &\quad - tg\left(p\right) - \left(1-t\right)g\left(q\right) \\ &= \left(e_n\left(p,q\right) - 1\right)\left(-g\left(x\right)\right) \\ &\quad + \left(e_n\left(p,q\right) - 1\right)tg\left(p\right) \\ &\quad + \left(e_n\left(p,q\right) - 1\right)\left(1-t\right)g\left(q\right) \\ &= \left(e_n\left(p,q\right) - 1\right) \\ &\quad \times \left[-g\left(x\right) + tg\left(p\right) + \left(1-t\right)g\left(q\right)\right]. \end{aligned} \quad (46)$$

This implies that $\lim_{n\to\infty} D_g(x, T^n x) = 0$. Thus for each $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$D_g(x, T^n x) \le \epsilon, \quad \forall n \ge n_0. \tag{47}$$

This means that the sequence $\{D_g(x, T^n x)\}_{n\in\mathbb{N}}$ is bounded. In view of Definition 4, we conclude that the sequence $\{T^n x\}_{n\in\mathbb{N}}$ is bounded. Then, by Lemma 8, we obtain $\lim_{n\to\infty}\|x - T^n x\| = 0$. Thus we have $T^{n+1}x \to x$; that is, $T(T^n x) \to x$. On the other hand, in view of three-point identity (see (26)), we deduce that

$$\begin{aligned} D_g(x, Tx) &= D_g\left(x, T^{n+1}x\right) + D_g\left(T^{n+1}x, Tx\right) \\ &\quad + \left\langle x - T^{n+1}x, \nabla g\left(T^{n+1}x\right) - \nabla g(Tx)\right\rangle \\ &\le D_g\left(x, T^{n+1}x\right) + a_1(T^n x) D_g(T^n x, x) \\ &\quad + \left\|x - T^{n+1}x\right\| \left\|\nabla g\left(T^{n+1}x\right) - \nabla g(Tx)\right\|. \end{aligned} \tag{48}$$

Letting $n \to \infty$ in the above inequalities we deduce that $D_g(x, Tx) = 0$ and hence by Lemma 8 we conclude that $Tx = x$, which completes the proof. □

Lemma 14. *Let E be a reflexive Banach space and let $g : E \to \mathbb{R}$ be a convex, continuous, strongly coercive, and Gâteaux differentiable function which is bounded on bounded subsets and uniformly convex on bounded subsets of E. Let C be a nonempty, bounded, closed, and convex subset of E and let $T \in \mathcal{BT}_r(C)$. If $\{x_n\}_{n\in\mathbb{N}}$ is a sequence of C such that $\lim_{n\to\infty}\|Tx_n - x_n\| = 0$, then, for any $m \in \mathbb{N}$, $\lim_{n\to\infty}\|T^m x_n - x_n\| = 0$.*

Proof. In view of (25), there exists a finite constant $M_1 > 0$ such that

$$\sum_{j=1}^{m-1} \sup\left\{a_j(x) : x \in C\right\} \le M_1. \tag{49}$$

It follows from three-point identity (see (26)) that

$$\begin{aligned} D_g(T^m x_n, x_n) &= D_g\left(T^m x_n, T^{m-1}x_n\right) \\ &\quad + D_g\left(T^{m-1}x_n, x_n\right) \\ &\quad + \left\langle T^m x_n - T^{m-1}x_n, \nabla g\left(T^{m-1}x_n\right) \right. \\ &\qquad \left. -\nabla g(x_n)\right\rangle \\ &\le a_{m-1}(x_n) D_g(Tx_n, x_n) \\ &\quad + D_g\left(T^{m-1}x_n, T^{m-2}x_n\right) \\ &\quad + D_g\left(T^{m-2}x_n, x_n\right) \\ &\quad + \left\langle T^{m-1}x_n - T^{m-2}x_n, \nabla g\left(T^{m-2}x_n\right) \right. \\ &\qquad \left. -\nabla g(x_n)\right\rangle \\ &\quad + M_2\left\|T^m x_n - T^{m-1}x_n\right\| \\ &\quad \vdots \\ &\le \left(\sum_{j=1}^{m-1} a_j(x_n) + 1\right) D_g(Tx_n, x_n) \\ &\quad + 2M_2 \sum_{j=1}^{m-1} \left\|T^m x_n - T^{m-1}x_n\right\|, \end{aligned} \tag{50}$$

where $M_2 := \sup\{\|\nabla g(T^j x_n)\| : j = 1, 2, \ldots, m \text{ and } n \in \mathbb{N}\}$. This, together with Lemma 8, implies that $\lim_{n\to\infty}\|T^m x_n - x_n\| = 0$. This completes the proof. □

Theorem 15. *Let E be a reflexive Banach space and let $g : E \to \mathbb{R}$ be a convex, continuous, strongly coercive, and Gâteaux differentiable function which is bounded on bounded subsets and uniformly convex and uniformly smooth on bounded subsets of E. Let C be a nonempty, bounded, closed, and convex subset of E and let $T \in \mathcal{BT}_r(C)$. If $\{x_n\}_{n\in\mathbb{N}}$ converges weakly to z and $\lim_{n\to\infty}\|Tx_n - x_n\| = 0$, then $Tz = z$. That is, $I - T$ is demiclosed at zero, where I is the identity mapping on E.*

Proof. Let the function $\phi : E \to [0, \infty)$ be defined by

$$\phi(x) = \limsup_{n\to\infty} D_g(x_n, x) \quad (x \in E). \tag{51}$$

For any $m, n \in \mathbb{N}$ with $m > 2$, in view of three-point identity (see (26)), we obtain

$$\begin{aligned} D_g(T^m x_n, x) &= D\left(T^m x_n, T^{m-1}x_n\right) + D_g\left(T^{m-1}x_n, x\right) \\ &\quad + \left\langle T^m x_n - T^{m-1}x_n, \nabla g\left(T^{m-1}x_n\right) \right. \\ &\qquad \left. -\nabla g(x)\right\rangle \\ &\le a_{m-1}(x_n) D_g(Tx_n, x_n) \\ &\quad + D\left(T^{m-1}x_n, T^{m-2}x_n\right) + D_g\left(T^{m-2}x_n, x\right) \\ &\quad + \left\langle T^{m-1}x_n - T^{m-2}x_n, \nabla g\left(T^{m-2}x_n\right)\right) \\ &\qquad \left. -\nabla g(x_n)\right\rangle \\ &\quad + M_2'\left\|T^m x_n - T^{m-1}x_n\right\| \\ &\quad \vdots \\ &\le \left(\sum_{j=1}^{m-1} a_j(x) + 1\right) D_g(Tx_n, x_n) \\ &\quad + D_g(Tx_n, x) + M_2' \sum_{j=1}^{m-1} \left\|T^m x_n - T^{m-1}x_n\right\|, \end{aligned} \tag{52}$$

where $M_2' := \sup\{\|\nabla g(T^j x_n)\| : j = 1, 2, \ldots, m \text{ and } n \in \mathbb{N}\}$. This, together with Lemma 8, implies that

$$\limsup_{n\to\infty} D_g(T^m x_n, x) \le \limsup_{n\to\infty} D_g(x_n, x) = \phi(x). \quad (53)$$

