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Research Article

Equivalency Relations between Continuous g-Frames and Stability of Alternate Duals of Continuous g-Frames in Hilbert C^*-Modules

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We introduce the modular continuous g-Riesz basis to improve one existing result for continuous g-Riesz basis in Hilbert C^*-modules, and then we study the equivalency relations between continuous g-frames in Hilbert C^*-modules, and, in particular, we obtain two necessary and sufficient conditions under which two continuous g-frames are similar. Finally, we generalize a stability result for alternate duals of g-frames in Hilbert spaces to alternate duals of continuous g-frames in Hilbert C^*-modules.

1. Introduction

Frames for Hilbert spaces were first introduced by Duffin and Schaeffer [1] in 1952 to study some deep problems in nonharmonic Fourier series, reintroduced in 1986 by Daubechies et al. [2] and popularized from then on. The theory of frames plays an important role in theoretics and applications, which has been extensively applied in signal processing, sampling theory, system modelling, and many other fields. We refer to [3–9] for an introduction to frame theory and its applications.

The theory of frames was rapidly generalized and, until 2006, various generalizations consisting of vectors in Hilbert spaces were developed. In 2006, Sun introduced the concept of g-frame in a Hilbert space in [10] and showed that this includes more of the other cases of generalizations of frame concept and proved that many basic properties can be derived within this more general context.

On the other hand, the concept of frames especially the g-frames was introduced in Hilbert C^*-modules, and some of their properties were investigated in [11–13]. As for Hilbert C^*-module, it is a generalization of Hilbert spaces by allowing the inner product to take values in a C^*-algebra rather than the field of complex numbers. Note that the theory of Hilbert C^*-modules is quite different from that of Hilbert spaces. Unlike Hilbert space cases, not every closed submodule of a Hilbert C^*-module is complemented. Moreover, the well-known Riesz representation theorem for continuous functionals in Hilbert spaces does not hold in Hilbert C^*-modules, which implies that not all bounded linear operators on Hilbert C^*-modules are adjointable. It should also be remarked that, due to the complexity of the C^*-algebras involved in the Hilbert C^*-modules and the fact that some useful techniques available in Hilbert spaces are either absent or unknown in Hilbert C^*-modules, the problems about frames and g-frames for Hilbert C^*-modules are more complicated than those for Hilbert spaces. This makes the study of the frames for Hilbert C^*-modules important and interesting. The properties of g-frames for Hilbert C^*-modules were further investigated in [14, 15].

The concept of a generalization of frames to a family indexed by some locally compact space endowed with a Radon measure was proposed by Kaiser [16] and independently by Ali et al. [17]. These frames are known as continuous frames. Gabardo and Han in [18] called these frames "Frames associated with measurable spaces"; Askari-Hemmat et al. in [19] called them generalized frames, and in mathematical physics they are referred to as coherent states [20].

The continuous g-frames in Hilbert C^*-modules, which were proposed by Kouchi and Nazari in [21], are an extension to g-frames in Hilbert C^*-modules and continuous frames in Hilbert spaces, and they made a discussion of some properties of continuous g-frames in Hilbert C^*-modules in some aspects. The purpose of this paper is to further investigate the properties of continuous g-frames in Hilbert C^*-modules.

The paper is organized in the following manner. We continue this introductory section with a review of the basic definitions and notations about Hilbert C^*-modules. Section 2 investigates some basic results of continuous g-frames in Hilbert C^*-modules and introduces the so-called modular continuous g-Riesz basis to improve one result for continuous g-Riesz basis obtained by Kouchi and Nazari plus a bit more. Equivalency relations between continuous g-frames are included in Section 3, where two necessary and sufficient conditions for two continuous g-frames to be similar are obtained. The last section of this paper generalizes a stability result for alternate duals of g-frames in Hilbert spaces to alternate duals of continuous g-frames in Hilbert C^*-modules.

Let us recall the definitions and some basic properties of Hilbert C^*-modules. For more details, the interested readers can refer to the books by Lance [22] and Wegge-Olsen [23]. Let A be a C^*-algebra with involution $*$. A pre-Hilbert C^*-module over A or, simply, a pre-Hilbert A-module, is a complex linear space $\mathscr{U}$ which is a left A-module with map $\langle\cdot,\cdot\rangle : \mathscr{U}\times\mathscr{U} \to A$, called an A-valued inner product, and it possesses the following properties:

(1) $\langle f, f\rangle \geq 0$ for all $f \in \mathscr{U}$ and $\langle f, f\rangle = 0$ if and only if $f = 0$;
(2) $\langle f, g\rangle = \langle g, f\rangle^*$ for all $f, g \in \mathscr{U}$;
(3) $\langle af + g, h\rangle = a\langle f, h\rangle + \langle g, h\rangle$ for all $a \in A$, $f, g, h \in \mathscr{U}$;
(4) $\langle \lambda f, g\rangle = \lambda\langle f, g\rangle$ whenever $\lambda \in \mathbb{C}$ and $f, g \in \mathscr{U}$.

For $f \in \mathscr{U}$, we define a norm on $\mathscr{U}$ by $\|f\|_{\mathscr{U}} = \|\langle f, f\rangle\|_A^{1/2}$. If $\mathscr{U}$ is complete with this norm, it is called a Hilbert C^*-module over A or a Hilbert A-module.

Let $(\mathscr{U}, \langle\cdot,\cdot\rangle_1)$ and $(\mathscr{V}, \langle\cdot,\cdot\rangle_2)$ be two Hilbert A-modules. A map $T : \mathscr{U} \to \mathscr{V}$ is said to be adjointable if there exists a map $S : \mathscr{V} \to \mathscr{U}$ such that $\langle Tf, g\rangle_2 = \langle f, Sg\rangle_1$ for all $f \in \mathscr{U}$ and $g \in \mathscr{V}$. We denote by $\mathrm{End}_A^*(\mathscr{U},\mathscr{V})$ the collection of all adjointable A-linear maps from $\mathscr{U}$ to $\mathscr{V}$. The following two lemmas will be used in the later section.

Lemma 1 (see [24]). *Let $\mathscr{M}$ and $\mathscr{N}$ be two Hilbert A-modules over a C^*-algebra A and let $T : \mathscr{M} \to \mathscr{N}$ be a linear map. Then the following conditions are equivalent:*

(1) *the operator T is bounded and A-linear;*
(2) *there exists a constant $K \geq 0$ such that the inequality $\langle Tx, Tx\rangle \leq K\langle x, x\rangle$ holds in A for all $x \in \mathscr{M}$.*

Lemma 2 (see [25]). *Let A be a C^*-algebra, let $\mathscr{U}$ and $\mathscr{V}$ be two Hilbert A-modules, and let $T \in End_A^*(\mathscr{U},\mathscr{V})$. The following statements are equivalent:*

(1) *T is surjective;*
(2) *T^* is bounded below with respect to norm; that is, there is $m > 0$ such that $\|T^* f\| \geq m\|f\|$ for all $f \in \mathscr{V}$;*
(3) *T^* is bounded below with respect to inner product; that is, there is $m' > 0$ such that $\langle T^* f, T^* f\rangle \geq m'\langle f, f\rangle$ for all $f \in \mathscr{V}$.*

Let $\mathscr{V}$ be a Hilbert A-module and $\{\mathscr{V}_m\}_{m\in\mathscr{M}}$ a sequence of closed submodules of $\mathscr{V}$. Set

$$\bigoplus_{m\in\mathscr{M}}\mathscr{V}_m = \left\{g = \{g_m\} : g_m \in \mathscr{V}_m, \left\|\int_{m\in\mathscr{M}} \langle g_m, g_m\rangle \, d\mu(m)\right\| < \infty\right\}. \tag{1}$$

For any $f = \{f_m : m \in \mathscr{M}\}$ and $g = \{g_m : m \in \mathscr{M}\}$, if the A-valued inner product is defined by $\langle f, g\rangle = \int_{m\in\mathscr{M}}\langle f_m, g_m\rangle d\mu(m)$ and the norm is defined by $\|f\| = \|\langle f, f\rangle\|^{1/2}$, then $\bigoplus_{m\in\mathscr{M}}\mathscr{V}_m$ is a Hilbert A-module (see [22]).

Throughout this paper, A is a unital C^*-algebra, $\mathscr{U}$ and $\mathscr{V}$ are Hilbert A-modules, and $\{\mathscr{V}_m\}_{m\in\mathscr{M}}$ is a sequence of closed submodules of $\mathscr{V}$. For $T \in \mathrm{End}_A^*(\mathscr{U},\mathscr{V})$, we use $\mathrm{Ran}(T)$ and $\mathscr{N}(T)$ to denote the range and the null space of T, respectively. As usual, we use $I_{\mathscr{U}}$ to denote the identity operator on $\mathscr{U}$.

2. Basic Results of Continuous g-Frames and Modular Continuous g-Riesz Bases

In this section, we recall some basic properties of continuous g-frames in Hilbert C^*-modules and, in particular, we obtain an equivalent condition under which a Hilbert C^*-module has a continuous g-frame. Moreover, we introduce the modular continuous g-Riesz basis to improve one result for continuous g-Riesz basis in Hilbert C^*-modules.

