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Research Article

The Relationship between Two Kinds of Generalized Convex Set-Valued Maps in Real Ordered Linear Spaces

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A new notion of the ic-cone convexlike set-valued map characterized by the algebraic interior and the vector closure is introduced in real ordered linear spaces. The relationship between the ic-cone convexlike set-valued map and the nearly cone subconvexlike set-valued map is established. The results in this paper generalize some known results in the literature from locally convex spaces to linear spaces.

1. Introduction

In optimization theory, the generalized convexity of set-valued maps plays an important role. Corley [1] introduced the cone convexity of set-valued maps. To extend the cone convexity of set-valued maps, some authors [2–5] introduced new generalized convexity such as cone convexlikeness, cone subconvexlikeness, generalized cone subconvexlikeness, nearly cone subconvexlikeness, and ic-cone-convexlikeness. The above generalized convexity set-valued maps mentioned were defined in topological spaces. Recently, Li [6] has introduced the cone subconvexlike set-valued map based on the algebraic interior in linear spaces. Very recently, Hernández et al. [7] have defined the cone subconvexlikeness of the set-valued map characterized by the relative algebraic interior. Xu and Song [8] gave the relationship between ic-cone convexity and nearly cone subconvexlikeness in locally convex spaces. In this paper, we will extend the results obtained by Xu and Song [8] from locally convex spaces to linear spaces.

This paper is organized as follows. In Section 2, we give some preliminaries, including notations and lemmas. In Section 3, we obtain the relationship between ic-cone convexity and nearly cone subconvexlikeness in linear spaces. Our results generalize and improve the ones obtained by Xu and Song [8].

2. Preliminaries

In this paper, we always suppose that A is a nonempty set and Y is a real ordered linear space. Let 0 denote the zero element for every space. Let K be a nonempty subset in Y. The affine hull of K is defined as $\operatorname{aff}(K) := \{k \mid k = \sum_{i=1}^{n} \lambda_i k_i, \forall i \in \{1, 2, \ldots, n\}, k_i \in K, \lambda_i \in \mathbb{R}, \sum_{i=1}^{n} \lambda_i = 1\}$. The generated cone of K is defined as $\operatorname{cone}(K) := \{\lambda k \mid k \in K, \lambda \geq 0\}$. Write $\operatorname{cone}_+(K) := \{\lambda k \mid k \in K, \lambda > 0\}$. Clearly, $\operatorname{cone}(K) = \operatorname{cone}_+(K) \cup \{0\}$. K is called a cone if and only if $\lambda K \subseteq K$ for any $\lambda \geq 0$. Note that some authors defined the cone in the following way: K is called a cone if and only if $\lambda K \subseteq K$ for any $\lambda > 0$ [5]. It is possible that $0 \notin K$ if K is a cone in the sense of the latter definition. Moreover, if K is a cone in the sense of the latter definition, then $K \cup \{0\}$ is a cone in the sense of the former definition. In this paper, if not specially specified, we suppose that all the cones mentioned are defined in the sense of the former definition. K is called a convex set if and only if

$$\lambda k_1 + (1-\lambda) k_2 \in K, \quad \forall \lambda \in [0, 1], \forall k_1, k_2 \in K. \tag{1}$$

Clearly, a cone K is convex if and only if $K + K \subseteq K$. K is said to be nontrivial if and only if $K \neq \{0\}$ and $K \neq Y$.

From now on, we suppose that C is a nontrivial convex cone in Y and C_+ satisfies the condition $C = C_+ \cup \{0\}$. We recall the following well-known concepts.

Definition 1 (see [9]). Let K be a nonempty subset in Y. The algebraic interior of K is the set

$$\operatorname{cor}(K) := \left\{k \in K \mid \forall h \in Y, \exists \lambda' > 0, \forall \lambda \in \left[0, \lambda'\right], k + \lambda h \in K\right\}. \tag{2}$$

Definition 2 (see [10]). Let K be a nonempty subset in Y. The relative algebraic interior of K is the set

$$\operatorname{icr}(K) := \left\{k \in K \mid \forall h \in \operatorname{aff}(K) - k, \exists \lambda' > 0, \forall \lambda \in \left[0, \lambda'\right], k + \lambda h \in K\right\}. \tag{3}$$

Remark 3. Clearly, $\operatorname{cor}(K) \subseteq \operatorname{icr}(K)$. Moreover, if $\operatorname{cor}(K) \neq \emptyset$, then $\operatorname{cor}(K) = \operatorname{icr}(K)$.

