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Research Article

An Iterative Method for Solving a System of Mixed Equilibrium Problems, System of Quasivariational Inclusions, and Fixed Point Problems of Nonexpansive Semigroups with Application to Optimization Problems

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We introduce a general implicit iterative scheme base on viscosity approximation method with a ϕ-strongly pseudocontractive mapping for finding a common element of the set of solutions for a system of mixed equilibrium problems, the set of common fixed point for a nonexpansive semigroup, and the set of solutions of system of variational inclusions with set-valued maximal monotone mapping and Lipschitzian relaxed cocoercive mappings in Hilbert spaces. Furthermore, we prove that the proposed iterative algorithm converges strongly to a common element of the above three sets, which is a solution of the optimization problem related to a strongly positive bounded linear operator.

1. Introduction

Throughout this paper we denoted by $\mathbb{N}$ and $\mathbb{R}^+$ the set of all positive integers and all positive real numbers, respectively. We always assume that H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$, respectively, C is a nonempty closed convex subset of H. Let $\varphi : C \to \mathbb{R}$ be a real-valued function and $\Theta : C \times C \to \mathbb{R}$ be an equilibrium bifunction. The *mixed equilibrium problem* (for short, MEP) is to find $x^* \in C$ such that

$$\Theta(x^*, y) + \varphi(y) - \varphi(x^*) \geq 0, \quad \forall y \in C. \tag{1.1}$$

The set of solutions of (1.1) is denoted by $\mathrm{MEP}(\Theta, \varphi)$, that is,

$$\mathrm{MEP}(\Theta, \varphi) = \{x^* \in C : \Theta(x^*, y) + \varphi(y) - \varphi(x^*) \geq 0, \ \forall y \in C\}. \tag{1.2}$$

In particular, if $\varphi = 0$, this problem reduces to the *equilibrium problem*, that is, to find $x^* \in C$ such that

$$\Theta(x^*, y) \geq 0, \quad \forall y \in C. \tag{1.3}$$

The set of solution of (1.3) is denoted by $\mathrm{EP}(\Theta)$.

Mixed equilibrium problems include fixed point problems, optimization problems, variational inequality problems, Nash equilibrium problems, and equilibrium problems as special cases (see, e.g., [1–6]). Some methods have been proposed to solve the equilibrium problem, see, for instance, [7–21].

Let A be a strongly positive bounded linear operator on H, that is, there exists a constant $\overline{\gamma} > 0$ such that

$$\langle Ax, x \rangle \geq \overline{\gamma} \|x\|^2, \quad \forall x \in H. \tag{1.4}$$

Recall that, a mapping $f : H \to H$ is said to be *contractive* if there exists a constant $\alpha \in (0,1)$ such that

$$\|f(x) - f(y)\| \leq \alpha \|x - y\|, \quad \forall x, y \in H. \tag{1.5}$$

A mapping $T : H \to H$ is said to be

(i) *nonexpansive* if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in H, \tag{1.6}$$

(ii) *pseudocontractive* if

$$\langle Tx - Ty, x - y \rangle \leq \|x - y\|^2, \quad \forall x, y \in H, \tag{1.7}$$

(iii) *ϕ-strongly pseudocontractive* if there exists a continuous and strictly increasing function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ with $\phi(0) = 0$ such that

$$\langle Tx - Ty, x - y \rangle \leq \|x - y\|^2 - \phi(\|x - y\|)\|x - y\|, \quad \forall x, y \in H. \tag{1.8}$$

It is obvious that pseudocontractive mapping is more general than ϕ-strongly pseudocontractive mapping. If $\phi(t) = \alpha t$ with $0 < \alpha < 1$, then ϕ-strongly pseudocontractive mapping reduces to β-strongly pseudocontractive mapping with $1 - \alpha = \beta \in (0,1)$, which is more general than contractive mapping.

Definition 1.1. A one-parameter family mapping $\mathcal{S} = \{T(t) : t \in \mathbb{R}^+\}$ from C into itself is said to be a *nonexpansive semigroup* on C if it satisfies the following conditions:

(i) $T(0)x = x$ *for all* $x \in C$,

(ii) $T(s+t) = T(s) \circ T(t)$ for all $s, t \in \mathbb{R}^+$,

(iii) for each $x \in C$ the mapping $t \mapsto T(t)x$ is continuous,

(iv) $\|T(t)x - T(t)y\| \leq \|x - y\|$ for all $x, y \in C$ and $t \in \mathbb{R}^+$.

Remark 1.2. We denote by $F(\mathcal{S})$ the set of all common fixed points of $\mathcal{S}$, that is, $F(\mathcal{S}) := \bigcap_{t \in \mathbb{R}^+} F(T(t)) = \{x \in C : T(t)x = x\}$.

Recall the following definitions of a nonlinear mapping $B : C \to H$, the following are mentioned.

Definition 1.3. The nonlinear mapping $B : C \to H$ is said to be

(i) *monotone* if

$$\langle Bx - By, x - y \rangle \geq 0, \quad \forall x, y \in C, \tag{1.9}$$

(ii) *β-strongly monotone* if there exists a constant $\beta > 0$ such that

$$\langle Bx - By, x - y \rangle \geq \beta \|x - y\|^2, \quad \forall x, y \in C, \tag{1.10}$$

(iii) *L-Lipschitz continuous* if there exists a constant $L > 0$ such that

$$\|Bx - By\| \leq L \|x - y\|, \quad \forall x, y \in C, \tag{1.11}$$

(iv) *ν-inverse-strongly monotone* if there exists a constant $\nu > 0$ such that

$$\langle Bx - By, x - y \rangle \geq \nu \|Bx - By\|^2, \quad \forall x, y \in C, \tag{1.12}$$

(v) *relaxed* (c, d)*-cocoercive* if there exists a constants $c, d > 0$ such that

$$\langle Bx - By, x - y \rangle \geq (-c) \|Bx - By\|^2 + d \|dx - y\|^2, \quad \forall x, y \in C. \tag{1.13}$$

The resolvent operator technique for solving variational inequalities and variational inclusions is interesting and important. The resolvent equation technique is used to develop powerful and efficient numerical techniques for solving various classes of variational inequalities, inclusions, and related optimization problems.