In view of (25), there exists a finite constant $M_3 > 0$ such that

$$\sum_{j=1}^{m-1} \sup\{a_j(x) : x \in C\} \le M_3, \quad (54)$$

where $M_4 := \sup\{\|\nabla g(T^{m-1} x_n)\| : m = 1, 2, \ldots, m \text{ and } n \in \mathbb{N}\}$. This, together with Lemma 8, implies that $\lim_{n\to\infty}\|T^m x_n - x_n\| = 0$. Employing Lemma 8, we conclude that

$$\begin{aligned}\phi(x) &= \limsup_{n\to\infty} D_g(x_n, x)\\ &\le \limsup_{n\to\infty} D_g(x_n, T^m x_n)\\ &\quad + \limsup_{n\to\infty} D_g(T^m x_n, x)\\ &\quad + \limsup_{n\to\infty} M_2' \|x_n - T^m x_n\|\\ &\le \limsup_{n\to\infty} D_g(T^m x_n, x) \le \phi(x).\end{aligned} \quad (55)$$

This means that

$$\phi(x) = \limsup_{n\to\infty} D_g(T^m x_n, x). \quad (56)$$

Since T is a Bregman asymptotic pointwise nonexpansive mapping, it follows that

$$\phi(T^m x) \le a_m(x_n)\phi(x), \quad \forall x \in C. \quad (57)$$

In view of (57), we deduce that

$$\limsup_{m\to\infty} \phi(T^m z) \le \phi(z). \quad (58)$$

By the Bregman Opial-like inequality ((34)) we obtain that for any $x \ne z$

$$\phi(z) = \limsup_{n\to\infty} D_g(x_n, z) < \limsup_{n\to\infty} D_g(x_n, x) = \phi(x). \quad (59)$$

This shows that $\phi(z) = \inf\{\phi(x) : x \in C\}$. Thus we have

$$\phi(T^m z) = \phi(z). \quad (60)$$

Put $s_2 = \sup\{\|\nabla g(z)\|, \|\nabla g(T^m z)\| : m \in \mathbb{N}\}$, $z_m = \nabla g^*((1/2)\nabla g(z) + (1/2)\nabla g(T^m z))$, and $u_m = \operatorname{proj}_C^g(z_m)$ for all $m \in \mathbb{N}$. Then we have $u_m \in C$. In view of Remark 1, we obtain a continuous strictly increasing convex function $\rho_{s_2}^* : [0, +\infty) \to [0, +\infty)$ with $\rho_{s_2}^*(0) = 0$ such that

$$\begin{aligned} D_g(x_n, u_m) &= D_g(x_n, \operatorname{proj}_C^g(z_m)) \le D_g(x_n, z_m)\\ &= g(x_n) - g(z_m) - \langle x_n - z_m, \nabla g(z_m)\rangle\\ &= g(x_n) + g^*(z_m) - \langle z_m, \nabla g(z_m)\rangle\\ &\quad - \langle x_n, \nabla g(z_m)\rangle + \langle z_m, \nabla g(z_m)\rangle\\ &= g(x_n) + g^*\left(\frac{1}{2}\nabla g(z) + \frac{1}{2}\nabla g(T^m z)\right)\\ &\quad - \left\langle x_n, \frac{1}{2}\nabla g(z) + \frac{1}{2}\nabla g(T^m z)\right\rangle\\ &\le \frac{1}{2}g(x_n) + \frac{1}{2}g(x_n) + \frac{1}{2}g^*(\nabla g(x_n))\\ &\quad + \frac{1}{2}g^*(\nabla g(T^m z)) - \frac{1}{4}\rho_{s_2}^*(\|\nabla g(z) - \nabla g(T^m z)\|)\\ &\quad - \frac{1}{2}\langle x_n, \nabla g(x_n)\rangle - \frac{1}{2}\langle x_n, \nabla g(T^m z)\rangle\\ &= \frac{1}{2}[g(x_n) + g^*(\nabla g(z)) - \langle x_n, \nabla g(x_n)\rangle]\\ &\quad + \frac{1}{2}[g(x_n) + g^*(\nabla g(T^m z)) - \langle x_n, \nabla g(T^m z)\rangle]\\ &\quad - \frac{1}{4}\rho_{s_2}^*(\|\nabla g(z) - \nabla g(T^m z)\|)\\ &= \frac{1}{2}[g(x_n) - g(z) + \langle x_n, \nabla g(z)\rangle - \langle x_n, \nabla g(z)\rangle]\\ &\quad + \frac{1}{2}[g(x_n) - g(T^m z) + \langle T^m z, \nabla g(T^m z)\rangle\\ &\qquad - \langle x_n, \nabla g(T^m z)\rangle]\\ &\quad - \frac{1}{4}\rho_{s_2}^*(\|\nabla g(z) - \nabla g(T^m z)\|)\\ &= \frac{1}{2}D_g(x_n, z) + \frac{1}{2}D_g(x_n, T^m z)\\ &\quad - \frac{1}{4}\rho_{s_2}^*(\|\nabla g(z) - \nabla g(T^m z)\|). \end{aligned} \quad (61)$$

Applying to both sides $\limsup_{n\to\infty}$ and remembering that $\phi(z) = \inf\{\phi(x) : x \in C\}$ we obtain

$$\phi(z) \le \frac{1}{2}\phi(z) + \frac{1}{2}\phi(T^m z) - \frac{1}{4}\rho_{s_2}^*(\|\nabla g(z) - \nabla g(T^m z)\|). \quad (62)$$