Definition 3 (see [21]). We call a family of adjointable A-linear operators $\{\Lambda_m \in \mathrm{End}_A^*(\mathscr{U},\mathscr{V}_m) : m \in \mathscr{M}\}$ a continuous generalized frame or simply a continuous g-frame for Hilbert C^*-module $\mathscr{U}$ with respect to $\{\mathscr{V}_m : m \in \mathscr{M}\}$ if

(1) for any $f \in \mathscr{U}$, the function $\tilde{f} : \mathscr{M} \to \mathscr{V}_m$ defined by $\tilde{f}(m) = \Lambda_m f$ is measurable;
(2) there is a pair of constants $A, B > 0$ such that, for any $f \in \mathscr{U}$,

$$A\langle f, f\rangle \leq \int_{m\in\mathscr{M}} \langle \Lambda_m f, \Lambda_m f\rangle \, d\mu(m) \leq B\langle f, f\rangle. \tag{2}$$

The constants A and B are called continuous g-frame bounds. We call $\{\Lambda_m : m \in \mathscr{M}\}$ a continuous tight g-frame if $A = B$ and a continuous Parseval g-frame if $A = B = 1$. If only the right-hand inequality of (2) is satisfied, we call $\{\Lambda_m : m \in \mathscr{M}\}$ a continuous g-Bessel sequence for $\mathscr{U}$ with respect to $\{\mathscr{V}_m : m \in \mathscr{M}\}$ with Bessel bound B.

We have the following equivalent definition for continuous g-Bessel sequences in Hilbert C^*-modules.

Proposition 4. *Let $\{\Lambda_m \in End_A^*(\mathscr{U},\mathscr{V}_m) : m \in \mathscr{M}\}$ be a sequence of adjointable A-linear operators on $\mathscr{U}$. Then $\{\Lambda_m : m \in \mathscr{M}\}$ is a continuous g-Bessel sequence with Bessel bound D if and only if, for all $f \in \mathscr{U}$,*

$$\left\|\int_{m\in\mathscr{M}} \langle \Lambda_m f, \Lambda_m f\rangle \, d\mu(m)\right\| \leq D\|f\|^2. \tag{3}$$

Proof. "⇒". It is obvious.

"⇐". Define a linear operator $T : \mathscr{U} \to \bigoplus_{m\in\mathscr{M}} \mathscr{V}_m$ by $Tf = \{\Lambda_m f : m \in \mathscr{M}\}$ for all $f \in \mathscr{U}$. Then

$$\begin{aligned} \|Tf\|^2 &= \|\langle Tf, Tf\rangle\| \\ &= \left\| \int_{m\in\mathscr{M}} \langle \Lambda_m f, \Lambda_m f\rangle \, d\mu(m) \right\| \leq D\|f\|^2, \end{aligned} \tag{4}$$

which implies that $\|Tf\| \leq \sqrt{D}\|f\|$. Hence, T is bounded. It is clear that T is A-linear. Then by Lemma 1, we have $\langle Tf, Tf\rangle \leq D\langle f, f\rangle$, equivalently, $\int_{m\in\mathscr{M}} \langle \Lambda_m f, \Lambda_m f\rangle d\mu(m) \leq D\langle f, f\rangle$, as desired. □

The following proposition gives an equivalent condition for a continuous g-Bessel sequence to be a continuous g-frame.

Proposition 5. *Let $\{\Lambda_m \in End_A^*(\mathscr{U}, \mathscr{V}_m) : m \in \mathscr{M}\}$ be a continuous g-Bessel sequence for $\mathscr{U}$ with respect to $\{\mathscr{V}_m : m \in \mathscr{M}\}$. Then $\{\Lambda_m : m \in \mathscr{M}\}$ is a continuous g-frame for $\mathscr{U}$ if and only if there exists a constant $C > 0$ such that*

$$C\|f\|^2 \leq \left\| \int_{m\in\mathscr{M}} \langle \Lambda_m f, \Lambda_m f\rangle \, d\mu(m) \right\|, \quad \forall f \in \mathscr{U}. \tag{5}$$

Proof. "⇒". It is straightforward.

"⇐". We define a linear operator as follows:

$$T : \mathscr{U} \longrightarrow \bigoplus_{m\in\mathscr{M}} \mathscr{V}_m, \qquad Tf = \{\Lambda_m f : m \in \mathscr{M}\}, \quad \forall f \in \mathscr{U}. \tag{6}$$

Then T is adjointable. Indeed,

$$\begin{aligned} \langle Tf, g\rangle &= \int_{m\in\mathscr{M}} \langle \Lambda_m f, g_m\rangle \, d\mu(m) \\ &= \left\langle f, \int_{m\in\mathscr{M}} \Lambda_m^* g_m d\mu(m) \right\rangle, \end{aligned} \tag{7}$$

for all $f \in \mathscr{U}, g = \{g_m\} \in \bigoplus_{m\in\mathscr{M}} \mathscr{V}_m$. It follows from (5) that

$$\begin{aligned} \|Tf\|^2 &= \|\langle Tf, Tf\rangle\| \\ &= \left\| \int_{m\in\mathscr{M}} \langle \Lambda_m f, \Lambda_m f\rangle \, d\mu(m) \right\| \geq C\|f\|^2. \end{aligned} \tag{8}$$

Thus, $\|Tf\| \geq \sqrt{C}\|f\|$ for all $f \in \mathscr{U}$. Then by Lemma 2, there exists $M > 0$ such that $\langle Tf, Tf\rangle \geq M\langle f, f\rangle$; that is, $M\langle f, f\rangle \leq \int_{m\in\mathscr{M}} \langle \Lambda_m f, \Lambda_m f\rangle d\mu(m)$. The proof is over. □

Using the above equivalent definition of continuous g-frames we can easily prove the following result that will be used in the proof of Lemma 20.

Proposition 6. *Let $\{\Lambda_m \in End_A^*(\mathscr{U}, \mathscr{V}_m) : m \in \mathscr{M}\}$ and $\{\Gamma_m \in End_A^*(\mathscr{U}, \mathscr{V}_m) : m \in \mathscr{M}\}$ be two continuous g-Bessel sequences for $\mathscr{U}$ with respect to $\{\mathscr{V}_m : m \in \mathscr{M}\}$. If $f = \int_{m\in\mathscr{M}} \Lambda_m^* \Gamma_m f d\mu(m)$ holds for all $f \in \mathscr{U}$, then both $\{\Lambda_m : m \in \mathscr{M}\}$ and $\{\Gamma_m : m \in \mathscr{M}\}$ are continuous g-frames for $\mathscr{U}$ with respect to $\{\mathscr{V}_m : m \in \mathscr{M}\}$.*

Proof. Let us denote the Bessel bound of $\{\Gamma_m : m \in \mathscr{M}\}$ by D. For all $f \in \mathscr{U}$, we have

$$\begin{aligned} \|f\|^4 &= \left\| \left\langle \int_{m\in\mathscr{M}} \Lambda_m^* \Gamma_m f d\mu(m), f \right\rangle \right\|^2 \\ &= \left\| \int_{m\in\mathscr{M}} \langle \Lambda_m f, \Gamma_m f\rangle \, d\mu(m) \right\|^2 \\ &\leq \left\| \int_{m\in\mathscr{M}} \langle \Lambda_m f, \Lambda_m f\rangle \, d\mu(m) \right\| \\ &\quad \times \left\| \int_{m\in\mathscr{M}} \langle \Gamma_m f, \Gamma_m f\rangle \, d\mu(m) \right\| \\ &\leq D\|f\|^2 \left\| \int_{m\in\mathscr{M}} \langle \Lambda_m f, \Lambda_m f\rangle \, d\mu(m) \right\|. \end{aligned} \tag{9}$$

It follows that

$$D^{-1}\|f\|^2 \leq \left\| \int_{m\in\mathscr{M}} \langle \Lambda_m f, \Lambda_m f\rangle \, d\mu(m) \right\|. \tag{10}$$

Similarly, we can show that $\{\Gamma_m : m \in \mathscr{M}\}$ is a continuous g-frame for $\mathscr{U}$. □

Let $\{\Lambda_m \in \mathrm{End}_A^*(\mathscr{U}, \mathscr{V}_m) : m \in \mathscr{M}\}$ be a continuous g-Bessel sequence for $\mathscr{U}$ with respect to $\{\mathscr{V}_m : m \in \mathscr{M}\}$, we define the synthesis operator $T_\Lambda : \bigoplus_{m\in\mathscr{M}} \mathscr{V}_m \to \mathscr{U}$ by

$$T_\Lambda g = \int_{m\in\mathscr{M}} \Lambda_m^* g_m d\mu(m), \quad \forall g = \{g_m\} \in \bigoplus_{m\in\mathscr{M}} \mathscr{V}_m. \tag{11}$$

It follows immediately from the observation that for all $f \in \mathscr{U}$, $g = \{g_m\} \in \bigoplus_{m\in\mathscr{M}} \mathscr{V}_m$, and

$$\begin{aligned} \langle T_\Lambda g, f\rangle &= \int_{m\in\mathscr{M}} \langle g_m, \Lambda_m f\rangle \, d\mu(m) \\ &= \langle g, \{\Lambda_m f : m \in \mathscr{M}\}\rangle, \end{aligned} \tag{12}$$

T_Λ is adjointable and its adjoint operator $T_\Lambda^* : \mathscr{U} \to \bigoplus_{m\in\mathscr{M}} \mathscr{V}_m$ is given by $T_\Lambda^* f = \{\Lambda_m f : m \in \mathscr{M}\}$ for all $f \in \mathscr{U}$. We call T_Λ^* the analysis operator. By composing T_Λ and T_Λ^*, we obtain the frame operator $S_\Lambda : \mathscr{U} \to \mathscr{U}$. Note that S_Λ is a positive, self-adjoint operator which is invertible if and only if $\{\Lambda_m : m \in \mathscr{M}\}$ is a continuous g-frame of $\mathscr{U}$. If $\{\Lambda_m : m \in \mathscr{M}\}$ is a continuous g-frame, then every $f \in \mathscr{U}$ has a representation of the form

$$f = \int_{m\in\mathscr{M}} \Lambda_m^* \Lambda_m S_\Lambda^{-1} f d\mu(m) = \int_{m\in\mathscr{M}} S_\Lambda^{-1} \Lambda_m^* \Lambda_m f d\mu(m). \tag{13}$$

We can characterize the continuous g-frames in Hilbert C^*-modules in terms of the associated synthesis and analysis operators.