Definition 1.4. Let $M : H \to 2^H$ be a multivalued maximal monotone mapping. The single-valued mapping $J_{(M,\rho)} : H \to H$, defined by

$$J_{(M,\rho)}(u) = (I + \rho M)^{-1}(u), \quad \forall u \in H, \tag{1.14}$$

is called *resolvent operator associated with* M, where ρ is any positive number and I is the identity mapping.

Next, we consider a system of quasivariational inclusions problem is to find $(x^*, y^*) \in H \times H$ such that

$$\begin{aligned} 0 &\in x^* - y^* + \rho_1 (B_1 y^* + M_1 x^*), \\ 0 &\in y^* - x^* + \rho_2 (B_2 x^* + M_2 y^*), \end{aligned} \tag{1.15}$$

where $B_i : H \to H$ and $M_i : H \to 2^H$ are nonlinear mappings for each $i = 1, 2$.

As special cases of the problem (1.15), we have the following results.

(1) If $B_1 = B_2 = B$ and $M_1 = M_2 = M$, then the problem (1.15) is reduces to the following. Find $(x^*, y^*) \in H \times H$ such that

$$\begin{aligned} 0 &\in x^* - y^* + \rho_1 (By^* + Mx^*), \\ 0 &\in y^* - x^* + \rho_2 (Bx^* + My^*). \end{aligned} \tag{1.16}$$

(2) Further, if $x^* = y^*$ in problem (1.16), then the problem (1.16) is reduces to the following. Find $x^* \in H$ such that

$$0 \in Bx^* + Mx^*. \tag{1.17}$$

The problem (1.17) is called *variational inclusion problem*. We denote by $\mathrm{VI}(H, B, M)$ the set of solutions of the variational inclusion problem (1.17). Next, we consider two special cases of the problem (1.17).

(1) $M = \partial\phi : H \to 2^H$, where $\phi : H \to \mathbb{R} \cup \{+\infty\}$ is a proper convex lower semicontinuous function and $\partial\phi$ is the subdifferential of ϕ then the quasivariational inclusion problem (1.17) is equivalent to finding $x^* \in H$ such that $\langle Bx^*, x - x^* \rangle + \phi(x) - \phi(x^*) \geq 0$, for all $x \in H$, which is said to be the *mixed quasivariational inequality*.

(2) If $M = \partial\delta_C$, where C is a nonempty closed convex subset of H, and $\delta_C : H \to [0, \infty)$ is the indicator function of C, that is,

$$\delta_C(x) = \begin{cases} 0, & x \in C, \\ +\infty, & x \notin C, \end{cases} \tag{1.18}$$

then the quasivariational inclusion problem (1.17) is equivalent to the classical variational inequality problem denoted by $\mathrm{VI}(C, B)$ which is to find $x^* \in C$ such that

$$\langle Bx^*, x - x^* \rangle \geq 0, \quad \forall x \in C. \tag{1.19}$$

This problem is called *Hartman-Stampacchia variational inequality problem* (see e.g., [22–24]).

It is known that problem (1.17) provides a convenient framework for the unified study of optimal solutions in many optimization related areas including mathematical programming, complementarity, variational inequalities, optimal control, mathematical economics, equilibria, and game theory. Also various types of variational inclusions problems have been extended and generalized (see [25–40] and the references therein).

On the other hand, the following optimization problem has been studied extensively by many authors:

$$\min_{x\in\overline{\Omega}} \frac{\mu}{2}\langle Ax, x\rangle + \frac{1}{2}\|x-u\|^2 - h(x), \tag{1.20}$$

where $\overline{\Omega} = \bigcap_{n=1}^{\infty} C_n, C_1, C_2, \ldots$ are infinitely many closed convex subsets of H such that $\bigcap_{n=1}^{\infty} C_n \neq \emptyset$, $u \in H$, $\mu \geq 0$ is a real number, A is a strongly positive linear bounded operator on H and h is a potential function for γf (i.e., $h'(x) = \gamma f(x)$ for all $x \in H$). This kind of optimization problem has been studied extensively by many authors (see, e.g. [41–44]) when $\overline{\Omega} = \bigcap_{n=1}^{\infty} C_n$ and $h(x) = \langle x, b\rangle$, where b is a given point in H.

Li et al. [45] introduced two steps of iterative procedures for the approximation of common fixed point of a nonexpansive semigroup $\mathcal{S} = \{T(t) : t \in \mathbb{R}^+\}$ on a nonempty closed convex subset C in a Hilbert space. Recently, Liu et al. [46] introduced a hybrid iterative scheme for finding a common element of the set of solutions of system of mixed equilibrium problems, the set of common fixed points for nonexpansive semigroup and the set of solution of quasivariational inclusions with multivalued maximal monotone mappings and inverse-strongly monotone mappings. Very recently, Hao [47] introduced a general iterative method for finding a common element of solution set of quasivariational inclusion problems and the set of common fixed points of an infinite family of nonexpansive mappings.

In this paper, motivated and inspired by Li et al. [45], Liu et al. [46], and Hao [47], we introduce a general implicit iterative algorithm base on viscosity approximation methods with a ϕ-strongly pseudocontractive mapping which is more general than a contraction mapping for finding a common element of the set of solutions for a system of mixed equilibrium problems, the set of common fixed point for a nonexpansive semigroup, and the set of solutions of system of variational inclusions (1.15) with set-valued maximal monotone mapping and Lipschitzian relaxed cocoercive mappings in Hilbert spaces. We prove that the proposed iterative algorithm converges strongly to a common element of the above three sets, which is a solution of the optimization problem related to a strongly positive bounded linear operator. The results obtained in this paper extend and improve several recent results in this area.