This implies that

$$\rho_{s_2}^*(\|\nabla g(z) - \nabla g(T^m z)\|) \le 2\phi(T^m z) - 2\phi(z). \quad (63)$$

Letting $m \to \infty$ in (63) we conclude that

$$\lim_{m\to\infty} \rho_{s_2}^* \left(\|\nabla g(z) - \nabla g(T^m z)\|\right) = 0. \quad (64)$$

From the properties of $\rho_{s_2}^*$, we deduce that $\lim_{m\to\infty}\|\nabla g(z) - \nabla g(T^m z)\|$. Since ∇g^* is uniformly norm-to-norm continuous on bounded subsets of E^*, we arrive at $\lim_{m\to\infty}\|z - T^m z\| = 0$. Thus we have $T^{m+1}z \to z$; that is, $T(T^m z) \to z$. On the other hand, in view of three-point identity (see (26)), we deduce that

$$\begin{aligned} D_g(z, Tz) &= D_g\left(z, T^{m+1}z\right) + D_g\left(T^{m+1}z, Tz\right) \\ &\quad + \left\langle z - T^{m+1}z, \nabla g\left(T^{m+1}z\right) - \nabla g(Tz)\right\rangle \\ &\le D_g\left(z, T^{m+1}z\right) + a_1(T^m z) D_g(T^m z, z) \\ &\quad + \left\|z - T^{m+1}z\right\| \left\|\nabla g\left(T^{m+1}z\right) - \nabla g(Tz)\right\|. \end{aligned} \quad (65)$$

Letting $m \to \infty$ in the above inequalities we deduce that $D_g(z, Tz) = 0$ and hence by Lemma 8 we conclude that $Tz = z$, which completes the proof. □

4. Weak Convergence Theorems of Generalized Mann Iteration Process

In this section, we prove weak convergence theorems of generalized Mann iteration process in a reflexive Banach space.

Definition 16. Let E be a reflexive Banach space and let $g : E \to \mathbb{R}$ be a convex, continuous, strongly coercive, and Gâteaux differentiable function which is bounded on bounded subsets and uniformly convex on bounded subsets of E. Let C be a nonempty, bounded, closed, and convex subset of E and let $T \in \mathcal{BT}_r(C)$. Let $\{n_k\}_{k\in\mathbb{N}}$ be an increasing sequence in $\mathbb{N}$ and let $\{\gamma_k\}_{k\in\mathbb{N}} \subset (0,1)$ be such that $\liminf_{k\to}\gamma_k(1-\gamma_k) > 0$. The generalized Mann iteration process generated by the mapping T, the sequence $\{\gamma_k\}_{k\in\mathbb{N}}$, and the sequence $\{n_k\}_{k\in\mathbb{N}}$, denoted by $gM(T, \{\gamma_k\}_{k\in\mathbb{N}}, \{n_k\}_{k\in\mathbb{N}})$, is defined by the following iterative formula:

$$\begin{gathered} x_{k+1} = \gamma_k T^{n_k}(x_k) + (1-\gamma_k) x_k, \\ \text{where } x_1 \in C \text{ is chosen arbitrarily.} \end{gathered} \quad (66)$$

Definition 17. We say that a generalized Mann iteration process $gM(T, \{\gamma_k\}_{k\in\mathbb{N}}, \{n_k\}_{k\in\mathbb{N}})$ is well defined if

$$\limsup_{k\to\infty} a_{n_k}(x_k) = 1. \quad (67)$$

Remark 18. Observe that by the definition of Bregman asymptotic pointwise nonexpansive, $\lim_{k\to\infty}a_{n_k}(x) = 1$ for every $x \in C$. Hence we can always select a subsequence $\{a_{n_k}\}_{k\in\mathbb{N}}$ of $\{a_n\}_{n\in\mathbb{N}}$ such that (67) holds. In other words, by a suitable choice of $\{n_k\}_{k\in\mathbb{N}}$ we can always make $gM(T, \{\gamma_k\}_{k\in\mathbb{N}}, \{n_k\}_{k\in\mathbb{N}})$ well defined.

We will prove a series of lemmas necessary for the proof of the generalized Mann process convergence theorem.

Lemma 19. *Let E be a reflexive Banach space and let $g : E \to \mathbb{R}$ be a convex, continuous, strongly coercive, and Gâteaux differentiable function which is bounded on bounded subsets and uniformly convex on bounded subsets of E. Let C be a nonempty, bounded, closed, and convex subset of E. Let $T \in \mathcal{BT}_r(C)$ and let $\{n_k\}_{k\in\mathbb{N}} \subset \mathbb{N}$. Let $\{\gamma_k\}_{k\in\mathbb{N}} \subset (0,1)$ such that $\liminf_{k\to\infty}\gamma_k(1-\gamma_k) > 0$. Let $w \in F(T)$ and let $gM(T, \{\gamma_k\}_{k\in\mathbb{N}}, \{n_k\}_{k\in\mathbb{N}})$ be a generalized Mann process. Then there exists $\lambda \in \mathbb{R}$ such that $\lim_{k\to\infty}D(x_k, w) = \lambda$.*

Proof. Let $w \in F(T)$ be arbitrary chosen. In view of (66), we obtain

$$\begin{aligned} D_g(x_{k+1}, w) &\le \gamma_k D_g\left(T^{n_k}(x_k), w\right) \\ &\quad + (1-\gamma_k) D_g(x_k, w) \\ &\le \gamma_k D_g\left(T^{n_k}(x_k), T^{n_k}(w)\right) \\ &\quad + (1-\gamma_k) D_g(x_k, w) \\ &\le \gamma_k \left(1 + b_{n_k}(x_k)\right) D_g(x_k, w) \\ &\quad + (1-\gamma_k) D_g(x_k, w) \\ &\le \gamma_k b_{n_k}(x_k) D_g(x_k, w) + D_g(x_k, w) \\ &\le b_{n_k}(x_k) B \operatorname{diam}(C) + D_g(x_k, w). \end{aligned} \quad (68)$$

This implies that for every $n \in \mathbb{N}$,

$$D_g(x_{k+n}, w) \le D_g(x_k, w) + B \operatorname{diam}(C) \sum_{i=k}^{k+n-1} b_{n_i}(x_i). \quad (69)$$