Proposition 7. *Let $\{\Lambda_m \in End_A^*(\mathscr{U}, \mathscr{V}_m) : m \in \mathscr{M}\}$ be a family of adjointable A-linear operators on $\mathscr{U}$. Then the following* *statements are equivalent:*

(1) $\{\Lambda_m : m \in \mathcal{M}\}$ *is a continuous g-frame for* $\mathcal{U}$ *with respect to* $\{\mathcal{V}_m : m \in \mathcal{M}\}$;

(2) *the synthesis operator* T_Λ *is well defined and surjective;*

(3) *the analysis operator* T_Λ^* *is bounded below with respect to norm.*

Proof. (1) $\Leftrightarrow$ (2). See [21, Theorem 4.3].

(2) $\Leftrightarrow$ (3). It follows directly from Lemma 2. □

We are now ready to present a necessary and sufficient condition for a Hilbert C^*-module to have a continuous g-frame.

Theorem 8. *A Hilbert A-module* $\mathcal{U}$ *has a continuous g-frame with respect to* $\{\mathcal{V}_m : m \in \mathcal{M}\}$ *if and only if there exists an adjointable and invertible map from* $\mathcal{U}$ *to a closed submodule of* $\bigoplus_{m\in\mathcal{M}}\mathcal{V}_m$.

Proof. "$\Rightarrow$". Assume that $\{\Lambda_m \in \mathrm{End}_A^*(\mathcal{U},\mathcal{V}_m) : m \in \mathcal{M}\}$ is a continuous g-frame for $\mathcal{U}$ with respect to $\{\mathcal{V}_m : m \in \mathcal{M}\}$ with synthesis operator T_Λ. It follows from Proposition 7 that the analysis operator T_Λ^* is bounded below with respect to norm; and, consequently, T_Λ^* is injective with closed range. Now, T_Λ^* is an adjointable and invertible map from $\mathcal{U}$ to $\mathrm{Ran}(T_\Lambda^*)$, which is a closed submodule of $\bigoplus_{m\in\mathcal{M}}\mathcal{V}_m$.

"$\Leftarrow$". Suppose that M is a closed submodule of $\bigoplus_{m\in\mathcal{M}}\mathcal{V}_m$ and $S : \mathcal{U} \to M$ is an adjointable and invertible map. We define a family of adjointable operators as follows:

$$P_m : \bigoplus_{m\in\mathcal{M}}\mathcal{V}_m \longrightarrow \mathcal{V}_m, \qquad P_m(\{F_m\}) = F_m, \quad \forall\{F_m\} \in \bigoplus_{m\in\mathcal{M}}\mathcal{V}_m. \tag{14}$$

Taking $\Lambda_m = P_m S$ for each $m \in \mathcal{M}$, then

$$\begin{aligned}\int_{m\in\mathcal{M}} \langle \Lambda_m f, \Lambda_m f\rangle\, d\mu(m) &= \int_{m\in\mathcal{M}} \langle P_m Sf, P_m Sf\rangle\, d\mu(m)\\ &= \langle Sf, Sf\rangle = \langle S^*Sf, f\rangle.\end{aligned} \tag{15}$$

Hence, by [22, Proposition 1.2], we have

$$\left\|S^{-1}\right\|^{-2}\langle f,f\rangle \le \int_{m\in\mathcal{M}} \langle \Lambda_m f, \Lambda_m f\rangle\, d\mu(m) \le \|S\|^2\langle f,f\rangle. \tag{16}$$

□

It is easy to see that a continuous g-Bessel sequence $\{\Lambda_m \in \mathrm{End}_A^*(\mathcal{U},\mathcal{V}_m) : m \in \mathcal{M}\}$ for $\mathcal{U}$ with respect to $\{\mathcal{V}_m : m \in \mathcal{M}\}$ is a continuous g-frame if and only if there exists a continuous g-Bessel sequence $\{\Gamma_m \in \mathrm{End}_A^*(\mathcal{U},\mathcal{V}_m) : m \in \mathcal{M}\}$ for $\mathcal{U}$ with respect to $\{\mathcal{V}_m : m \in \mathcal{M}\}$ such that

$$f = \int_{m\in\mathcal{M}} \Lambda_m^*\Gamma_m f\, d\mu(m), \quad \forall f \in \mathcal{U}. \tag{17}$$

In this case, we call $\{\Gamma_m : m \in \mathcal{M}\}$ a dual continuous g-frame of $\{\Lambda_m : m \in \mathcal{M}\}$. If S_Λ is the frame operator of $\{\Lambda_m \in \mathrm{End}_A^*(\mathcal{U},\mathcal{V}_m) : m \in \mathcal{M}\}$, a continuous g-frame for $\mathcal{U}$ with respect to $\{\mathcal{V}_m : m \in \mathcal{M}\}$, then, a direct calculation yields that $\{\Lambda_m S_\Lambda^{-1} : m \in \mathcal{M}\}$ is a dual continuous g-frame of $\{\Lambda_m : m \in \mathcal{M}\}$; it is called the canonical dual. A dual which is not the canonical dual is called an alternate dual or simply a dual.

Our next result is a generalization of Lemma 2.1 in [10].

Proposition 9. *Let* $\{\Lambda_m \in End_A^*(\mathcal{U},\mathcal{V}_m) : m \in \mathcal{M}\}$ *be a continuous g-frame for* $\mathcal{U}$ *with respect to* $\{\mathcal{V}_m : m \in \mathcal{M}\}$ *which possesses more than one dual, and let* S_Λ *be the frame operator for* $\{\Lambda_m : m \in \mathcal{M}\}$. *Then for any dual continuous g-frame* $\{\Gamma_m : m \in \mathcal{M}\}$ *of* $\{\Lambda_m : m \in \mathcal{M}\}$, *the inequality*

$$\begin{aligned}&\int_{m\in\mathcal{M}} \left\langle \Lambda_m S_\Lambda^{-1} f, \Lambda_m S_\Lambda^{-1} f\right\rangle d\mu(m)\\ &\quad\le \int_{m\in\mathcal{M}} \langle \Gamma_m f, \Gamma_m f\rangle\, d\mu(m)\end{aligned} \tag{18}$$

is valid for all $f \in \mathcal{U}$. *Besides, the quality holds precisely if* $\Gamma_m = \Lambda_m S_\Lambda^{-1}$ *for all* $m \in \mathcal{M}$.

More generally, whenever $f = \int_{m\in\mathcal{M}} \Lambda_m^* g_m d\mu(m)$ *for certain* $g = \{g_m\} \in \bigoplus_{m\in\mathcal{M}}\mathcal{V}_m$, *we have*

$$\begin{aligned}&\int_{m\in\mathcal{M}} \langle g_m, g_m\rangle\, d\mu(m)\\ &\quad= \int_{m\in\mathcal{M}} \left\langle \Lambda_m S_\Lambda^{-1} f, \Lambda_m S_\Lambda^{-1} f\right\rangle d\mu(m)\\ &\qquad+ \int_{m\in\mathcal{M}} \left\langle g_m - \Lambda_m S_\Lambda^{-1} f, g_m - \Lambda_m S_\Lambda^{-1} f\right\rangle d\mu(m).\end{aligned} \tag{19}$$