2. Preliminaries

In the sequel, we use $x_n \rightharpoonup x$ and $x_n \to x$ to denote the weak convergence and strong convergence of the sequence $\{x_n\}$ in H, respectively.

This collects some results that will be used in the proofs of our main results.

Proposition 2.1 (see [21]). *(i) The resolvent operator $J_{(M,\rho)}$ associated with M is single-valued and nonexpansive for all $\rho > 0$, that is,*

$$\left\|J_{(M,\rho)}(x) - J_{(M,\rho)}(y)\right\| \leq \|x-y\|, \quad \forall x, y \in H, \forall \rho > 0. \tag{2.1}$$

(ii) *The resolvent operator* $J_{(M,\rho)}$ *is 1-inverse-strongly monotone, that is,*

$$\|J_{(M,\rho)}(x) - J_{(M,\rho)}(y)\|^2 \leq \langle x - y, J_{(M,\rho)}(x) - J_{(M,\rho)}(y)\rangle, \quad \forall x, y \in H. \tag{2.2}$$

Obviously, this immediately implies that

$$\|(x - y) - (J_{(M,\rho)}(x) - J_{(M,\rho)}(y))\|^2 \leq \|x - y\|^2 - \|J_{(M,\rho)}(x) - J_{(M,\rho)}(y)\|^2, \quad \forall x, y \in H. \tag{2.3}$$

For solving the equilibrium problem for bifunction $\Theta : H \times H \to \mathbb{R}$, let us assume that satisfies the following conditions:

(H1) $\Theta(x, x) = 0$ for all $x \in H$;

(H2) Θ is monotone, that is, $\Theta(x, y) + \Theta(y, x) \leq 0$ for all $x, y \in H$;

(H3) for each $y \in H$, $x \mapsto \Theta(x, y)$ is concave and upper semicontinuous;

(H4) for each $y \in H$, $x \mapsto \Theta(x, y)$ is convex;

(H5) for each $y \in H$, $x \mapsto \Theta(x, y)$ is lower semicontinuous.

Definition 2.2. A map $\eta : H \times H \to H$ *is* called Lipschitz continuous, if there exists a constant $L > 0$ such that

$$\|\eta(x, y)\| \leq L\|x - y\|, \quad \forall x, y \in H. \tag{2.4}$$

A differentiable function $K : H \to \mathbb{R}$ on a convex set H is called

(i) η—*convex* [7] if

$$K(y) - K(x) \geq \langle K'(x), \eta(y, x)\rangle, \quad \forall x, y \in H, \tag{2.5}$$

where $K'(x)$ is the *Fréchet* differentiable of K at x,

(i) η—*strongly convex* [7] if there exists a constant $\nu > 0$ such that

$$K(y) - K(x) - \langle K'(x), \eta(x, y)\rangle \geq \left(\frac{\nu}{2}\right)\|x - y\|^2, \quad \forall x, y \in H. \tag{2.6}$$

Let $\Theta : H \times H \to \mathbb{R}$ be an equilibrium bifunction satisfying the conditions (H1)–(H5). Let r be any given positive number. For a given point $x \in H$, consider the following *auxiliary problem* for MEP (for short, MEP (x, y)) to find $y \in H$ such that

$$\Theta(y, z) + \varphi(z) - \varphi(y) + \frac{1}{r}\langle K'(y) - K'(x), \eta(z, y)\rangle \geq 0, \quad \forall z \in H, \tag{2.7}$$

where $\eta : H \times H \to H$ is a mapping, and $K'(x)$ is the *Fréchet* derivative of a functional $K : H \to \mathbb{R}$ at x. Let $V_r^{(\Theta,\varphi)} : H \to H$ be the mapping such that for each $x \in H$, $V_r^{(\Theta,\varphi)}(x)$ is the set of solutions of MEP (x, y), that is,

$$V_r^{(\Theta,\varphi)}(x) = \Big\{ y \in H : \Theta(y,z) + \varphi(z) - \varphi(y) + \frac{1}{r}\langle K'(y) - K'(x), \eta(z,y)\rangle \geq 0, \forall z \in H \Big\}, \quad \forall x \in H. \tag{2.8}$$

Then the following conclusion holds.

Proposition 2.3 (see [7]). *Let H be a real Hilbert space, $\varphi : H \to \mathbb{R}$ be a a lower semicontinuous and convex functional. Let $\Theta : H \times H \to \mathbb{R}$ be an equilibrium bifunction satisfying conditions (H1)–(H5). Assume that*

(i) *$\eta : H \times H \to \mathbb{R}$ is Lipschitz continuous with constant $\sigma > 0$ such that*

   (a) *$\eta(x, y) + \eta(y, x) = 0$ for all $x, y \in H$;*
   (b) *$\eta(\cdot, \cdot)$ is affine in the first variable;*
   (c) *for each fixed $y \in H$, $x \mapsto \eta(y, x)$ is continuous from the weak topology to the weak topology;*

(ii) *$K : H \to \mathbb{R}$ is η-strongly convex with constant $\mu > 0$, and its derivative K' is continuous from the weak topology to the strong topology;*

(iii) *for each $x \in H$, there exists a bounded subset $D_x \subset H$ and $z_x \in H$ such that for all $y \notin D_x$,*

$$\Theta(y, z_x) + \varphi(z_x) - \varphi(y) + \frac{1}{r}\langle K'(y) - K'(x), \eta(z_x, y)\rangle < 0. \tag{2.9}$$

Then the following hold:

(i) *$V_r^{(\Theta,\varphi)}$ is single valued;*

(ii) $F(V_r^{(\Theta,\varphi)}) = \mathrm{MEP}(\Theta, ')$;

(iii) $\mathrm{MEP}(\Theta, \varphi)$ is closed and convex.