Put $r_p = D_g(x_p, w)$ for every $p \in \mathbb{N}$ and $d_{k,n} = B\operatorname{diam}(C)\sum_{i=k}^{k+n-1} b_{n_i}(x_i)$. Since $T \in \mathcal{BT}_r(C)$, we obtain that $\limsup_{k\to\infty}\limsup_{n\to\infty}d_{k,n} = 0$. In view of Lemma 11, there exists $\lambda \in \mathbb{R}$ such that $\lim_{k\to\infty}D_g(x_k, w) = \lambda$. This completes the proof. □

Lemma 20. *Let E be a reflexive Banach space and let $g : E \to \mathbb{R}$ be a convex, continuous, strongly coercive, and Gâteaux differentiable function which is bounded on bounded subsets and uniformly convex on bounded subsets of E. Let C be a nonempty, bounded, closed, and convex subset of E. Let $T \in \mathcal{BT}_r(C)$ and let $\{n_k\}_{k\in\mathbb{N}} \subset \mathbb{N}$. Let $\{\gamma_k\}_{k\in\mathbb{N}} \subset (0,1)$ such that $\liminf_{k\to\infty}\gamma_k(1-\gamma_k) > 0$ and let $gM(T, \{\gamma_k\}_{k\in\mathbb{N}}, \{n_k\}_{k\in\mathbb{N}})$ be a generalized Mann process. Then*

$$\lim_{k\to\infty} D_g\left(T^{n_k}(x_k), x_k\right) = 0, \qquad \lim_{k\to\infty} D_g(x_{k+1}, x_k) = 0. \quad (70)$$

Proof. In view of Proposition 13, we conclude that $F(T) \neq \emptyset$. Let $w \in F(T)$ be fixed. It follows from Lemma 19 that there exists $\lambda \in \mathbb{R}$ such that $\lim_{k\to\infty}D_g(x_k, w) = \lambda$. Let $s_3 = \sup\{\|T^{n_k}(x_k)\|, \|x_k\| : k \in \mathbb{N}\}$ and let $\rho_{s_3} : E \to \mathbb{R}$ be the gauge of uniform convexity of g. By the definition of T, we obtain

$$\begin{aligned}
D_g(x_{k+1}, w) &= D_g(\gamma_k T^{n_k}(x_k) + (1-\gamma_k) x_k, w) \\
&\le \gamma_k D_g(T^{n_k}(x_k), w) + (1-\gamma_k) D_g(x_k, w) \\
&= \gamma_k D_g(T^{n_k}(x_k), T^{n_k}(w)) + (1-\gamma_k) D_g(x_k, w) \\
&\le \gamma_k a_{n_k}(x_k) D_g(x_k, w) + (1-\gamma_k) D_g(x_k, w) \\
&\quad - \gamma_k (1-\gamma_k) \rho_{s_3}(\|T^{n_k}(x_k) - x_k\|) \\
&= \gamma_k \left(a_{n_k}(x_k) - 1\right) D_g(x_k, w) + D_g(x_k, w) \\
&\quad - \gamma_k (1-\gamma_k) \rho_{s_3}(\|T^{n_k}(x_k) - x_k\|).
\end{aligned} \tag{71}$$

This implies that

$$\begin{aligned}
\gamma_k (1-\gamma_k) \rho_{s_3}(\|T^{n_k}(x_k) - x_k\|) &\le \left(a_{n_k}(x_k) - 1\right) D_g(x_k, w) \\
&\quad + D_g(x_k, w) - D_g(x_{k+1}, w).
\end{aligned} \tag{72}$$

Letting $k \to \infty$ in (72) we conclude that

$$\lim_{k\to\infty} \rho_{s_3}(\|T^{n_k}(x_k) - x_k\|) = 0. \tag{73}$$

From the properties of ρ_{s_3}, we deduce that $\lim_{k\to\infty}\|T^{n_k}(x_k) - x_k\| = 0$. Employing Lemma 8, we conclude that

$$\lim_{k\to\infty} D_g(T^{n_k}(x_k), x_k) = 0, \qquad \lim_{k\to\infty} D_g(x_{k+1}, x_k) = 0. \tag{74}$$

This completes the proof. □

In the next lemma, we prove that under suitable assumption the sequence $\{x_k\}_{k\in\mathbb{N}}$ becomes an approximate fixed point sequence, which will provide an important step in the proof of the generalized Mann iteration process convergence. First, we need to recall the following notions.

Definition 21. A strictly increasing sequence $\{n_i\}_{i\in\mathbb{N}} \subset \mathbb{N}$ is called quasiperiodic if the sequence $\{n_{i+1} - n_i\}_{i\in\mathbb{N}}$ is bounded or equivalently if there exists a number $p \in \mathbb{N}$ such that any block of consecutive natural numbers must contain a term of the sequence $\{n_i\}_{i\in\mathbb{N}}$. The smallest of such numbers will be called a quasi period of $\{n_i\}_{i\in\mathbb{N}}$.

Lemma 22. *Let E be a reflexive Banach space and let $g : E \to \mathbb{R}$ be a convex, continuous, strongly coercive, and Gâteaux differentiable function which is bounded on bounded subsets and uniformly convex on bounded subsets of E. Let C be a nonempty, bounded, closed, and convex subset of E. Let $T \in \mathcal{BT}_r(C)$ and let $\{n_k\}_{k\in\mathbb{N}} \subset \mathbb{N}$. Let $\{\gamma_k\}_{k\in\mathbb{N}} \subset (0,1)$ such that $\liminf_{k\to\infty}\gamma_k(1-\gamma_k) > 0$. Let $\{n_k\}_{k\in\mathbb{N}} \subset \mathbb{N}$ be such that the generalized Mann process $gM(T, \{\gamma_k\}_{k\in\mathbb{N}}, \{n_k\}_{k\in\mathbb{N}})$ is well defined. If, in addition, the set of indices $\mathcal{J} = \{j \in \mathbb{N} : n_{j+1} = 1 + n_j\}$ is quasi-periodic, then $\{x_k\}_{k\in\mathbb{N}}$ is an approximate fixed point sequence; that is,*