Definition 4 (see [11]). Let K be a nonempty subset in Y. The vector closure of K is the set

$$\operatorname{vcl}(K) := \left\{k \in Y \mid \exists h \in Y, \forall \lambda' > 0, \exists \lambda \in \left]0, \lambda'\right], k + \lambda h \in K\right\}. \tag{4}$$

Let $F : A \rightrightarrows Y$ be a set-valued map on A. $F(A) := \bigcup_{x \in A} F(x)$.

Definition 5 (see [12]). A set-valued map $F : A \rightrightarrows Y$ is called nearly C-subconvexlike on A if and only if $\operatorname{vcl}(\operatorname{cone}(F(A) + C))$ is a convex set in Y.

Remark 6. When the set-valued map $F : A \rightrightarrows Y$ becomes a vector-valued map $f : A \to Y$, Definition 5 reduces to Definition 4.1 in [13]. When the linear spaces Y becomes a topological space, Definition 5 becomes Definition 2.2 in [4].

In locally convex spaces, Sach [5] introduced the ic-C_+-convexlikeness of the set-valued map. Now, we use the vector closure and the algebraic interior to introduce the ic-C_+-convexlikeness of the set-valued map in linear spaces.

Definition 7. A set-valued map $F : A \rightrightarrows Y$ is called ic-C_+-convexlike on A if and only if $\operatorname{cor}(\operatorname{cone}_+(F(A) + C_+))$ is a convex set in Y and $\operatorname{cone}_+(F(A)+C_+) \subseteq \operatorname{vcl}(\operatorname{cor}(\operatorname{cone}_+(F(A)+C_+)))$.

Lemma 8. *Let U_1 and U_2 be two nonempty sets in Y. Then, $\operatorname{vcl}(U_1 \cup U_2) = \operatorname{vcl}(U_1) \cup \operatorname{vcl}(U_2)$.*

Proof. Since $U_1 \subseteq U_1 \cup U_2$ and $U_2 \subseteq U_1 \cup U_2$, $\operatorname{vcl}(U_1) \cup \operatorname{vcl}(U_2) \subseteq \operatorname{vcl}(U_1 \cup U_2)$. Now, we prove

$$\operatorname{vcl}(U_1 \cup U_2) \subseteq \operatorname{vcl}(U_1) \cup \operatorname{vcl}(U_2). \tag{5}$$

Suppose that $y \notin \operatorname{vcl}(U_1) \cup \operatorname{vcl}(U_2)$. Then, $y \notin \operatorname{vcl}(U_1)$ and $y \notin \operatorname{vcl}(U_2)$. For any $h \in Y$, there exists $\lambda_1 > 0$ such that

$$y + \lambda h \notin U_1, \quad \forall \lambda \in \left]0, \lambda_1\right]. \tag{6}$$

For the above $h \in Y$, there exists $\lambda_2 > 0$ such that

$$y + \lambda h \notin U_2, \quad \forall \lambda \in \left]0, \lambda_2\right]. \tag{7}$$

It follows from (6) and (7) that, for the above $h \in Y$, there exists $\lambda_3 = \min\{\lambda_1, \lambda_2\} > 0$ such that

$$y + \lambda h \notin U_1 \cup U_2, \quad \forall \lambda \in \left]0, \lambda_3\right], \tag{8}$$

which implies that $y \notin \operatorname{vcl}(U_1 \cup U_2)$. Therefore, (5) holds. Thus, we obtain $\operatorname{vcl}(U_1 \cup U_2) = \operatorname{vcl}(U_1) \cup \operatorname{vcl}(U_2)$. □

Lemma 9 (see [11]). *If K is a nonempty convex set in Y and $\operatorname{icr}(K) \neq \emptyset$, then*

(a) *$\operatorname{vcl}(K)$ is a convex set in Y;*

(b) *$\operatorname{vcl}(\operatorname{vcl}(K)) = \operatorname{vcl}(K)$, namely, $\operatorname{vcl}(K)$ is vectorially closed;*

(c) *$\operatorname{vcl}(K) = \operatorname{vcl}(\operatorname{icr}(K))$.*

Lemma 10 (see [11]). *Let K be a nonempty subset of Y, and let C be a nontrivial and convex cone with $\operatorname{cor}(C) \neq \emptyset$. Then, $\operatorname{cor}(K + \operatorname{cor}(C)) = K + \operatorname{cor}(C) = \operatorname{cor}(\operatorname{vcl}(K + C)) = \operatorname{cor}(K + C)$.*

Remark 11. The conclusions of Lemma 10 are true when C is replaced by C_+.