Lemma 2.4 (see [48]). *Let C be a nonempty bounded closed and convex subset of a real Hilbert space H. Let $\mathcal{S} = \{T(t) : t \in \mathbb{R}^+\}$ be a nonexpansive semigroup on C, then for all $h > 0$,*

$$\limsup_{t\to\infty} \sup_{x \in C} \left\| \frac{1}{t}\int_0^t T(s)x\, ds - T(h)\left(\frac{1}{t}\int_0^t T(s)x\, ds\right) \right\| = 0. \tag{2.10}$$

Lemma 2.5 (see [49]). *Let X be a uniformly convex Banach space, C be a nonempty closed and convex subset of X, and $T : C \to X$ be a nonexpansive mapping. Then $I - T$ is demiclosed at zero.*

Lemma 2.6 (see [50]). *Assume that A is a strongly positive linear bounded operator on H with coefficient $\overline{\gamma} > 0$ and $0 < \rho \leq \|A\|^{-1}$. Then $\|I - \rho A\| \leq 1 - \rho\overline{\gamma}$.*

Lemma 2.7 (see [51]). *Let X be a Banach space and $f : X \to X$ be a ϕ-strongly pseudocontractive and continuous mapping. Then f has a unique fixed point in X.*

Lemma 2.8. *In a real Hilbert space H, the following inequality holds:*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle, \quad \forall x, y \in H. \tag{2.11}$$

The following lemma can be found in [52, 53] (see also Lemma 2.2 in [54]).

Lemma 2.9. *Let C be a nonempty closed and convex subset of a real Hilbert space H and $g : C \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous differentiable convex function differentiable convex function. If x^* is a solution to the minimization problem*

$$g(x^*) = \inf_{x \in C} g(x), \tag{2.12}$$

then

$$\langle g'(x), x - x^*\rangle \geq 0, \quad x \in C. \tag{2.13}$$

In particular, if x^* solves the optimization problem

$$\min_{x \in \Omega} \frac{\mu}{2}\langle Ax, x\rangle + \frac{1}{2}\|x - u\|^2 - h(x), \tag{2.14}$$

then

$$\langle u + (\gamma f - (I + \mu A))x^*, x - x^*\rangle \leq 0, \quad x \in C, \tag{2.15}$$

where h is a potential function for γf.

The following lemmas can be found in ([55, 56]). For the sake of the completeness, one includes its proof in a Hilbert space version. Without loss of generality, one assumes that $c, d \in (0, 1)$ and $L_B \in [1, \infty)$.

Lemma 2.10. *Let H be a real Hilbert space, $B : H \to H$ be an L_B-Lipschitzian and relaxed (c, d)-cocoercive mapping. Then, one has*

$$\|(I - \rho B)x - (I - \rho B)y\|^2 \leq \left(1 + 2\rho c L_B^2 - 2\rho d + \rho^2 L_B^2\right)\|x - y\|^2, \tag{2.16}$$

where $\rho > 0$. In particular, if $0 < \rho \leq 2(d - cL_B^2)/L_B^2$, then $I - \rho B$ is nonexpansive.

Proof. For all $x, y \in H$, we have

$$\begin{aligned}\|(I-\rho B)x-(I-\rho B)y\|^2 &= \|(x-y)-(\rho Bx-\rho By)\|^2\\ &= \|x-y\|^2-2\rho\langle Bx-By, x-y\rangle+\rho^2\|Bx-By\|^2\\ &\le \|x-y\|^2-2\rho\left[-c\|Bx-By\|^2+d\|x-y\|^2\right]+\rho^2\|Bx-By\|^2\\ &= \|x-y\|^2-2\rho d\|x-y\|^2+2\rho c\|Bx-By\|^2+\rho^2\|Bx-By\|^2\\ &\le \left(1+2\rho cL_B^2-2\rho d+\rho^2L_B^2\right)\|x-y\|^2.\end{aligned} \tag{2.17}$$

It is clear that, if $0<\rho\le 2(d-cL_B^2)/L_B^2$, then $I-\rho B$ is nonexpansive. This completes the proof. □

Lemma 2.11. *Let H be a real Hilbert space, $M_i : H \to 2^H$ be the a maximal monotone mapping and $B_i : H \to H$ be an L_i-Lipschitzian and relaxed (c_i, d_i)-cocoercive mapping for all $i = 1, 2$. Let $Q : H \to H$ be a mapping defined by*

$$Qx := J_{(M_1,\rho_1)}\left[J_{(M_2,\rho_2)}(x-\rho_2B_2x)-\rho_1B_1J_{(M_2,\rho_2)}(x-\rho_2B_2x)\right],\quad \forall x\in H. \tag{2.18}$$

If $0<\rho_i\le 2(d_i-c_iL_i^2)/L_i^2$ for all $i=1,2$, then $Q : H \to H$ is nonexpansive.

Proof. By Lemma 2.10, we know that $(I-\rho_2B_2)$ and $(I-\rho_1B_1)$ are nonexpansive, for all $x, y \in H$, we have

$$\begin{aligned}\|Qx-Qy\| &= \|J_{(M_1,\rho_1)}\left[J_{(M_2,\rho_2)}(x-\rho_2B_2x)-\rho_1B_1J_{(M_2,\rho_2)}(x-\rho_2B_2x)\right]\\ &\qquad -J_{(M_1,\rho_1)}\left[J_{(M_2,\rho_2)}(y-\rho_2B_2y)-\rho_1B_1J_{(M_2,\rho_2)}(y-\rho_2B_2y)\right]\|\\ &\le \|\left[J_{(M_2,\rho_2)}(x-\rho_2B_2x)-\rho_1B_1J_{(M_2,\rho_2)}(x-\rho_2B_2x)\right]\\ &\qquad -\left[J_{(M_2,\rho_2)}(y-\rho_2B_2y)-\rho_1B_1J_{(M_2,\rho_2)}(y-\rho_2B_2y)\right]\|\\ &= \|J_{(M_2,\rho_2)}(I-\rho_2B_2)(I-\rho_1B_1)x-J_{(M_2,\rho_2)}(I-\rho_2B_2)(I-\rho_1B_1)y\|\\ &\le \|(I-\rho_2B_2)(I-\rho_1B_1)x-(I-\rho_2B_2)(I-\rho_1B_1)y\|\\ &\le \|(I-\rho_2B_2)x-(I-\rho_2B_2)y\|\\ &\le \|x-y\|,\end{aligned} \tag{2.19}$$