$$\lim_{k\to\infty} \|T(x_k) - x_k\| = 0. \tag{75}$$

Proof. In view of (66), we have

$$x_{k+1} - x_k = \gamma_k (T^{n_k}(x_k) - x_k). \tag{76}$$

This, together with Lemmas 8 and 20, implies that

$$\lim_{k\to\infty} \|x_{k+1} - x_k\| = 0. \tag{77}$$

In view of Lemma 8, we conclude that

$$\lim_{m\to\infty} D_g(x_{k+1}, x_k) = 0. \tag{78}$$

Let $p \in \mathbb{N}$ be a quasi-period of $\mathcal{J}$. We first prove that $\|x_k - T(x_k)\| \to 0$ as $k \to \infty$ through $\mathcal{J}$. Since $n_{k+1} = n_k + 1$ for such k, we obtain

$$\begin{aligned}
D_g(T^{n_{k+1}}(x_{k+1}), T^{n_{k+1}}(x_k)) &= D_g\left(T^{n_k+1}(x_{k+1}), T^{n_k+1}(x_k)\right) \\
&\le a_{n_k+1}(x_{k+1}) D_g(x_{k+1}, x_k), \\
D_g(TT^{n_k}(x_k), T(x_k)) &\le M_5 D_g(T^{n_k}(x_k), x_k),
\end{aligned} \tag{79}$$

where $M_5 = \sup\{a_1(x) : x \in C\} < \infty$. This implies that

$$\begin{aligned}
&\lim_{k\to\infty} D_g\left(T^{n_k+1}(x_{k+1}), T^{n_k+1}(x_k)\right) = 0, \\
&\lim_{k\to\infty} D_g(TT^{n_k}(x_k), T(x_k)) = 0.
\end{aligned} \tag{80}$$

It follows from Lemma 8 that

$$\begin{aligned}
&\lim_{k\to\infty} \left\|T^{n_k+1}(x_{k+1}) - T^{n_k+1}(x_k)\right\| = 0, \\
&\lim_{k\to\infty} \|TT^{n_k}(x_k) - T(x_k)\| = 0.
\end{aligned} \tag{81}$$

On the other hand, we have

$$\begin{aligned}
\|x_k - T(x_k)\| &\le \|x_k - x_{k+1}\| + \left\|x_{k+1} - T^{n_k+1}(x_{k+1})\right\| \\
&\quad + \left\|T^{n_{k+1}}(x_{k+1}) - T^{n_{k+1}}(x_k)\right\| \\
&\quad + \|TT^{n_k}(x_k) - T(x_k)\|.
\end{aligned} \tag{82}$$

Thus, we obtain $\|x_k - T(x_k)\| \to 0$ as $k \to \infty$ through $\mathcal{J}$. In view of Lemma 8, we conclude that $D_g(x_k, T(x_k)) \to 0$ as $k \to \infty$ through $\mathcal{J}$. Now, let $\epsilon > 0$ be fixed. It follows from $D_g(x_k, T(x_k)) \to 0$ as $k \to \infty$ through $\mathcal{J}$ that there exists $N_0 \in \mathbb{N}$ such that

$$D_g(x_k, T(x_k)) < \frac{\epsilon}{3}, \quad \forall k \ge N_0. \tag{83}$$

Since $\mathcal{J}$ is quasi-periodic, for any $k \in \mathbb{N}$ there exists $j_k \in \mathcal{J}$ such that $|k - j_k| \le p$. Assume that $k - p \le j_k \le k$ (the proof of the other case is identical). Since T is a Bregman M_6-Lipschitzian mapping where $M_6 = \sup\{a_1(x) : x \in C\}$, there exists $0 < \delta < \epsilon/3$ such that

$$D_g(Tx, Ty) < \frac{\epsilon}{3} \quad \text{if } D_g(x, y) < \delta. \tag{84}$$

In view of (66) and (83), there exists $N_1 \in \mathbb{N}$ such that

$$\|x_{k+1} - x_k\| < \frac{\delta}{p}, \quad \forall k \geq N_1. \tag{85}$$

We also obtain

$$\begin{aligned} \|x_k - x_{j_k}\| &\leq \|x_k - x_{k-1}\| + \|x_{k-1} - x_{k-2}\| \\ &\quad + \cdots + \|x_{j_{k-1}} - x_{j_k}\| \leq p\frac{\delta}{p} = \delta. \end{aligned} \tag{86}$$

This implies that

$$\lim_{k\to\infty} D_g(x_k, x_{j_k}) = 0, \qquad \lim_{k\to\infty} D_g(T(x_k), T(x_{j_k})) = 0. \tag{87}$$

If follows from (77)-(78) that

$$\begin{aligned} \|x_k - Tx_k\| &\leq \|x_k - x_{j_k}\| + \|x_{j_k} - T(x_{j_k})\| \\ &\quad + \|T(x_{j_k}) - T(x_k)\| \leq \delta + \frac{\epsilon}{3} + \frac{\epsilon}{3} < \epsilon. \end{aligned} \tag{88}$$

This completes the proof. □

Theorem 23. *Let E be a reflexive Banach space and let $g : E \to \mathbb{R}$ be a convex, continuous, strongly coercive and Gâteaux differentiable function which is bounded on bounded subsets and uniformly convex on bounded subsets. Let C be a nonempty, bounded, closed and convex subset of E. Let $T \in \mathscr{BT}_r(C)$ and let $\{n_k\}_{k\in\mathbb{N}} \subset \mathbb{N}$. Let $\{\gamma_k\}_{k\in\mathbb{N}} \subset (0,1)$ such that $\liminf_{k\to\infty}\gamma_k(1-\gamma_k) > 0$. Let $\{n_k\}_{k\in\mathbb{N}} \subset \mathbb{N}$ be such that the generalized Mann process $gM(T, \{\gamma_k\}_{k\in\mathbb{N}}, \{n_k\}_{k\in\mathbb{N}})$ is well defined. If, in addition, the set of indices $\mathscr{J} = \{j \in \mathbb{N} : k_{j+1} = 1 + k_j\}$ is quasi-periodic, then the sequence $\{x_k\}_{k\in\mathbb{N}}$ generated by $gM(T, \{\gamma_k\}_{k\in\mathbb{N}}, \{n_k\}_{k\in\mathbb{N}})$ converges weakly to a fixed point of T.*