Proof. We begin with showing the first statement. Since $\{\Gamma_m : m \in \mathcal{M}\}$ is a dual continuous g-frame of $\{\Lambda_m : m \in \mathcal{M}\}$, it follows that $\int_{m\in\mathcal{M}}(\Lambda_m^*\Gamma_m f - \Lambda_m^*\Lambda_m S_\Lambda^{-1} f)d\mu(m) = 0$ for all $f \in \mathcal{U}$. Therefore,

$$\begin{aligned}&\int_{m\in\mathcal{M}} \langle \Gamma_m f, \Gamma_m f\rangle\, d\mu(m)\\ &\quad= \int_{m\in\mathcal{M}} \left\langle \Gamma_m f - \Lambda_m S_\Lambda^{-1} f + \Lambda_m S_\Lambda^{-1} f, \Gamma_m f \right.\\ &\qquad\qquad \left. - \Lambda_m S_\Lambda^{-1} f + \Lambda_m S_\Lambda^{-1} f\right\rangle d\mu(m)\\ &\quad= \int_{m\in\mathcal{M}} \left\langle \Lambda_m S_\Lambda^{-1} f, \Lambda_m S_\Lambda^{-1} f\right\rangle d\mu(m)\\ &\qquad+ \int_{m\in\mathcal{M}} \left\langle \Gamma_m f - \Lambda_m S_\Lambda^{-1} f, \Gamma_m f - \Lambda_m S_\Lambda^{-1} f\right\rangle d\mu(m),\end{aligned} \tag{20}$$

showing that the first part of the assertion holds since

$$\int_{m\in\mathcal{M}} \left\langle \Gamma_m f - \Lambda_m S_\Lambda^{-1} f, \Gamma_m f - \Lambda_m S_\Lambda^{-1} f\right\rangle d\mu(m) \ge 0. \tag{21}$$

Now, suppose that $f \in \mathcal{U}$ has two decompositions

$$\begin{aligned}f &= \int_{m\in\mathcal{M}} \Lambda_m^*\Lambda_m S_\Lambda^{-1} f\, d\mu(m) = \int_{m\in\mathcal{M}} \Lambda_m^* g_m\, d\mu(m),\\ &\qquad g = \{g_m\} \in \bigoplus_{m\in\mathcal{M}}\mathcal{V}_m.\end{aligned} \tag{22}$$

Since

$$\begin{aligned}\int_{m\in\mathcal{M}} \left\langle g_m, \Lambda_m S_\Lambda^{-1} f\right\rangle d\mu(m) &= \left\langle f, S_\Lambda^{-1} f\right\rangle \\ &= \int_{m\in\mathcal{M}} \left\langle \Lambda_m S_\Lambda^{-1} f, g_m\right\rangle d\mu(m),\end{aligned} \tag{23}$$

it follows that

$$\begin{aligned}\int_{m\in\mathcal{M}} \langle g_m, g_m\rangle d\mu(m) &= \int_{m\in\mathcal{M}} \langle \Lambda_m S_\Lambda^{-1} f, \Lambda_m S_\Lambda^{-1} f\rangle d\mu(m) \\ &\quad + \int_{m\in\mathcal{M}} \langle g_m - \Lambda_m S_\Lambda^{-1} f, g_m - \Lambda_m S_\Lambda^{-1} f\rangle d\mu(m).\end{aligned} \tag{24}$$

□

Definition 10 (see [21]). A continuous g-frame $\{\Lambda_m \in \mathrm{End}_A^*(\mathcal{U}, \mathcal{V}_m) : m \in \mathcal{M}\}$ for Hilbert C^*-module $\mathcal{U}$ with respect to $\{\mathcal{V}_m : m \in \mathcal{M}\}$ is said to be a continuous g-Riesz basis if it satisfies the following:

(1) $\Lambda_m \neq 0$ for any $m \in \mathcal{M}$;

(2) if $\int_{m\in\mathcal{K}} \Lambda_m^* g_m d\mu(m) = 0$, then $\Lambda_m^* g_m$ is equal to zero for each $m \in \mathcal{K}$, where $\{g_m\}_{m\in\mathcal{K}} \in \bigoplus_{m\in\mathcal{M}} \mathcal{V}_m$ and $\mathcal{K}$ is a measurable subset of $\mathcal{M}$.

By using the synthesis operator, Kouchi and Nazari gave a characterization for continuous g-Riesz basis as follows.

Theorem 11 (see [21]). *A family of adjointable A-linear operators $\{\Lambda_m \in End_A^*(\mathcal{U}, \mathcal{V}_m) : m \in \mathcal{M}\}$ is a continuous g-Riesz basis for $\mathcal{U}$ with respect to $\{\mathcal{V}_m : m \in \mathcal{M}\}$ if and only if the synthesis operator T_Λ is a homeomorphism.*

We note, however, that in the proof of the above theorem, they said that "$\Lambda_m^* f_m = 0$ for any $m \in \mathcal{M}$ and $\Lambda_m \neq 0$, so $f_m = 0$", which is not true, because if Λ_m has a dense range, then Λ_m^* is one-to-one. We can improve their result by introducing the following modular continuous g-Riesz basis.

Definition 12 (see [26]). We call a family $\{\Lambda_m \in \mathrm{End}_A^*(\mathcal{U}, \mathcal{V}_m) : m \in \mathcal{M}\}$ of adjointable A-linear operators on $\mathcal{U}$ a modular continuous g-Riesz basis if

(1) $\{f \in \mathcal{U} : \Lambda_m f = 0, m \in \mathcal{M}\} = \{0\}$;

(2) there exist constants $A, B > 0$ such that for any $g = \{g_m\} \in \bigoplus_{m\in\mathcal{M}} \mathcal{V}_m$,

$$A\|g\|^2 \leq \left\|\int_{m\in\mathcal{M}} \Lambda_m^* g_m d\mu(m)\right\|^2 \leq B\|g\|^2. \tag{25}$$

Theorem 13 (see [26]). *A sequence $\{\Lambda_m \in End_A^*(\mathcal{U}, \mathcal{V}_m) : m \in \mathcal{M}\}$ is a modular continuous g-Riesz basis for $\mathcal{U}$ with respect to $\{\mathcal{V}_m : m \in \mathcal{M}\}$ if and only if the synthesis operator T_Λ is a homeomorphism.*

Proof. Suppose first that $\{\Lambda_m \in \mathrm{End}_A^*(\mathcal{U}, \mathcal{V}_m) : m \in \mathcal{M}\}$ is a modular continuous g-Riesz basis for $\mathcal{U}$ with synthesis operator T_Λ. Then (25) turns to be

$$A\|g\|^2 \leq \|T_\Lambda g\|^2 \leq B\|g\|^2, \quad \forall g = \{g_m\} \in \bigoplus_{m\in\mathcal{M}} \mathcal{V}_m, \tag{26}$$

showing that T_Λ is bounded below with respect to norm. Hence, by Lemma 2, its adjoint operator T_Λ^* is surjective. Since condition (1) in Definition 12 implies that T_Λ^* is injective, it follows that T_Λ^* is invertible, and so T_Λ is invertible.

Conversely, let T_Λ be a homeomorphism. Then T_Λ^* is injective. So condition (1) in Definition 12 holds. Now, for any $g = \{g_m\} \in \bigoplus_{m\in\mathcal{M}} \mathcal{V}_m$,

$$\begin{aligned}\left\|T_\Lambda^{-1}\right\|^{-2}\|g\|^2 &\leq \left\|\int_{m\in\mathcal{M}} \Lambda_m^* g_m d\mu(m)\right\|^2 \\ &= \|T_\Lambda g\|^2 \leq \|T_\Lambda\|^2\|g\|^2.\end{aligned} \tag{27}$$

Therefore, $\{\Lambda_m \in \mathrm{End}_A^*(\mathcal{U}, \mathcal{V}_m) : m \in \mathcal{M}\}$ is a modular continuous g-Riesz basis for $\mathcal{U}$ with respect to $\{\mathcal{V}_m : m \in \mathcal{M}\}$. □

The following is an immediate consequence of Theorem 13.

Corollary 14. *Let $\{\Lambda_m \in End_A^*(\mathcal{U}, \mathcal{V}_m) : m \in \mathcal{M}\}$ be a continuous g-frame for $\mathcal{U}$ with respect to $\{\mathcal{V}_m : m \in \mathcal{M}\}$ with synthesis operator T_Λ, then it is a modular continuous g-Riesz basis for $\mathcal{U}$ with respect to $\{\mathcal{V}_m : m \in \mathcal{M}\}$ if and only if T_Λ^* is surjective.*

Let $\{\Lambda_m \in \mathrm{End}_A^*(\mathcal{U}, \mathcal{V}_m) : m \in \mathcal{M}\}$ and $\{\Gamma_m \in \mathrm{End}_A^*(\mathcal{U}, \mathcal{V}_m) : m \in \mathcal{M}\}$ be continuous g-Bessel sequences for $\mathcal{U}$ with respect to $\{\mathcal{V}_m : m \in \mathcal{M}\}$. In [21], the authors defined an adjointable operator L about them as follows:

$$L : \mathcal{U} \longrightarrow \mathcal{U}, \qquad Lf = \int_{m\in\mathcal{M}} \Gamma_m^* \Lambda_m f d\mu(m). \tag{28}$$

Theorem 15. *Let $\{\Lambda_m \in End_A^*(\mathcal{U}, \mathcal{V}_m) : m \in \mathcal{M}\}$ be a continuous g-frame for $\mathcal{U}$ with respect to $\{\mathcal{V}_m : m \in \mathcal{M}\}$ with bounds A, B and frame operator S_Λ, and $\{\Gamma_m \in End_A^*(\mathcal{U}, \mathcal{V}_m) : m \in \mathcal{M}\}$ is a continuous g-Bessel sequence for $\mathcal{U}$ with respect to $\{\mathcal{V}_m : m \in \mathcal{M}\}$. Suppose that there exists a number $0 < \lambda < A$ such that for all $f \in \mathcal{U}$,*

$$\|Lf - S_\Lambda f\| \leq \lambda \|f\|. \tag{29}$$

Then $\{\Lambda_m : m \in \mathcal{M}\}$ is a modular continuous g-Riesz basis for $\mathcal{U}$ if and only if $\{\Gamma_m : m \in \mathcal{M}\}$ is a modular continuous g-Riesz basis for $\mathcal{U}$.