which implies that Q is nonexpansive. This completes the proof. □

Lemma 2.12. *For all $(x^*, y^*) \in H\times H$, where $y^* = J_{(M_2,\rho_2)}(x^*-\rho_2B_2x^*)$, (x^*, y^*) is a solution of the problem* (1.15) *if and only if x^* is a fixed point of the mapping $Q : H \to H$ defined as in Lemma 2.11.*

Proof. Let $(x^*, y^*) \in H \times H$ be a solution of the problem (1.15). Then, we have

$$\begin{aligned} y^* - \rho_1 B_1 y^* &\in (I + \rho_1 M_1) x^*, \\ x^* - \rho_2 B_2 x^* &\in (I + \rho_2 M_2) y^*, \end{aligned} \tag{2.20}$$

which implies that

$$\begin{aligned} x^* &= J_{(M_1,\rho_1)}(y^* - \rho_1 B_1 y^*), \\ y^* &= J_{(M_2,\rho_2)}(x^* - \rho_2 B_2 x^*). \end{aligned} \tag{2.21}$$

We can deduce that (2.21) is equivalent to

$$x^* = J_{(M_1,\rho_1)}\left[J_{(M_2,\rho_2)}(x^* - \rho_2 B_2 x^*) - \rho_1 B_1 J_{(M_2,\rho_2)}(x^* - \rho_2 B_2 x^*)\right]. \tag{2.22}$$

This completes the proof. □

3. Main Results

Now, in this section, we prove our main results of this article. Before proving the main result we need the following lemma.

Lemma 3.1. *Let H be a real Hilbert space. Let $\mathcal{S} = \{T(t) : t \in \mathbb{R}^+\}$ be a nonexpansive semigroup from H into itself. Then $I - \sigma_t(\cdot)$ is monotone, where $\sigma_t(x) := (1/t)\int_0^t T(s)x\,ds$ for all $x \in H$ and $t \geq 0$.*

Proof. For all $x, y \in H$, we have

$$\begin{aligned} \langle x - y, (I - \sigma_t(\cdot))x - (I - \sigma_t(\cdot))y \rangle &= \left\langle x - y, \left(I - \frac{1}{t}\int_0^t T(s)ds\right)x - \left(I - \frac{1}{t}\int_0^t T(s)ds\right)y \right\rangle \\ &= \|x - y\|^2 - \left\langle x - y, \frac{1}{t}\int_0^t T(s)x\,ds - \frac{1}{t}\int_0^t T(s)y\,ds \right\rangle \\ &\geq \|x - y\|^2 - \|x - y\|\frac{1}{t}\int_0^t \|T(s)x - T(s)y\|ds \\ &\geq \|x - y\|^2 - \|x - y\|^2 \\ &= 0, \end{aligned} \tag{3.1}$$

which implies that $I - \sigma_t(\cdot)$ is monotone. This completes the proof. □

Theorem 3.2. *Let H be a real Hilbert space. Let $\varphi_i : H \to \mathbb{R}$ $(i = 1, 2, \ldots, N)$ be a finite family of lower semicontinuous and convex function, $\Theta_i : H \times H \to \mathbb{R}$ $(i = 1, 2, \ldots, N)$ be a finite family of* *bifunctions satisfying (H1)–(H5), and* $\eta_i : H \times H \to H$ *be a finite family of Lipschitz continuous mappings with a constant* σ_i $(i = 1, 2, \ldots, N)$*. Let* $\mathcal{S} = \{T(t) : t \in \mathbb{R}^+\}$ *be a nonexpansive semigroup from* H *into itself,* $B_i : H \to H (i = 1, 2)$ *be an* L_i*-Lipschitzian and relaxed* (c_i, d_i)*-cocoercive mapping with* $\rho_i \in (0, 2(d_i - c_i L_i^2)/L_i^2]$ *for all* $i = 1, 2$ *and* $M_i : H \to 2^H (i = 1, 2)$ *be a maximal monotone mapping. Assume that* $\Omega := F(\mathcal{S}) \cap \bigcap_{k=1}^{N} MEP\,(\Theta_k, \varphi_k) \cap F(Q) \neq \emptyset$*, where* Q *is defined as in Lemma 2.11. Let* $f : H \to H$ *be a* ϕ*-strongly pseudocontractive mapping with* $\lim_{t \to +\infty} Œ(t) = +\infty$ *and* A *be a strongly positive linear bounded operator on* H *with a coefficient* $\overline{\gamma} > 0$*. Let* $\mu > 0$ *and* $\gamma > 0$ *be two constants such that* $0 < \gamma < 1 + \mu\overline{\gamma}$*. Let* $\{r_{i,n}\} (i = 1, 2, \ldots, N)$ *be a finite family of positive real sequence such that* $\liminf_{n \to \infty} r_{i,n} > 0$*,* $\{\alpha_n\}$ *and* $\{\beta_n\}$ *be two sequences in* $[0, 1]$*, and* $\{t_n\}$ *be a positive real divergent sequence. For any fixed* $u \in H$*, let* $\{x_n\}$ *be the sequence defined by*