3. The Relationship between Two Kinds of Generalized Convexity

In this section, we will give the relationship between two kinds of generalized convexity in real ordered linear spaces.

Theorem 12. *Let $F : A \rightrightarrows Y$ be a set-valued map on A and $\operatorname{icr}(\operatorname{cor}(\operatorname{cone}_+(F(A) + C_+))) \neq \emptyset$. If F is ic-C_+-convexlike on A, then F is nearly C-subconvexlike on A.*

Proof. Since F is ic-C_+-convexlike on A, $\operatorname{cor}(\operatorname{cone}_+(F(A) + C_+))$ is a convex set in Y and $\operatorname{cone}_+(F(A) + C_+) \subseteq \operatorname{vcl}(\operatorname{cor}(\operatorname{cone}_+(F(A) + C_+)))$, which implies that

$$\operatorname{vcl}(\operatorname{cone}_+(F(A) + C_+)) \subseteq \operatorname{vcl}(\operatorname{vcl}(\operatorname{cor}(\operatorname{cone}_+(F(A) + C_+)))). \tag{9}$$

Using the convexity of $\operatorname{cor}(\operatorname{cone}_+(F(A) + C_+))$ and (b) of Lemma 9, we have

$$\operatorname{vcl}(\operatorname{vcl}(\operatorname{cor}(\operatorname{cone}_+(F(A) + C_+)))) = \operatorname{vcl}(\operatorname{cor}(\operatorname{cone}_+(F(A) + C_+))). \tag{10}$$

It follows from (9) and (10) that

$$\operatorname{vcl}(\operatorname{cone}_+(F(A) + C_+)) \subseteq \operatorname{vcl}(\operatorname{cor}(\operatorname{cone}_+(F(A) + C_+))). \tag{11}$$

Clearly,

$$\operatorname{vcl}(\operatorname{cor}(\operatorname{cone}_+(F(A) + C_+))) \subseteq \operatorname{vcl}(\operatorname{cone}_+(F(A) + C_+)). \tag{12}$$

By (11) and (12), we obtain

$$\operatorname{vcl}\left(\operatorname{cone}_{+}\left(F(A)+C_{+}\right)\right)=\operatorname{vcl}\left(\operatorname{cor}\left(\operatorname{cone}_{+}\left(F(A)+C_{+}\right)\right)\right). \tag{13}$$

Since $\operatorname{cor}(\operatorname{cone}_{+}(F(A)+C_{+}))$ is a convex set in Y, it follows from (13) and (a) of Lemma 9 that $\operatorname{vcl}(\operatorname{cone}_{+}(F(A)+C_{+}))$ is a convex set in Y. Using Lemma 8, we have

$$\begin{aligned}
&\operatorname{vcl}(\operatorname{cone}(F(A)+C))\\
&\quad=\operatorname{vcl}\left(\operatorname{cone}_{+}(F(A)+C)\cup\{0\}\right)\\
&\quad=\operatorname{vcl}\left(\operatorname{cone}_{+}(F(A)+C)\right)\cup\operatorname{vcl}\{0\}\\
&\quad=\operatorname{vcl}\left(\operatorname{cone}_{+}(F(A)+C)\right)\\
&\quad=\operatorname{vcl}\left(\operatorname{cone}_{+}(F(A))+C\right)\\
&\quad=\operatorname{vcl}\left(\operatorname{cone}_{+}(F(A))+C_{+}\cup\{0\}\right)\\
&\quad=\operatorname{vcl}\left(\left(\operatorname{cone}_{+}(F(A))+C_{+}\right)\cup\operatorname{cone}_{+}(F(A))\right)\\
&\quad=\operatorname{vcl}\left(\operatorname{cone}_{+}(F(A))+C_{+}\right)\cup\operatorname{vcl}\left(\operatorname{cone}_{+}(F(A))\right).
\end{aligned} \tag{14}$$

Now, we prove that

$$\operatorname{vcl}\left(\operatorname{cone}_{+}(F(A))\right)\subseteq\operatorname{vcl}\left(\operatorname{cone}_{+}(F(A))+C_{+}\right). \tag{15}$$