Proof. For any $f \in \mathcal{U}$, we have

$$\begin{aligned}\|Lf\| &= \|Lf - S_\Lambda f + S_\Lambda f\| \\ &\geq \|S_\Lambda f\| - \|Lf - S_\Lambda f\| \geq (A - \lambda)\|f\|.\end{aligned} \tag{30}$$

So, L is bounded below with respect to norm. On the other hand, since

$$\|L^* f - S_\Lambda f\| \le \|L^* - S_\Lambda\| \|f\| = \left\|(L - S_\Lambda)^*\right\| \|f\| \le \lambda \|f\|, \quad \forall f \in \mathscr{U}, \tag{31}$$

by the above result, L^* is also bounded below with respect to norm, and hence, by Lemma 2, both L and L^* are surjective, and furthermore, L is invertible. Let T_Λ and T_Γ be the synthesis operators of $\{\Lambda_m : m \in \mathscr{M}\}$ and $\{\Gamma_m : m \in \mathscr{M}\}$, respectively. It is easy to check that $L = T_\Gamma T_\Lambda^*$. Thus, T_Λ^* is invertible if and only if T_Γ^* is invertible, and consequently, $\{\Lambda_m : m \in \mathscr{M}\}$ is a modular continuous g-Riesz basis for $\mathscr{U}$ if and only if $\{\Gamma_m : m \in \mathscr{M}\}$ is a modular continuous g-Riesz basis for $\mathscr{U}$. □

3. The Equivalency Relations between Continuous g-Frames in Hilbert C^*-Modules

The definitions of similar and unitary equivalent frames give rise to definitions of similar and unitary equivalent continuous g-frames in Hilbert C^*-modules.

Definition 16. Let $\{\Lambda_m \in \mathrm{End}_A^*(\mathscr{U}, \mathscr{V}_m) : m \in \mathscr{M}\}$ and $\{\Gamma_m \in \mathrm{End}_A^*(\mathscr{U}, \mathscr{V}_m) : m \in \mathscr{M}\}$ be two continuous g-frames for $\mathscr{U}$ with respect to $\{\mathscr{V}_m : m \in \mathscr{M}\}$. One has the following.

(1) They are said to be similar or equivalent if there is an adjointable and invertible operator $T : \mathscr{U} \to \mathscr{U}$ such that $\Gamma_m = \Lambda_m T$ for each $m \in \mathscr{M}$.

(2) They are said to be unitary equivalent if there exists an adjointable and unitary linear operator $U : \mathscr{U} \to \mathscr{U}$ such that $\Gamma_m = \Lambda_m U$ for each $m \in \mathscr{M}$.

Theorem 17. *Let $\{\Lambda_m \in End_A^*(\mathscr{U}, \mathscr{V}_m) : m \in \mathscr{M}\}$ and $\{\Gamma_m \in End_A^*(\mathscr{U}, \mathscr{V}_m) : m \in \mathscr{M}\}$ be two continuous g-frames for $\mathscr{U}$ with respect to $\{\mathscr{V}_m : m \in \mathscr{M}\}$ with synthesis operators T_Λ and T_Γ, respectively. Then the following statements are equivalent:*

(1) *there is an adjointable and invertible operator $T : \mathscr{U} \to \mathscr{U}$ such that $\Gamma_m = \Lambda_m T^*$ for each $m \in \mathscr{M}$; that is, $\{\Lambda_m : m \in \mathscr{M}\}$ and $\{\Gamma_m : m \in \mathscr{M}\}$ are similar;*

(2) *there exists a constant $M > 0$ such that*

$$\begin{aligned} &\langle (T_\Lambda - T_\Gamma) g, (T_\Lambda - T_\Gamma) g \rangle \\ &\quad \le M \cdot \min\{\langle T_\Lambda g, T_\Lambda g \rangle, \langle T_\Gamma g, T_\Gamma g \rangle\} \end{aligned} \tag{32}$$

for all $g = \{g_m\} \in \bigoplus_{m\in\mathscr{M}} \mathscr{V}_m$. Moreover, if (2) holds, then

$$\begin{aligned} \frac{1}{\left(1 + \sqrt{M}\right)^2} \langle f, f \rangle &\le \langle Tf, Tf \rangle \\ &\le \left(1 + \sqrt{M}\right)^2 \langle f, f \rangle, \quad \forall f \in \mathscr{U}. \end{aligned} \tag{33}$$

Proof. (1) ⇒ (2). Suppose that $T : \mathscr{U} \to \mathscr{U}$ is an adjointable and invertible operator such that $\Gamma_m = \Lambda_m T^*$ for each $m \in \mathscr{M}$. If $f = \int_{m\in\mathscr{M}} \Lambda_m^* g_m d\mu(m)$ for certain $g = \{g_m\} \in \bigoplus_{m\in\mathscr{M}} \mathscr{V}_m$, then we have

$$\begin{aligned} Tf &= \int_{m\in\mathscr{M}} T\Lambda_m^* g_m d\mu(m) = \int_{m\in\mathscr{M}} (\Lambda_m T^*)^* g_m d\mu(m) \\ &= \int_{m\in\mathscr{M}} \Gamma_m^* g_m d\mu(m). \end{aligned} \tag{34}$$

Therefore,

$$\begin{aligned} \langle f - Tf, f - Tf \rangle &= \langle f, f \rangle + \langle Tf, Tf \rangle + \langle (-T^* - T) f, f \rangle \\ &\le \langle f, f \rangle + \|T\|^2 \langle f, f \rangle + \|T^* + T\| \langle f, f \rangle \\ &\le (1 + \|T\|)^2 \langle f, f \rangle. \end{aligned} \tag{35}$$

On the other hand,

$$\begin{aligned} \langle f - Tf, f - Tf \rangle &= \langle f, f \rangle + \langle Tf, Tf \rangle \\ &\quad + \langle (-T^* - T) f, f \rangle \\ &= \left\langle T^{-1}Tf, T^{-1}Tf \right\rangle + \langle Tf, Tf \rangle \\ &\quad + \left\langle (-T^* - T) T^{-1}Tf, T^{-1}Tf \right\rangle \\ &\le \left(\left\|T^{-1}\right\|^2 + 1\right) \langle Tf, Tf \rangle \\ &\quad + \left\langle \left(T^{-1}\right)^* (-T^* - T) T^{-1}Tf, Tf \right\rangle \\ &\le \left(\left\|T^{-1}\right\|^2 + 1\right) \langle Tf, Tf \rangle \\ &\quad + \left\|\left(T^{-1}\right)^* (-T^* - T) T^{-1}\right\| \langle Tf, Tf \rangle \\ &\le \left(1 + \left\|T^{-1}\right\|\right)^2 \langle Tf, Tf \rangle. \end{aligned} \tag{36}$$

Hence, (32) follows.

(2) ⇒ (1). For each $f = \int_{m\in\mathscr{M}} \Lambda_m^* g_m d\mu(m) \in \mathscr{U}$, we define an operator $T : \mathscr{U} \to \mathscr{U}$ as follows:

$$Tf = T\left(\int_{m\in\mathscr{M}} \Lambda_m^* g_m d\mu(m)\right) = \int_{m\in\mathscr{M}} \Gamma_m^* g_m d\mu(m). \tag{37}$$

It is clear that T is well defined, and furthermore, T is adjointable. A simple calculation shows that its adjoint operator T^* is given by

$$T^* h = \int_{m\in\mathscr{M}} S_\Lambda^{-1} \Lambda_m^* \Gamma_m h d\mu(m), \quad \forall h \in \mathscr{U}, \tag{38}$$

where S_Λ is the frame operator of $\{\Lambda_m : m \in \mathscr{M}\}$. Since T_Γ is surjective by Proposition 7, it follows that T is also surjective. And (32) implies that T is injective, and so T is invertible.

It remains to establish that $\Gamma_m = \Lambda_m T^*$ for each $m \in \mathscr{M}$. For all $f \in \mathscr{U}$, $g = \{g_m\} \in \bigoplus_{m\in\mathscr{M}} \mathscr{V}_m$, we have

$$\begin{aligned}\int_{m\in\mathscr{M}} \langle g_m, \Lambda_m T^* f\rangle d\mu(m) &= \left\langle \int_{m\in\mathscr{M}} T\Lambda_m^* g_m d\mu(m), f\right\rangle \\ &= \left\langle T\int_{m\in\mathscr{M}} \Lambda_m^* g_m d\mu(m), f\right\rangle \\ &= \left\langle \int_{m\in\mathscr{M}} \Gamma_m^* g_m d\mu(m), f\right\rangle \\ &= \int_{m\in\mathscr{M}} \langle g_m, \Gamma_m f\rangle d\mu(m).\end{aligned} \tag{39}$$

That is, $\langle g, \{\Lambda_m T^* f - \Gamma_m f : m \in \mathscr{M}\}\rangle = 0$. Hence, $\Gamma_m = \Lambda_m T^*$ for each $m \in \mathscr{M}$.