$$
\begin{gathered}
\Theta_1\left(u_n^{(1)}, x\right) + \varphi_1(x) - \varphi_1\left(u_n^{(1)}\right) + \frac{1}{r_{1,n}}\left\langle K_1'\left(u_n^{(1)}\right) - K_1'(x_n), \eta_1\left(x, u_n^{(1)}\right)\right\rangle \geq 0, \quad \forall x \in H, \\
\Theta_2\left(u_n^{(2)}, x\right) + \varphi_2(x) - \varphi_2\left(u_n^{(2)}\right) + \frac{1}{r_{2,n}}\left\langle K_2'\left(u_n^{(2)}\right) - K_2'\left(u_n^{(1)}\right), \eta_1\left(x, u_n^{(2)}\right)\right\rangle \geq 0, \quad \forall x \in H, \\
\vdots \\
\Theta_N\left(u_n^{(N)}, x\right) + \varphi_N(x) - \varphi_N\left(u_n^{(N)}\right) + \frac{1}{r_{N,n}}\left\langle K_N'\left(u_n^{(N)}\right) - K_N'\left(u_n^{(N-1)}\right), \eta_N\left(x, u_n^{(N)}\right)\right\rangle \geq 0, \ \forall x \in H, \\
y_n = J_{(M_2, \rho_2)}\left(u_n^{(N)} - \rho_2 B_2 u_n^{(N)}\right), \\
x_n = \alpha_n (u + \gamma f(x_n)) + \beta_n x_n + ((1 - \beta_n) I - \alpha_n (I + \mu A)) \frac{1}{t_n} \int_0^{t_n} T(s) J_{(M_1, \rho_1)}(y_n - \rho_1 B_1 y_n)\, ds,
\end{gathered}
\tag{3.2}
$$

where

$$
\begin{aligned}
u_n^{(1)} &= V_{r_{1,n}}^{(\Theta_1, \varphi_1)} x_n, \\
u_n^{(i)} &= V_{r_{i,n}}^{(\Theta_i, \varphi_i)} u_n^{(i-1)} = V_{r_{i,n}}^{(\Theta_i, \varphi_i)} V_{r_{i-1,n}}^{(\Theta_{i-1}, \varphi_{i-1})} u_n^{(i-2)} = V_{r_{i,n}}^{(\Theta_i, \varphi_i)} \cdots V_{r_{2,n}}^{(\Theta_2, \varphi_2)} u_n^{(1)} \\
&= V_{r_{i,n}}^{(\Theta_i, \varphi_i)} \cdots V_{r_{2,n}}^{(\Theta_2, \varphi_2)} V_{r_{1,n}}^{(\Theta_1, \varphi_1)} x_n, \quad i = 2, 3, \ldots, N,
\end{aligned}
\tag{3.3}
$$

and $V_{r_{i,n}}^{(\Theta_i, \varphi_i)} : H \to H$*,* $i = 1, 2, \ldots, N$ *is the mapping defined by* (2.8)*. Assume the following.*

(i) $\eta_i : H \times H \to H$ *is Lipschitz continuous with constant* $\sigma_i > 0$ $(i = 1, 2, \ldots, N)$ *such that*

(a) $\eta_i(x, y) + \eta_i(y, x) = 0$ *for all* $x, y \in H$*,*

(b) $\eta_i(\cdot, \cdot)$ *is affine in the first variable,*

(c) *for each fixed* $y \in H$*,* $x \mapsto \eta_i(y, x)$ *is sequentially continuous from the weak topology to the weak topology.*

(ii) $K_i : H \to \mathbb{R}$ *is* η_i*-strongly convex with constant* $\mu_i > 0$*, and its derivative* K_i' *is not only continuous from the weak topology to the strong topology but also Lipschitz continuous with constant* ν_i *such that* $\mu_i \geq \sigma_i \nu_i$*.*

(iii) *For all* $i = 1, 2, \ldots, N$ *and for all* $x \in H$, *there exists a bounded subset* $D_x \subset H$ *and* $z_x \in H$ *such that for all* $y \notin D_x$,

$$\Theta_i(y, z_n) + \varphi_i(z_x) - \varphi_i(y) + \frac{1}{r_{i,n}} \langle K_i'(y) - K_i'(x), \eta_i(z_x, y) \rangle < 0. \tag{3.4}$$

If the following conditions are satisfied:

(C1) $\lim_{n\to\infty} \alpha_n = 0$,

(C2) $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$,

then the sequence $\{x_n\}$ *defined by* (3.2) *converges strongly to* $x^* \in \Omega := F(\mathcal{S}) \cap \bigcap_{k=1}^{N} \mathrm{MEP}(\Theta_k, \varphi_k) \cap F(Q)$, provided $V_{r_{i,n}}^{(\Theta_i, \varphi_i)}$ is firmly nonexpansive, where x^* is the unique solution of the variational inequality

$$\left\langle u + (\gamma f - (I + \mu A))x^*, z - x^* \right\rangle \le 0, \quad \forall z \in \Omega, \tag{3.5}$$

or, equivalently, x^* is the unique solution of the optimization problem

$$\min_{x\in\Omega} \frac{\mu}{2} \langle Ax, x \rangle + \frac{1}{2} \|x - u\|^2 - h(x), \tag{3.6}$$

where h is a potential function for γf and (x^*, y^*) is the solution of the problem (1.15), where $y^* = J_{(M_2, \rho_2)}(x^* - \rho_2 B_2 x^*)$.