Let $y\in\operatorname{vcl}(\operatorname{cone}_{+}(F(A)))$. Then, $\exists h\in Y$, for all $\lambda'>0$, $\exists\lambda\in\,]0,\lambda']$, and we have

$$y+\lambda h\in\operatorname{cone}_{+}(F(A)). \tag{16}$$

Take $c\in C_{+}$ in Y. By (16), $\exists h+c\in Y$, for all $\lambda'>0$, $\exists\lambda\in\,]0,\lambda']$, and we have

$$y+\lambda(h+c)\in\operatorname{cone}_{+}(F(A))+C_{+}, \tag{17}$$

which implies $y\in\operatorname{vcl}(\operatorname{cone}_{+}(F(A))+C_{+})$. Therefore, (15) holds. It follows from (14) and (15) that

$$\begin{aligned}
\operatorname{vcl}(\operatorname{cone}(F(A)+C))&=\operatorname{vcl}\left(\operatorname{cone}_{+}(F(A))+C_{+}\right)\\
&=\operatorname{vcl}\left(\operatorname{cone}_{+}\left(F(A)+C_{+}\right)\right).
\end{aligned} \tag{18}$$

Since $\operatorname{vcl}(\operatorname{cone}_{+}(F(A)+C_{+}))$ is a convex set in Y, it follows from (18) that $\operatorname{vcl}(\operatorname{cone}(F(A)+C))$ is a convex set in Y. Therefore, F is nearly C-subconvexlike on A. □

Remark 13. If Y is a locally convex space or a finite dimensional linear space, then the condition $\operatorname{icr}(\operatorname{cor}(\operatorname{cone}_{+}(F(A)+C_{+})))\neq\emptyset$ can be dropped. Thus, Theorem 12 generalizes Theorem 3.2 in [8] from locally convex spaces to linear spaces.

The following example shows that the converse of Theorem 12 is not true.

Example 14. Let $Y=\mathbb{R}^2$, $C=\{(y_1,y_2)\mid y_1\geq 0, y_2=0\}$, $C_{+}=\{(y_1,y_2)\mid y_1>0, y_2=0\}$, and $A=\{(1,0),(0,1)\}$. The set-valued map $F:A\rightrightarrows Y$ is defined as follows:

$$\begin{aligned}
F(1,0)&=\left\{(y_1,y_2)\mid 1\leq y_1\leq 2-y_2,\ y_2>0\right\},\\
F(0,1)&=\left\{(y_1,y_2)\mid 1\leq y_1\leq 2+y_2,\ y_2<0\right\}.
\end{aligned} \tag{19}$$

It is easy to check that $\operatorname{icr}(\operatorname{cor}(\operatorname{cone}_{+}(F(A)+C_{+})))\neq\emptyset$. Moreover, $\operatorname{vcl}(\operatorname{cone}(F(A)+C))$ is a convex set in Y. Therefore, F is nearly C-subconvexlike on A. However, $\operatorname{cor}(\operatorname{cone}_{+}(F(A)+C_{+}))$ is not a convex set in Y. Therefore, F is not ic-C_{+}-convexlike on A.

In Theorem 12, we do not suppose that $\operatorname{cor}(C)\neq\emptyset$. If $\operatorname{cor}(C)\neq\emptyset$, we have the following result.

Theorem 15. *Let $F:A\rightrightarrows Y$ be a set-valued map on A. If $\operatorname{cor}(C)\neq\emptyset$, then F is ic-C_{+}-convexlike on A if and only if F is nearly C-subconvexlike on A.*

Proof. Necessity. Suppose that F is ic-C_{+}-convexlike on A. Clearly,

$$\begin{aligned}
&\operatorname{icr}\left(\operatorname{cor}\left(\operatorname{cone}_{+}\left(F(A)+C_{+}\right)\right)\right)\\
&\qquad=\operatorname{icr}\left(\operatorname{cor}\left(\operatorname{cone}_{+}(F(A))+C_{+}\right)\right).
\end{aligned} \tag{20}$$

Since $\operatorname{cor}(C)\neq\emptyset$, $\operatorname{cor}(C_{+})\neq\emptyset$. It follows from Lemma 10 that

$$\begin{aligned}
&\operatorname{cor}\left(\operatorname{cor}\left(\operatorname{cone}_{+}(F(A))+C_{+}\right)\right)\\
&\qquad=\operatorname{cor}\left(\operatorname{cone}_{+}(F(A))+\operatorname{cor}\left(C_{+}\right)\right)\\
&\qquad=\operatorname{cone}_{+}(F(A))+\operatorname{cor}\left(C_{+}\right)\neq\emptyset,
\end{aligned} \tag{21}$$

which implies that

$$\operatorname{icr}\left(\operatorname{cor}\left(\operatorname{cone}_{+}(F(A))+C_{+}\right)\right)\neq\emptyset. \tag{22}$$