Proof. In view of Lemma 22, we obtain

$$\lim_{k\to\infty} \|Tx_k - x_k\| = 0. \tag{89}$$

Let $y, z \in C$ be weak cluster points of the sequence $\{x_k\}_{k\in\mathbb{N}}$. Then there exist subsequences $\{y_k\}_{k\in\mathbb{N}}$ and $\{z_k\}_{k\in\mathbb{N}}$ of $\{x_k\}_{k\in\mathbb{N}}$ such that $y_k \rightharpoonup y$ and $z_k \rightharpoonup z$ as $k \to \infty$. In view of (89) and Theorem 15, we conclude that $Ty = y$ and $Tz = z$. It follows from Lemma 19 that there exist real numbers λ_1 and λ_2 such that

$$\lim_{k\to\infty} D_g(y_k, y) = \lambda_1, \qquad \lim_{k\to\infty} D_g(z_k, z) = \lambda_2. \tag{90}$$

We claim that $y = z$. Assume on the contrary that $y \neq z$. By the Bregman Opial-like property we obtain

$$\begin{aligned} \lambda_1 &= \limsup_{k\to\infty} D_g(y_k, y) < \limsup_{k\to\infty} D_g(y_k, z) \\ &= \limsup_{k\to\infty} D_g(z_k, z) < \limsup_{k\to\infty} D_g(z_k, y) \\ &= \limsup_{k\to\infty} D_g(y_k, y) = \lambda_1. \end{aligned} \tag{91}$$

This is a contradiction and hence there exists $w \in C$ such that $x_k \rightharpoonup w$ as $k \to \infty$. Since C is weakly sequentially compact, such a weak cluster point w is unique. In view of Theorem 15, we conclude that $T(w) = w$, which completes the proof. □

5. Weak Convergence of Generalized Ishikawa Iteration Process

The two-step Ishikawa iteration process is a generalization of the one-step Mann iteration process. The Ishikawa iteration process provides more flexibility in defining the algorithm parameters which is important from numerical implementation perspective.

Definition 24. Let E be a reflexive Banach space and let $g : E \to \mathbb{R}$ be a convex, continuous, strongly coercive, and Gâteaux differentiable function which is bounded on bounded subsets and uniformly convex on bounded subsets. Let C be a nonempty, bounded, closed, and convex subset of E. Let $T \in \mathscr{BT}_r(C)$ and let $\{n_k\}_{k\in\mathbb{N}} \subset \mathbb{N}$ be an increasing sequence. Let $\{\beta_k\}_{k\in\mathbb{N}} \subset (0,1)$ and $\{\gamma_k\}_{k\in\mathbb{N}} \subset (0,1)$ be sequences of real numbers such that $\liminf_{k\to\infty}\beta_k(1-\beta_k) > 0$ and $\liminf_{k\to\infty}\gamma_k(1-\gamma_k) > 0$. The generalized Ishikawa iteration process generated by the mapping T, the sequences $\{\beta_k\}_{k\in\mathbb{N}} \subset (0,1)$, and $\{\gamma_k\}_{k\in\mathbb{N}} \subset (0,1)$, and the sequence $\{n_k\}_{k\in\mathbb{N}}$ denoted by $gI(T, \{\beta_k\}_{k\in\mathbb{N}}, \{\gamma_k\}_{k\in\mathbb{N}}, \{n_k\}_{k\in\mathbb{N}})$ is defined by the following iterative scheme:

$$\begin{aligned} &x_1 \in C \text{ chosen arbitrarily}, \\ &y_k = \beta_k T^{n_k}(x_k) + (1-\beta_k)x_k, \\ &x_{k+1} = \gamma_k T^{n_k}(y_k) + (1-\gamma_k)x_k. \end{aligned} \tag{92}$$

Definition 25. We say that a generalized Ishikawa iteration process $gI(T, \{\beta_k\}_{k\in\mathbb{N}}, \{\gamma_k\}_{k\in\mathbb{N}}, \{n_k\}_{k\in\mathbb{N}})$ is well defined if

$$\limsup_{k\to\infty} a_{n_k}(x_k) = \limsup_{k\to\infty} a_{n_k}(y_k) = 1. \tag{93}$$

Lemma 26. *Let E be a reflexive Banach space and let $g : E \to \mathbb{R}$ be a convex, continuous, strongly coercive, and Gâteaux differentiable function which is bounded on bounded subsets and uniformly convex on bounded subsets of E. Let C be a nonempty, bounded, closed, and convex subset of E. Let $T \in \mathscr{BT}_r(C)$ and let $\{n_k\}_{k\in\mathbb{N}} \subset \mathbb{N}$. Let $\{\beta_k\}_{k\in\mathbb{N}} \subset (0,1)$ and $\{\gamma_k\}_{k\in\mathbb{N}} \subset (0,1)$ be sequences of real numbers such that $\liminf_{k\to\infty}\beta_k(1-\beta_k) > 0$ and $\liminf_{k\to\infty}\gamma_k(1-\gamma_k) > 0$. Let $w \in F(T)$ and $gI(T, \{\beta_k\}_{k\in\mathbb{N}}, \{\gamma_k\}_{k\in\mathbb{N}}, \{n_k\}_{k\in\mathbb{N}})$ be a generalized Ishikawa iteration process. Then there exists $\theta \in \mathbb{R}$ such that $\lim_{k\to\infty} D_g(x_k, w) = \theta$.*

Proof. Let $M_7 > 1$ be fixed. Since $\lim_{k\to\infty} a_{n_k}(y_k) = 1$, there exists $k_0 \in \mathbb{N}$ such that for any $k > k_0$, $a_{n_k}(y_k) \leq M_7$. Let $s_4 = \sup\{\|T^{n_k}(x_k)\|, \|x_k\| : k \in \mathbb{N}\}$ and let $\rho_{s_4} : E \to \mathbb{R}$ be the gauge of uniform convexity of g. By the definition of T and in view of (93), we obtain