Proof. By the conditions $(C1)$ and $(C2)$, we may assume, without loss of generality, that $\alpha_n \le (1 - \beta_n)(1 + \mu\|A\|)^{-1}$ for all $n \in \mathbb{N}$. Since A is a linear bounded self-adjoint operator on H, by (1.4), we have

$$\|A\| = \sup\{|\langle Au, u\rangle| : u \in H, \|u\| = 1\}. \tag{3.7}$$

Observe that

$$\begin{aligned} \left\langle ((1 - \beta_n)I - \alpha_n(I + \mu A))u, u \right\rangle &= 1 - \beta_n - \alpha_n - \alpha_n \mu \langle Au, u\rangle \\ &\ge 1 - \beta_n - \alpha_n - \alpha_n \mu \|A\| \\ &\ge 0. \end{aligned} \tag{3.8}$$

This shows that $(1 - \beta_n)I - \alpha_n(I + \mu A)$ is positive. It follows that

$$\begin{aligned} \|(1 - \beta_n)I - \alpha_n(I + \mu A)\| &= \sup\left\{ \left| \left\langle ((1 - \beta_n)I - \alpha_n(I + \mu A))u, u \right\rangle \right| : u \in H, \|u\| = 1 \right\} \\ &= \sup\{1 - \beta_n - \alpha_n - \alpha_n \mu \langle Au, u\rangle : u \in H, \|u\| = 1\} \\ &\le 1 - \beta_n - \alpha_n - \alpha_n \mu \overline{\gamma}. \end{aligned} \tag{3.9}$$

In fact, by the assumption that for all $k \in \{1,2,\ldots,N\}$, $V_{r_{k,n}}^{(\Theta_k,\varphi_k)}$ is nonexpansive. Setting $V_n^k := V_{r_{k,n}}^{(\Theta_k,\varphi_k)} \cdots V_{r_{2,n}}^{(\Theta_2,\varphi_2)} V_{r_{1,n}}^{(\Theta_1,\varphi_1)}$ for $k \in \{1,2,\ldots,N\}$ and $V_n^0 := I$. Define a mapping $W_n : H \to H$ by

$$W_n x := \frac{1}{t_n}\int_0^{t_n} T(s)QV_n^N x\,ds, \quad \forall x \in H. \tag{3.10}$$

Hence, by Lemma 2.11 and nonexpansiveness (semigroup) of $T(s)$ and V_n^k, for all $x, y \in C$, we have

$$\begin{aligned}\|W_n x - W_n y\| &= \left\| \frac{1}{t_n}\int_0^{t_n} T(s)QV_n^N x\,ds - \frac{1}{t_n}\int_0^{t_n} T(s)QV_n^N y\,ds \right\| \\ &\le \left\| QV_n^N x - QV_n^N y \right\| \\ &\le \left\| V_n^N x - V_n^N y \right\| \\ &\le \|x - y\|,\end{aligned} \tag{3.11}$$

which implies that W_n is nonexpansive.

First, we show that $\{x_n\}$ defined by (3.2) is well defined. Define a mapping $T_n^f : H \to H$ by

$$T_n^f x := \alpha_n(u + \gamma f(x)) + \beta_n x + ((1-\beta_n)I - \alpha_n(I + \mu A))W_n x, \quad \forall x \in H. \tag{3.12}$$

Indeed, by Lemma 2.6, and from (3.11), for all $x, y \in H$, we have

$$\begin{aligned}\left\langle T_n^f x - T_n^f y, x - y \right\rangle &= \alpha_n\gamma\langle f(x) - f(y), x - y\rangle + \beta_n\langle x - y, x - y\rangle \\ &\quad + \langle ((1-\beta_n)I - \alpha_n(I + \mu A))(Wx - Wy), x - y\rangle \\ &\le \alpha_n\gamma\left(\|x - y\|^2 - \phi(\|x - y\|)\|x - y\|\right) \\ &\quad + \beta_n\|x - y\|^2 + (1 - \beta_n - \alpha_n - \alpha_n\mu\overline{\gamma})\|x - y\|^2 \\ &= [1 + \alpha_n(\gamma - (1 + \mu\overline{\gamma}))]\|x - y\|^2 - \alpha_n\gamma\phi(\|x - y\|)\|x - y\| \\ &\le \|x - y\|^2 - \alpha_n\gamma\phi(\|x - y\|)\|x - y\|.\end{aligned} \tag{3.13}$$

This shows that T_n^f is a ϕ-strongly pseudocontractive and strongly continuous. It follows from Lemma 2.7 that T_n^f has a unique fixed point $x_n \in H$, that is, $\{x_n\}$ defined by (3.2) is well defined.

Next, we show that uniqueness of the solution of the variational inequality (3.5). Suppose that $\tilde{x}, x^* \in \Omega$ satisfy (3.5), then

$$\begin{aligned}&\langle u + (\gamma f - (I+\mu A))x^*, \tilde{x} - x^*\rangle \le 0,\\&\langle u + (\gamma f - (I+\mu A))\tilde{x}, x^* - \tilde{x}\rangle \le 0.\end{aligned} \tag{3.14}$$

Adding up (3.14), we have

$$\begin{aligned}0 &\ge \langle (I+\mu A)x^* - (I+\mu A)\tilde{x} - \gamma(f(x^*) - f(\tilde{x})), x^* - \tilde{x}\rangle\\&= \langle (I+\mu A)x^* - (I+\mu A)\tilde{x}, x^* - \tilde{x}\rangle - \gamma\langle f(x^*) - f(\tilde{x}), x^* - \tilde{x}\rangle\\&\ge (1+\mu\overline{\gamma})\|x^* - \tilde{x}\|^2 - \gamma\|x^* - \tilde{x}\|^2 + \phi(\|x^* - \tilde{x}\|)(\|x^* - \tilde{x}\|)\\&= (1+\mu\overline{\gamma} - \gamma)\|x^* - \tilde{x}\|^2 + \phi(\|x^* - \tilde{x}\|)(\|x^* - \tilde{x}\|).\end{aligned} \tag{3.15}$$

It follows that

$$(1+\mu\overline{\gamma} - \gamma)\|x^* - \tilde{x}\| + \phi(\|x^* - \tilde{x}\|) \le 0, \tag{3.16}$$

which is a contradiction. Hence, $\tilde{x} = x^*$ and the uniqueness is proved.