By (20) and (22), we have $\operatorname{icr}(\operatorname{cor}(\operatorname{cone}_{+}(F(A)+C_{+})))\neq\emptyset$. Since F is ic-C_{+}-convexlike on A, it follows from Theorem 12 that F is nearly C-subconvexlike on A.

Sufficiency. We suppose that F is nearly C-subconvexlike on A. Since $\operatorname{cor}(C)\neq\emptyset$, it follows from Lemma 10 and (18) that

$$\begin{aligned}
\operatorname{cor}\left(\operatorname{cone}_{+}\left(F(A)+C_{+}\right)\right)&=\operatorname{cor}\left(\operatorname{cone}_{+}(F(A))+C_{+}\right)\\
&=\operatorname{cor}\left(\operatorname{vcl}\left(\operatorname{cone}_{+}(F(A))+C_{+}\right)\right)\\
&=\operatorname{cor}\left(\operatorname{vcl}\left(\operatorname{cone}_{+}\left(F(A)+C_{+}\right)\right)\right)\\
&=\operatorname{cor}(\operatorname{vcl}(\operatorname{cone}(F(A)+C))).
\end{aligned} \tag{23}$$

Since F is nearly C-subconvexlike on A, $\operatorname{cor}(\operatorname{vcl}(\operatorname{cone}(F(A)+C)))$ is a convex set in Y. Hence, $\operatorname{cor}(\operatorname{cone}_{+}(F(A)+C_{+}))$ is a convex set in Y.

Clearly,

$$\begin{aligned}
\operatorname{cone}_{+}\left(F(A)+C_{+}\right)&\subseteq\operatorname{vcl}\left(\operatorname{cone}_{+}\left(F(A)+C_{+}\right)\right)\\
&\subseteq\operatorname{vcl}\left(\operatorname{vcl}\left(\operatorname{cone}_{+}\left(F(A)+C_{+}\right)\right)\right).
\end{aligned} \tag{24}$$

Since $\operatorname{cor}(C)\neq\emptyset$ implies $\operatorname{cor}(\operatorname{cone}_{+}(F(A)+C_{+}))\neq\emptyset$, $\operatorname{cor}(\operatorname{vcl}(\operatorname{cone}_{+}(F(A)+C_{+})))\neq\emptyset$. By the near C-subconvexlikeness of F, it is easy to check that $\operatorname{vcl}((\operatorname{cone}_{+}(F(A)+C_{+}))$ is a convex set in Y. It follows from (c) of Lemma 9 that

$$\begin{aligned}
&\operatorname{vcl}\left(\operatorname{vcl}\left(\operatorname{cone}_{+}\left(F(A)+C_{+}\right)\right)\right)\\
&\qquad=\operatorname{vcl}\left(\operatorname{cor}\left(\operatorname{vcl}\left(\operatorname{cone}_{+}\left(F(A)+C_{+}\right)\right)\right)\right).
\end{aligned} \tag{25}$$

By Lemma 10, we have

$$\begin{aligned}&\operatorname{vcl}\left(\operatorname{cor}\left(\operatorname{vcl}\left(\operatorname{cone}_{+}\left(F\left(A\right)+C_{+}\right)\right)\right)\right)\\&\quad=\operatorname{vcl}\left(\operatorname{cor}\left(\operatorname{cone}_{+}\left(F\left(A\right)+C_{+}\right)\right)\right).\end{aligned}\tag{26}$$

By (24), (25), and (26), we have $\operatorname{cone}_{+}(F(A)+C_{+}) \subseteq \operatorname{vcl}(\operatorname{cor}(\operatorname{cone}_{+}(F(A)+C_{+})))$. Therefore, F is ic-C_{+}-convexlike on A. □

Remark 16. Theorem 15 generalizes Theorem 3.1 in [8] from locally convex spaces to linear spaces.

Remark 17. Xu and Song used Lemma 2.2 in [8] to prove Theorems 3.1 and 3.2 in [8]. However, in this paper, our methods are different from those in [8].

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