For the last statement, the assumptions implies that $\|f - Tf\| \le \sqrt{M}\|f\|$ and $\|f - Tf\| \le \sqrt{M}\|Tf\|$ for all $f \in \mathscr{U}$. If we replace f by $T^{-1}f$ in the last inequality, we have $\|(T^{-1} - I_{\mathscr{U}})f\| \le \sqrt{M}\|f\|$. Therefore,

$$\begin{aligned}\langle f, f\rangle &= \langle f - Tf, f - Tf\rangle + \langle Tf, Tf\rangle \\ &\quad + (\langle f - Tf, Tf\rangle + \langle Tf, f - Tf\rangle) \\ &\le (M+1)\langle Tf, Tf\rangle \\ &\quad + \left\langle \left(T^{-1}\right)^* \left(T^*(I_{\mathscr{U}} - T) + (I_{\mathscr{U}} - T)^* T\right) \right. \\ &\qquad \left. \times T^{-1} Tf, Tf\right\rangle \\ &\le (M+1)\langle Tf, Tf\rangle \\ &\quad + \left\|\left(T^{-1}\right)^* \left(T^*(I_{\mathscr{U}} - T) + (I_{\mathscr{U}} - T)^* T\right) T^{-1}\right\| \\ &\quad \times \langle Tf, Tf\rangle \\ &\le \left(1 + M + 2\left\|T^{-1} - I_{\mathscr{U}}\right\|\right)\langle Tf, Tf\rangle \\ &\le \left(1 + \sqrt{M}\right)^2 \langle Tf, Tf\rangle, \\ \langle Tf, Tf\rangle &= \langle Tf - f, Tf - f\rangle + \langle f, f\rangle \\ &\quad + \langle Tf - f, f\rangle + \langle f, Tf - f\rangle \\ &\le (M+1)\langle f, f\rangle \\ &\quad + \left\langle \left((T - I_{\mathscr{U}}) + (T - I_{\mathscr{U}})^*\right) f, f\right\rangle \\ &\le \left((M+1) + \left\|(T - I_{\mathscr{U}}) + (T - I_{\mathscr{U}})^*\right\|\right) \\ &\quad \times \langle f, f\rangle \\ &\le (1 + M + 2\|T - I_{\mathscr{U}}\|)\langle f, f\rangle \\ &\le \left(1 + \sqrt{M}\right)^2 \langle f, f\rangle.\end{aligned} \tag{40}$$

This completes the proof. □

To complete this section, we generalize the results in [27] for g-frames in Hilbert spaces to continuous g-frames in Hilbert C^*-modules.

Proposition 18. *Let $\{\Lambda_m \in End_A^*(\mathscr{U}, \mathscr{V}_m) : m \in \mathscr{M}\}$ and $\{\Gamma_m \in End_A^*(\mathscr{U}, \mathscr{V}_m) : m \in \mathscr{M}\}$ be two continuous Parseval g-frames for $\mathscr{U}$ with respect to $\{\mathscr{V}_m : m \in \mathscr{M}\}$ with synthesis operators T_Λ and T_Γ, respectively. Then*

(1) *$Ran(T_\Gamma^*) \subseteq Ran(T_\Lambda^*)$ if and only if there exists an adjointable operator $U : \mathscr{U} \to \mathscr{U}$ which preserves inner product such that $\Gamma_m = \Lambda_m U$ for each $m \in \mathscr{M}$. Conversely, if $U : \mathscr{U} \to \mathscr{U}$ is an adjointable operator which preserves inner product such that $\Gamma_m = \Lambda_m U$ for each $m \in \mathscr{M}$, then*

$$Ran(T_\Lambda^*) = T_\Lambda^*(\mathscr{N}(U^*)) \oplus Ran(T_\Gamma^*); \tag{41}$$

(2) *$Ran(T_\Gamma^*) = Ran(T_\Lambda^*)$ if and only if $\{\Lambda_m : m \in \mathscr{M}\}$ and $\{\Gamma_m : m \in \mathscr{M}\}$ are unitary equivalent.*

Proof. (1) "⇒". Assume that $\mathrm{Ran}(T_\Gamma^*) \subseteq \mathrm{Ran}(T_\Lambda^*)$. Let us denote $P = T_\Lambda^* T_\Lambda$ and $Q = T_\Gamma^* T_\Gamma$. Since both T_Λ and T_Γ are surjective, we know that $\mathrm{Ran}(P) = \mathrm{Ran}(T_\Lambda^*)$ and $\mathrm{Ran}(Q) = \mathrm{Ran}(T_\Gamma^*)$. Since $\{\Lambda_m : m \in \mathscr{M}\}$ and $\{\Gamma_m : m \in \mathscr{M}\}$ are two continuous Parseval g-frames for $\mathscr{U}$, it follows that P and Q are orthogonal projections from $\bigoplus_{m\in\mathscr{M}} \mathscr{V}_m$ onto $\mathrm{Ran}(T_\Lambda^*)$ and $\mathrm{Ran}(T_\Gamma^*)$, respectively. Let $U = T_\Lambda T_\Gamma^*$, then, for an arbitrary element f of $\mathscr{U}$, recalling that $T_\Gamma^* f \in \mathrm{Ran}(T_\Lambda^*)$, we have

$$U^* U f = T_\Gamma T_\Lambda^* T_\Lambda T_\Gamma^* f = T_\Gamma T_\Gamma^* f = f. \tag{42}$$

Thus, U preserves inner product. Also,

$$U^* f = T_\Gamma T_\Lambda^* f = \int_{m\in\mathscr{M}} \Gamma_m^* \Lambda_m f d\mu(m), \tag{43}$$

and so,

$$\begin{aligned}\int_{m\in\mathscr{M}} \langle \Lambda_m U f, \Gamma_m f\rangle d\mu(m) &= \left\langle \int_{m\in\mathscr{M}} \Gamma_m^* \Lambda_m U f d\mu(m), f\right\rangle \\ &= \langle U^* U f, f\rangle = \langle f, f\rangle.\end{aligned} \tag{44}$$

Note that

$$\begin{gathered}\int_{m\in\mathscr{M}} \langle \Gamma_m f, \Gamma_m f\rangle d\mu(m) = \langle f, f\rangle, \\ \int_{m\in\mathscr{M}} \langle \Lambda_m U f, \Lambda_m U f\rangle d\mu(m) = \langle Uf, Uf\rangle = \langle f, f\rangle;\end{gathered} \tag{45}$$

it follows that

$$\begin{aligned}&\langle \{(\Gamma_m - \Lambda_m U) f : m \in \mathscr{M}\}, \{(\Gamma_m - \Lambda_m U) f : m \in \mathscr{M}\}\rangle \\ &\quad = \int_{m\in\mathscr{M}} \langle (\Gamma_m - \Lambda_m U) f, (\Gamma_m - \Lambda_m U) f\rangle d\mu(m)\end{aligned}$$

$$\begin{aligned}
&= \int_{m\in\mathscr{M}} \langle \Gamma_m f, \Gamma_m f\rangle \, d\mu(m) \\
&\quad - \int_{m\in\mathscr{M}} \langle \Gamma_m f, \Lambda_m U f\rangle \, d\mu(m) \\
&\quad - \int_{m\in\mathscr{M}} \langle \Lambda_m U f, \Gamma_m f\rangle \, d\mu(m) \\
&\quad + \int_{m\in\mathscr{M}} \langle \Lambda_m U f, \Lambda_m U f\rangle \, d\mu(m) \\
&= \langle f, f\rangle - \langle f, f\rangle - \langle f, f\rangle + \langle f, f\rangle = 0.
\end{aligned} \tag{46}$$

Hence, $\{(\Gamma_m - \Lambda_m U) f : m \in \mathscr{M}\} = 0$ for each $f \in \mathscr{U}$, and $\Gamma_m = \Lambda_m U$ for each $m \in \mathscr{M}$ as a consequence.

"$\Leftarrow$". It is obvious.

For the second part of (1), since T_Λ^* is an isometry, it follows that

$$\begin{aligned}
\operatorname{Ran}(T_\Lambda^*) &= T_\Lambda^* \left(\mathscr{N}(U^*) \oplus (\mathscr{N}(U^*))^\perp \right) \\
&= T_\Lambda^* \left(\mathscr{N}(U^*) \oplus \operatorname{Ran}(U) \right) \\
&= T_\Lambda^* \left(\mathscr{N}(U^*) \right) \oplus \operatorname{Ran}(T_\Lambda^* U) \\
&= T_\Lambda^* \left(\mathscr{N}(U^*) \right) \oplus \operatorname{Ran}(T_\Gamma^*).
\end{aligned} \tag{47}$$

(2) Suppose that $\operatorname{Ran}(T_\Gamma^*) = \operatorname{Ran}(T_\Lambda^*)$, then (41) implies that $T_\Lambda^*(\mathscr{N}(U^*)) = 0$, and hence, $\mathscr{N}(U^*) = 0$. Thus, U^* is injective, and so, U is invertible. Since $U^* U = I_{\mathscr{U}}$, it follows that U is unitary. For the other implication, let $U : \mathscr{U} \to \mathscr{U}$ be a unitary linear operator such that $\Gamma_m = \Lambda_m U$ for each $m \in \mathscr{M}$. Then $T_\Gamma^* = T_\Lambda^* U$, and so, $\operatorname{Ran}(T_\Gamma^*) = \operatorname{Ran}(T_\Lambda^*)$. □

For the general case, we have the following proposition.