$$\begin{aligned}
D_g(x_{k+1}, w) &= D_g(\gamma_k T^{n_k}(y_k) + (1-\gamma_k)x_k, w) \\
&\le \gamma_k D_g(T^{n_k}(y_k), w) + (1-\gamma_k) D_g(x_k, w) \\
&= \gamma_k D_g(T^{n_k}(y_k), T^{n_k}(w)) + (1-\gamma_k) D_g(x_k, w) \\
&\le \gamma_k a_{n_k}(y_k) D_g(y_k, w) + (1-\gamma_k) D_g(x_k, w) \\
&\le \gamma_k a_{n_k}(y_k)\big[\beta_k D_g(T^{n_k}(x_k), w) + (1-\beta_k) D_g(x_k, w) - \rho_{s_4}(\|T^{n_k}(x_k) - x_k\|)\big] \\
&\quad + (1-\gamma_k) D_g(x_k, w) \\
&= \gamma_k a_{n_k}(w)\big[\beta_k D_g(T^{n_k}(x_k), T^{n_k}(w)) + (1-\beta_k) D_g(x_k, w) - \rho_{s_4}(\|T^{n_k}(x_k) - x_k\|)\big] \\
&\quad + (1-\gamma_k) D_g(x_k, w) \\
&\le \gamma_k a_{n_k}(y_k)\big[\beta_k a_{n_k}(x_k) D_g(x_k, w) + (1-\beta_k) D_g(x_k, w) - \rho_{s_4}(\|T^{n_k}(x_k) - x_k\|)\big] \\
&\quad + (1-\gamma_k) D_g(x_k, w) \\
&= \gamma_k a_{n_k}(y_k)\big[\beta_k (a_{n_k}(x_k) - 1) D_g(x_k, w) + \beta_k D_g(x_k, w) + (1-\beta_k) D_g(x_k, w) - \rho_{s_4}(\|T^{n_k}(x_k) - x_k\|)\big] \\
&\quad + (1-\gamma_k) D_g(x_k, w) \\
&= \gamma_k \beta_k a_{n_k}(y_k)(a_{n_k}(x_k) - 1) D_g(x_k, w) \\
&\quad + \gamma_k a_{n_k}(y_k) D_g(x_k, w) \\
&\quad - \gamma_k \rho_{s_4}(\|T^{n_k}(x_k) - x_k\|) + (1-\gamma_k) D_g(x_k, w) \\
&\le \gamma_k a_{n_k}(y_k)(a_{n_k}(x_k) - 1) D_g(x_k, w) \\
&\quad + \gamma_k a_{n_k}(y_k) D_g(x_k, w) + (1-\gamma_k) D_g(x_k, w) \\
&\quad - \gamma_k \rho_{s_4}(\|T^{n_k}(x_k) - x_k\|) \\
&= \gamma_k a_{n_k}(y_k) a_{n_k}(x_k) D_g(x_k, w) + (1-\gamma_k) D_g(x_k, w) \\
&\quad - \gamma_k \rho_{s_4}(\|T^{n_k}(x_k) - x_k\|) \\
&= \gamma_k a_{n_k}(y_k) a_{n_k}(x_k) D_g(x_k, w) \\
&\quad - D_g(x_k, w) + \gamma_k D_g(x_k, w) \\
&\quad + (1-\gamma_k) D_g(x_k, w) - \gamma_k \beta_k \rho_{s_4}(\|T^{n_k}(x_k) - x_k\|) \\
&\le \big[a_{n_k}(y_k) a_{n_k}(x_k) - 1\big] D_g(x_k, w) \\
&\quad + D_g(x_k, w) - \gamma_k \rho_{s_4}(\|T^{n_k}(x_k) - x_k\|) \\
&= a_{n_k}(y_k) a_{n_k}(x_k) D_g(x_k, w) \\
&\quad - \gamma_k \rho_{s_4}(\|T^{n_k}(x_k) - x_k\|) \\
&= a_{n_k}(y_k)\big[a_{n_k}(x_k) - 1\big] D_g(x_k, w) \\
&\quad + \big[a_{n_k}(x_k) - 1\big] D_g(x_k, w) \\
&\quad + D_g(x_k, w) - \gamma_k \rho_{s_4}(\| T^{n_k}(x_k) - x_k \|) \\
&\le M_7 b_{n_k}(x_k) B \operatorname{diam}(C) + D_g(x_k, w) \\
&\quad - \gamma_k \rho_{s_4}(\|T^{n_k}(x_k) - x_k\|).
\end{aligned} \tag{94}$$

This implies that for every $n \in \mathbb{N}$,

$$D_g(x_{k+n}, w) \le D_g(x_k, w) + M_7 B \operatorname{diam}(C) \sum_{i=k}^{k+n-1} b_{n_i}(x_i). \tag{95}$$

Put $r_p = D_g(x_p, w)$ for every $p \in \mathbb{N}$ and $d_{k,n} = M_7 B \operatorname{diam}(C) \sum_{i=k}^{k+n-1} b_{n_i}(x_i)$. Since $T \in \mathscr{BT}_r(C)$, we obtain that $\limsup_{k\to\infty} \limsup_{n\to\infty} d_{k,n} = 0$. In view of Lemma 11, there exists $\theta \in \mathbb{R}$ such that $\lim_{k\to\infty} D_g(x_k, w) = \theta$. Let $M_8 > 1$ be fixed. Since $\lim_{k\to\infty} a_{n_k}(x_k) = 1$, there exists $k_0 \in \mathbb{N}$ such that for any $k > k_0$, $a_{n_k}(x_k) \le M_8$. Therefore, by the same argument, as in the proof of Lemma 19, we conclude that for $k > k_0$ and $n > 1$

$$\begin{aligned}
D_g(x_{k+n}, w) &\le D_g(x_k, w) + M_7 B \operatorname{diam}(C) \sum_{i=k}^{k+n-1} b_{n_i}(x_i) \\
&\le D_g(x_k, w) + M_7 B \operatorname{diam}(C) \sum_{i=k}^{k+n-1} b_{n_i}(x_i).
\end{aligned} \tag{96}$$

By the same manner as in the proof of Lemma 19, we deduce that there exists $\theta \in \mathbb{R}$ such that $\lim_{k\to\infty} D_g(x_k, w) = \theta$, which completes the proof. □

Lemma 27. *Let E be a reflexive Banach space and let $g : E \to \mathbb{R}$ be a convex, continuous, strongly coercive, and Gâteaux differentiable function which is bounded on bounded subsets and uniformly convex on bounded subsets. Let C be a nonempty, bounded, closed, and convex subset of E. Let $T \in \mathscr{BT}_r(C)$ and let $\{n_k\}_{k\in\mathbb{N}} \subset \mathbb{N}$. Let $\{\beta_k\}_{k\in\mathbb{N}} \subset (0,1)$ and $\{\gamma_k\}_{k\in\mathbb{N}} \subset (0,1)$ be sequences of real numbers such that $\liminf_{k\to\infty} \gamma_k(1-\gamma_k) > 0$ and $\liminf_{k\to\infty} \beta_k(1 -$*