Proposition 19. *Let $\{\Lambda_m \in End_A^*(\mathscr{U}, \mathscr{V}_m) : m \in \mathscr{M}\}$ and $\{\Gamma_m \in End_A^*(\mathscr{U}, \mathscr{V}_m) : m \in \mathscr{M}\}$ be two continuous g-frames for $\mathscr{U}$ with respect to $\{\mathscr{V}_m : m \in \mathscr{M}\}$ with synthesis operators T_Λ and T_Γ and frame operators S_Λ and S_Γ, respectively. Then*

(1) *$Ran(T_\Gamma^*) \subseteq Ran(T_\Lambda^*)$ if and only if there exists an adjointable operator $U : \mathscr{U} \to \mathscr{U}$ such that $\Gamma_m = \Lambda_m U$ for each $m \in \mathscr{M}$;*

(2) *$Ran(T_\Gamma^*) = Ran(T_\Lambda^*)$ if and only if $\{\Lambda_m : m \in \mathscr{M}\}$ and $\{\Gamma_m : m \in \mathscr{M}\}$ are similar.*

Proof. (1) "$\Rightarrow$". Assume that $\operatorname{Ran}(T_\Gamma^*) \subseteq \operatorname{Ran}(T_\Lambda^*)$. We already know that $\operatorname{Ran}(T_\Lambda^*)$ and $\operatorname{Ran}(T_\Gamma^*)$ are closed submodules of $\bigoplus_{m\in\mathscr{M}} \mathscr{V}_m$. Then $\operatorname{Ran}(T_\Lambda^*) = (\mathscr{N}(T_\Lambda))^\perp$ and $\operatorname{Ran}(T_\Gamma^*) = (\mathscr{N}(T_\Gamma))^\perp$, and thus, $\mathscr{N}(T_\Lambda) \subseteq \mathscr{N}(T_\Gamma)$. It is easy to check that $\Lambda' = \{\Lambda_m S_\Lambda^{-1/2} : m \in \mathscr{M}\}$ and $\Gamma' = \{\Gamma_m S_\Gamma^{-1/2} : m \in \mathscr{M}\}$ are both continuous Parseval g-frames. Let us denote by $T_{\Lambda'}$ and $T_{\Gamma'}$ the synthesis operators of Λ' and Γ', respectively. Then $T_{\Lambda'} = S_\Lambda^{-1/2} T_\Lambda$ and $T_{\Gamma'} = S_\Gamma^{-1/2} T_\Gamma$. Therefore, $\mathscr{N}(T_{\Lambda'}) = \mathscr{N}(T_\Lambda)$ and $\mathscr{N}(T_{\Gamma'}) = \mathscr{N}(T_\Gamma)$. By Proposition 18, there exists an adjointable operator $S : \mathscr{U} \to \mathscr{U}$ such that $\Gamma_m S_\Gamma^{-1/2} = \Lambda_m S_\Lambda^{-1/2} S$ for each $m \in \mathscr{M}$. Hence, the result follows by letting $U = S_\Lambda^{-1/2} S S_\Gamma^{1/2}$.

"$\Leftarrow$". It is straightforward.

(2) "$\Rightarrow$". If $\operatorname{Ran}(T_\Gamma^*) = \operatorname{Ran}(T_\Lambda^*)$, then $\operatorname{Ran}(T_{\Gamma'}^*) = \operatorname{Ran}(T_{\Lambda'}^*)$. By part (2) of Proposition 18, S is unitary, and consequently, $U = S_\Lambda^{-1/2} S S_\Gamma^{1/2}$ is invertible.

"$\Leftarrow$". It is obvious. □

4. Stability of Duals of Continuous g-Frames in Hilbert C^*-Modules

The stability of frames is important in practice and is therefore studied widely by many authors. The stability of dual frames is also needed in practice. However, most of the known results on this topic are stated about canonical dual; see [28] for frames in Hilbert spaces and [29, 30] for g-frames in Hilbert spaces. Fortunately, Arefijamaal and Ghasemi [31] presented a stability result for alternate duals of g-frames in Hilbert spaces by observing the difference between an alternate dual and the canonical dual. In what follows, we will generalize their result to alternate duals of continuous g-frames in Hilbert C^*-modules. We start with the following lemma, which shows that the difference between an alternate dual and the canonical dual can be considered as an adjointable operator.

Lemma 20. *Let $\{\Lambda_m \in End_A^*(\mathscr{U}, \mathscr{V}_m) : m \in \mathscr{M}\}$ be a continuous g-frame for $\mathscr{U}$ with respect to $\{\mathscr{V}_m : m \in \mathscr{M}\}$ with bounds A, B and the synthesis operator T_Λ. Then there exists a one-to-one correspondence between the duals of $\{\Lambda_m : m \in \mathscr{M}\}$ and operator $\psi \in End_A^*(\mathscr{U}, \bigoplus_{m\in\mathscr{M}} \mathscr{V}_m)$ such that $T_\Lambda \psi = 0$.*

Proof. Assume first that $\{\Gamma_m \in \operatorname{End}_A^*(\mathscr{U}, \mathscr{V}_m) : m \in \mathscr{M}\}$ is a dual continuous g-frame of $\{\Lambda_m : m \in \mathscr{M}\}$ with bounds A_1 and B_1, and let S_Λ be the frame operator of $\{\Lambda_m : m \in \mathscr{M}\}$. Define $\psi : \mathscr{U} \to \bigoplus_{m\in\mathscr{M}} \mathscr{V}_m$, $f \mapsto \psi f$ by

$$(\psi f)_m = \Gamma_m f - \Lambda_m S_\Lambda^{-1} f, \quad m \in \mathscr{M}. \tag{48}$$

Then ψ is adjointable, that is; $\psi \in \operatorname{End}_A^*(\mathscr{U}, \bigoplus_{m\in\mathscr{M}} \mathscr{V}_m)$. Indeed,

$$\begin{aligned}
\langle \psi f, g\rangle &= \int_{m\in\mathscr{M}} \langle (\psi f)_m, g_m\rangle \, d\mu(m) \\
&= \int_{m\in\mathscr{M}} \left\langle \Gamma_m f - \Lambda_m S_\Lambda^{-1} f, g_m \right\rangle d\mu(m) \\
&= \int_{m\in\mathscr{M}} \langle f, \Gamma_m^* g_m\rangle \, d\mu(m) \\
&\quad - \int_{m\in\mathscr{M}} \left\langle f, S_\Lambda^{-1} \Lambda_m^* g_m \right\rangle d\mu(m) \\
&= \int_{m\in\mathscr{M}} \left\langle f, \Gamma_m^* g_m - S_\Lambda^{-1} \Lambda_m^* g_m \right\rangle d\mu(m) \\
&= \left\langle f, \int_{m\in\mathscr{M}} \left(\Gamma_m^* g_m - S_\Lambda^{-1} \Lambda_m^* g_m \right) d\mu(m) \right\rangle,
\end{aligned} \tag{49}$$

for all $f \in \mathscr{U}$, $g = \{g_m\} \in \bigoplus_{m\in\mathscr{M}} \mathscr{V}_m$. Moreover, we have

$$\begin{aligned} T_\Lambda \psi f &= \int_{m\in\mathscr{M}} \Lambda_m^* (\psi f)_m d\mu(m) \\ &= \int_{m\in\mathscr{M}} \Lambda_m^* \left(\Gamma_m f - \Lambda_m S_\Lambda^{-1} f\right) d\mu(m) \\ &= \int_{m\in\mathscr{M}} \Lambda_m^* \Gamma_m f d\mu(m) - \int_{m\in\mathscr{M}} \Lambda_m^* \Lambda_m S_\Lambda^{-1} f d\mu(m) \\ &= f - f = 0. \end{aligned} \tag{50}$$

Conversely, let $\psi \in \mathrm{End}_A^*(\mathscr{U}, \bigoplus_{m\in\mathscr{M}} \mathscr{V}_m)$ and $T_\Lambda \psi = 0$. Take

$$\Gamma_m f = (\psi f)_m + \Lambda_m S_\Lambda^{-1} f, \quad f \in \mathscr{U},\ m \in \mathscr{M}. \tag{51}$$

Since

$$\begin{aligned} &\left\| \int_{m\in\mathscr{M}} \langle \Gamma_m f, \Gamma_m f\rangle d\mu(m) \right\|^{1/2} \\ &\quad = \left\| \{\Gamma_m f\}_{m\in\mathscr{M}} \right\| \le \left\| \{(\psi f)_m\}_{m\in\mathscr{M}} \right\| + \left\| \{\Lambda_m S_\Lambda^{-1} f\}_{m\in\mathscr{M}} \right\| \\ &\quad \le \|\psi f\| + \left\| \int_{m\in\mathscr{M}} \left\langle \Lambda_m S_\Lambda^{-1} f, \Lambda_m S_\Lambda^{-1} f\right\rangle d\mu(m) \right\|^{1/2} \\ &\quad \le \|\psi\| \|f\| + \frac{1}{\sqrt{A}} \|f\|, \end{aligned} \tag{52}$$

it follows that $\{\Gamma_m : m \in \mathscr{M}\}$ is a continuous g-Bessel sequence for $\mathscr{U}$ with respect to $\{\mathscr{V}_m : m \in \mathscr{M}\}$. Furthermore,

$$\begin{aligned} \int_{m\in\mathscr{M}} \Lambda_m^* \Gamma_m f d\mu(m) &= \int_{m\in\mathscr{M}} \Lambda_m^* (\psi f)_m d\mu(m) \\ &\quad + \int_{m\in\mathscr{M}} \Lambda_m^* \Lambda_m S_\Lambda^{-1} f d\mu(m) \\ &= T_\Lambda \psi f + f = f. \end{aligned} \tag{53}$$

Thus, $\{\Gamma_m : m \in \mathscr{M}\}$ is a dual continuous g-frame of $\{\Lambda_m : m \in \mathscr{M}\}$, by Proposition 6. □

Theorem 21. *Let $\{\Lambda_m \in End_A^*(\mathscr{U}, \mathscr{V}_m) : m \in \mathscr{M}\}$ and $\{\Gamma_m \in End_A^*(\mathscr{U}, \mathscr{V}_m) : m \in \mathscr{M}\}$ be two continuous g-frames for $\mathscr{U}$ with respect to $\{\mathscr{V}_m : m \in \mathscr{M}\}$ with bounds A_1, B_1 and A_2, B_2, respectively. Also, let $\{\Lambda'_m : m \in \mathscr{M}\}$ be a fixed dual of $\{\Lambda_m : m \in \mathscr{M}\}$ with frame bounds A_3, B_3. If $\{\Lambda_m - \Gamma_m : m \in \mathscr{M}\}$ is a continuous g-Bessel sequence with Bessel bound ϵ, then there exists a dual $\{\Gamma'_m : m \in \mathscr{M}\}$ of $\{\Gamma_m : m \in \mathscr{M}\}$ such that $\{\Lambda'_m - \Gamma'_m : m \in \mathscr{M}\}$ is also a continuous g-Bessel sequence.*

Proof. Let us denote by T_Λ, T_Γ and S_Λ, S_Γ the synthesis operators and frame operators of $\{\Lambda_m : m \in \mathscr{M}\}$ and $\{\Gamma_m : m \in \mathscr{M}\}$, respectively. By the proof of Lemma 20 we know that there exists $\psi \in \mathrm{End}_A^*(\mathscr{U}, \bigoplus_{m\in\mathscr{M}} \mathscr{V}_m)$ with

$$(\psi f)_m = \frac{1}{2\sqrt{\epsilon}} \frac{1}{\left(\sqrt{B_3} + 1/\sqrt{A_1}\right)} \left(\Lambda'_m f - \Lambda_m S_\Lambda^{-1} f\right) \tag{54}$$

such that $T_\Lambda \psi = 0$ for all $f \in \mathscr{U}$, $m \in \mathscr{M}$. A simple calculation shows that

$$\|\psi\| \le \frac{1}{2\sqrt{\epsilon}} \frac{1}{\left(\sqrt{B_3} + 1/\sqrt{A_1}\right)} \left(\sqrt{B_3} + \frac{1}{\sqrt{A_1}}\right) = \frac{1}{2\sqrt{\epsilon}}. \tag{55}$$

Let

$$\Theta_m f = \Gamma_m S_\Gamma^{-1} f + (\psi f)_m, \quad \forall f \in \mathscr{U},\ m \in \mathscr{M}. \tag{56}$$

It is easy to see that $\{\Theta_m : m \in \mathscr{M}\}$ is a continuous g-Bessel sequence. Let T_Θ be the synthesis operator of $\{\Theta_m : m \in \mathscr{M}\}$, then $\|T_\Theta\| = \|T_\Theta^*\| \le 1/\sqrt{A_2} + 1/(2\sqrt{\epsilon})$ and

$$\begin{aligned} \|f - T_\Gamma T_\Theta^* f\| &= \left\| f - \int_{m\in\mathscr{M}} \Gamma_m^* \Theta_m f d\mu(m) \right\| \\ &= \left\| f - \int_{m\in\mathscr{M}} \Gamma_m^* \Gamma_m S_\Gamma^{-1} f d\mu(m) \right. \\ &\qquad \left. - \int_{m\in\mathscr{M}} \Gamma_m^* (\psi f)_m d\mu(m) \right\| \\ &= \|T_\Gamma \psi f\| = \|T_\Gamma \psi f - T_\Lambda \psi f\| \\ &\le \|T_\Gamma - T_\Lambda\| \|\psi\| \|f\| \\ &\le \sqrt{\epsilon} \|\psi\| \|f\| \\ &\le \sqrt{\epsilon} \frac{1}{2\sqrt{\epsilon}} \|f\| = \frac{1}{2} \|f\|. \end{aligned} \tag{57}$$

Hence, $\|I_{\mathscr{U}} - T_\Gamma T_\Theta^*\| < 1$, and furthermore, $T_\Gamma T_\Theta^*$ is invertible. Therefore, every $f \in \mathscr{U}$ can be represented by

$$f = T_\Gamma T_\Theta^* \left(T_\Gamma T_\Theta^*\right)^{-1} f = \int_{m\in\mathscr{M}} \Gamma_m^* \Theta_m \left(T_\Gamma T_\Theta^*\right)^{-1} f d\mu(m), \tag{58}$$

showing that $\{\Gamma'_m : m \in \mathscr{M}\} = \{\Theta_m (T_\Gamma T_\Theta^*)^{-1} : m \in \mathscr{M}\}$ is a dual of $\{\Gamma_m : m \in \mathscr{M}\}$. In what follows, we will show that $\{\Gamma'_m : m \in \mathscr{M}\}$ is the desired continuous g-frame.

If we take $T = (T_\Gamma T_\Theta^*)^{-1}$, then

$$\|T\| \le \frac{1}{1 - \left\|I_{\mathscr{U}} - T^{-1}\right\|} \le \frac{1}{1 - \sqrt{\epsilon}\|\psi\|}, \tag{59}$$

and so,

$$\left\|I_{\mathscr{U}} - T\right\| \le \|T\| \left\|I_{\mathscr{U}} - T^{-1}\right\| \le \frac{\sqrt{\epsilon}\|\psi\|}{1 - \sqrt{\epsilon}\|\psi\|}. \tag{60}$$

Denoting by $T_{\Lambda'}$ the synthesis operator of $\{\Lambda'_m : m \in \mathscr{M}\}$, we have

$$\begin{aligned}
&\left\| \int_{m\in\mathscr{M}} \left(\Lambda'_m\right)^* g_m d\mu(m) \right. \\
&\quad \left. - \int_{m\in\mathscr{M}} \left(\Gamma'_m\right)^* g_m d\mu(m) \right\| \\
&\quad = \left\| \int_{m\in\mathscr{M}} \left(\Lambda'_m\right)^* g_m d\mu(m) \right. \\
&\qquad \left. - \int_{m\in\mathscr{M}} T^* \Theta_m^* g_m d\mu(m) \right\| \\
&\quad = \left\| T_{\Lambda'} g - T^* T_\Theta g \right\| \\
&\quad = \left\| T_{\Lambda'} g - T^* T_{\Lambda'} g + T^* T_{\Lambda'} g - T^* T_\Theta g \right\| \\
&\quad \le \left\| I_{\mathscr{U}} - T \right\| \left\| T_{\Lambda'} g \right\| \\
&\qquad + \|T\| \left\| T_{\Lambda'} g - T_\Theta g \right\| \\
&\quad \le \left(\left\| I_{\mathscr{U}} - T \right\| + \|T\| \right) \left\| T_{\Lambda'} g \right\| \\
&\qquad + \|T\| \left\| T_\Theta g \right\| \\
&\quad \le \frac{1 + \sqrt{\epsilon}\|\psi\|}{1 - \sqrt{\epsilon}\|\psi\|} \sqrt{B'} \|g\| \\
&\qquad + \frac{1}{1 - \sqrt{\epsilon}\|\psi\|} \left(\frac{1}{\sqrt{A_2}} + \frac{1}{2\sqrt{\epsilon}} \right) \|g\| \\
&\quad = \frac{\left(1 + \sqrt{\epsilon}\|\psi\|\right)\sqrt{B_1} + 1/\sqrt{A_2} + 1/2\sqrt{\epsilon}}{1 - \sqrt{\epsilon}\|\psi\|} \|g\|,
\end{aligned} \tag{61}$$

where B' is the upper frame bound of $\{\Lambda'_m : m \in \mathscr{M}\}$. The proof is completed. □

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