$\beta_k) > 0$. *Let* $w \in F(T)$ *and* $gI(T, \{\gamma_k\}_{k\in\mathbb{N}}, \{\beta_k\}_{k\in\mathbb{N}}, \{n_k\}_{k\in\mathbb{N}})$ *be a generalized Ishikawa iteration process. Then*

$$\lim_{k\to\infty} D\left(T^{n_k}\left(y_k\right), x_k\right) = 0, \qquad \lim_{k\to\infty} D_g\left(x_{k+1}, x_k\right) = 0. \tag{97}$$

Proof. In view of Proposition 13, $F(T) \neq \emptyset$. Take any $w \in F(T)$ arbitrarily chosen. Then, by Lemma 26, there exists $\theta \in \mathbb{R}$ such that $\lim_{k\to\infty} D_g(x_k, w) = \theta$. By the same arguments, as in the proof of Lemma 26, we conclude that

$$D_g\left(x_{k+1}, w\right) \le M_7 b_{n_k}\left(x_k\right) B \operatorname{diam}(C) + D_g\left(x_k, w\right) - \gamma_k \rho_{s_4}\left(\left\|T^{n_k}\left(x_k\right) - x_k\right\|\right). \tag{98}$$

This implies that

$$\gamma_k \rho_{s_4}\left(\left\|T^{n_k}\left(x_k\right) - x_k\right\|\right) \le M_7 b_{n_k}\left(x_k\right) B \operatorname{diam}(C) + D_g\left(x_k, w\right) - D_g\left(x_{k+1}, w\right). \tag{99}$$

This implies that

$$\lim_{k\to\infty} \rho_{s_4}\left(\left\|T^{n_k}\left(x_k\right) - x_k\right\|\right) = 0. \tag{100}$$

Therefore, from the property of ρ_{s_4} we deduce that

$$\lim_{k\to\infty} \left\|T^{n_k}\left(x_k\right) - x_k\right\| = 0. \tag{101}$$

In a similar way, as in the proof of Lemma 20, we can prove that

$$\lim_{k\to\infty} \left\|T\left(x_k\right) - x_k\right\| = 0. \tag{102}$$

This completes the proof. □

Lemma 28. *Let* E *be a reflexive Banach space and let* $g : E \to \mathbb{R}$ *be a convex, continuous, strongly coercive, and Gâteaux differentiable function which is bounded on bounded subsets and uniformly convex on bounded subsets. Let* C *be a nonempty, bounded, closed, and convex subset of* E. *Let* $T \in \mathscr{BT}_r(C)$ *and let* $\{n_k\}_{k\in\mathbb{N}} \subset \mathbb{N}$. *Let* $\{\gamma_k\}_{k\in\mathbb{N}} \subset (0,1)$ *such that* $\liminf_{k\to\infty} \gamma_k(1-\gamma_k) > 0$. *Let* $\{n_k\}_{k\in\mathbb{N}} \subset \mathbb{N}$ *be such that the generalized Ishikawa process* $gI(T, \{\gamma_k\}_{k\in\mathbb{N}}, \{\beta_k\}_{k\in\mathbb{N}}, \{n_k\}_{k\in\mathbb{N}})$ *is well defined. If, in addition, the set of indices* $\mathcal{J} = \{j \in \mathbb{N} : n_{j+1} = 1 + n_j\}$ *is quasi-periodic, then* $\{x_k\}_{k\in\mathbb{N}}$ *is an approximate fixed point sequence; that is,*

$$\lim_{k\to\infty} \left\|T\left(x_k\right) - x_k\right\| = 0. \tag{103}$$

Theorem 29. *Let* E *be a reflexive Banach space and let* $g : E \to \mathbb{R}$ *be a convex, continuous, strongly coercive, and Gâteaux differentiable function which is bounded on bounded subsets and uniformly convex on bounded subsets. Let* C *be a nonempty, bounded, closed, and convex subset of* E. *Let* $T \in \mathscr{BT}_r(C)$ *and let* $\{n_k\}_{k\in\mathbb{N}} \subset \mathbb{N}$. *Let* $\{\gamma_k\}_{k\in\mathbb{N}} \subset (0,1)$ *such that* $\liminf_{k\to\infty} \gamma_k(1-\gamma_k) > 0$. *Let* $\{n_k\}_{k\in\mathbb{N}} \subset \mathbb{N}$ *be such that the generalized Ishikawa process* $gI(T, \{\gamma_k\}_{k\in\mathbb{N}}, \{\beta_k\}_{k\in\mathbb{N}}, \{n_k\}_{k\in\mathbb{N}})$ *is well defined. If, in addition, the set of indices* $\mathcal{J} = \{j \in \mathbb{N} : k_{j+1} = 1 + k_j\}$ *is quasi-periodic, then the sequence* $\{n_k\}_{k\in\mathbb{N}}$ *generated by* $gI(T, \{\gamma_k\}_{k\in\mathbb{N}}, \{\beta_k\}_{k\in\mathbb{N}}, \{n_k\}_{k\in\mathbb{N}})$ *converges weakly to a fixed point of* T.

Remark 30. Theorem 29 improves Theorems 3.1 and 4.1 of [18] in the following aspects.

(1) For the structure of Banach spaces, we extend the duality mapping to more general case, that is, a convex, continuous, strongly coercive Bregman function which is bounded on bounded sets and uniformly convex and uniformly smooth on bounded sets.

(2) For the mappings, we extend the mapping from an asymptotic pointwise nonexpansive mapping to a Bregman asymptotic pointwise nonexpansive mapping.

(3) Since we do not need the weak sequential continuity of the duality mapping in Theorems 23 and 29 as was the case in [18], we can apply Theorem 29 in the Lebesgue space L^p where $1 < p < \infty$ and $p \neq 2$ while this space is not applicable for Theorems 3.1, 4.1, and 5.1 of [18].

Authors' Contribution

All authors read and approved the final paper.

Conflict of Interests

The authors declare that they have no competing interests.

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