Next, we show that $\{x_n\}$ is bounded. Taking $\overline{x} \in \Omega$, it follows from Lemma 2.11 that

$$\overline{x} = J_{(M_1,\rho_1)}\left[J_{(M_2,\rho_2)}(\overline{x} - \rho_2 B_2\overline{x}) - \rho_1 B_1 J_{(M_2,\rho_2)}(\overline{x} - \rho_2 B_2 \overline{x})\right]. \tag{3.17}$$

Putting $\overline{y} = J_{(M_2,\rho_2)}(\overline{x} - \rho_2 B_2\overline{x})$, we have $\overline{x} = J_{(M_1,\rho_1)}(\overline{y} - \rho_1 B_1\overline{y})$. Setting $z_n := V_n^N x_n$ and $v_n := J_{(M_1,\rho_1)}(y_n - \rho_1 B_1 y_n)$, then

$$x_n = \alpha_n(u + \gamma f(x_n)) + \beta_n x_n + ((1-\beta_n)I - \alpha_n(I+\mu A))\frac{1}{t_n}\int_0^{t_n} T(s)v_n\,ds. \tag{3.18}$$

Since for all $k \in \{1,2,\ldots,N\}$, $V_{r_{k,n}}^{(\Theta_k,\varphi_k)}$ is nonexpansive, we also have that V_n^N is nonexpansive and $\overline{x} = V_n^N\overline{x}$, then

$$\|z_n - \overline{x}\| = \left\|V_n^N x_n - V_n^N \overline{x}\right\| \le \|x_n - \overline{x}\|, \quad \forall n \in \mathbb{N}. \tag{3.19}$$

By nonexpansiveness of $J_{(M_i,\rho_i)}$ and $I - \rho_i B_i$ $(i = 1,2)$, we have

$$\begin{aligned}\|v_n - \overline{x}\| &= \left\|J_{(M_1,\rho_1)}(y_n - \rho_1 B_1 y_n) - J_{(M_1,\rho_1)}(\overline{y} - \rho_1 B_1\overline{y})\right\|\\&\le \left\|(y_n - \rho_1 B_1 y_n) - (\overline{y} - \rho_1 B_1 \overline{y})\right\|\\&\le \left\|y_n - \overline{y}\right\|\\&= \left\|J_{(M_2,\rho_2)}(z_n - \rho_2 B_2 z_n) - J_{(M_2,\rho_2)}(\overline{x} - \rho_2 B_2\overline{x})\right\|\\&\le \left\|(z_n - \rho_2 B_2 z_n) - (\overline{x} - \rho_2 B_2\overline{x})\right\|\end{aligned}$$

$$\begin{aligned}&\leq \|z_n - \overline{x}\|\\&\leq \|x_n - \overline{x}\|.\end{aligned} \tag{3.20}$$

It follows from (3.20) that

$$\begin{aligned}\|x_n - \overline{x}\|^2 &= \langle x_n - \overline{x}, x_n - \overline{x}\rangle\\&= \alpha_n\langle u + \gamma f(x_n) - (I + \mu A)\overline{x}, x_n - \overline{x}\rangle + \beta_n\langle x_n - \overline{x}, x_n - \overline{x}\rangle\\&\quad + \left\langle ((1-\beta_n)I - \alpha_n(I + \mu A))\left(\frac{1}{t_n}\int_0^{t_n}(T(s)v_n - \overline{x})ds\right), x_n - \overline{x}\right\rangle\\&= \alpha_n\gamma\langle f(x_n) - f(\overline{x}), x_n - \overline{x}\rangle + \alpha_n\langle u + \gamma f(\overline{x}) - (I + \mu A)\overline{x}, x_n - \overline{x}\rangle\\&\quad + \beta_n\langle x_n - \overline{x}, x_n - \overline{x}\rangle\\&\quad + \left\langle ((1-\beta_n)I - \alpha_n(I + \mu A))\left(\frac{1}{t_n}\int_0^{t_n}(T(s)v_n - \overline{x})ds\right), x_n - \overline{x}\right\rangle\\&\leq \alpha_n\gamma\left(\|x_n - \overline{x}\|^2 - \phi(\|x_n - \overline{x}\|)\|x_n - \overline{x}\|\right)\\&\quad + \alpha_n\langle u + \gamma f(\overline{x}) - (I + \mu A)\overline{x}, x_n - \overline{x}\rangle\beta_n\|x_n - \overline{x}\|^2\\&\quad + (1 - \beta_n - \alpha_n - \alpha_n\mu\overline{\gamma})\|x_n - \overline{x}\|^2,\end{aligned} \tag{3.21}$$

and, so

$$\gamma\phi(\|x_n - \overline{x}\|)(\|x_n - \overline{x}\|) + (1 - \gamma + \mu\overline{\gamma})\|x_n - \overline{x}\|^2 \leq \langle u + \gamma f(\overline{x}) - (I + \mu A)\overline{x}, x_n - \overline{x}\rangle. \tag{3.22}$$

It follows that

$$\begin{aligned}\gamma\phi(\|x_n - \overline{x}\|)\|x_n - \overline{x}\| &\leq \langle u + \gamma f(\overline{x}) - (I + \mu A)\overline{x}, x_n - \overline{x}\rangle\\&\leq \|u + \gamma f(\overline{x}) - (I + \mu A)\overline{x}\|\|x_n - \overline{x}\|.\end{aligned} \tag{3.23}$$

Hence

$$\|x_n - \overline{x}\| \leq \phi^{-1}\left(\frac{\|u + \gamma f(\overline{x}) - (I + \mu A)\overline{x}\|}{\gamma}\right), \tag{3.24}$$

which implies that $\{x_n\}$ is bounded, so are $\{z_n\}$, $\{y_n\}$, and $\{v_n\}$. Since f is ϕ-strongly pseudocontractive, we have

$$\begin{aligned}\langle f(x_n) - f(\overline{x}), x_n - \overline{x}\rangle &\leq \|x_n - \overline{x}\|^2 - \phi(\|x_n - \overline{x}\|)\|x_n - \overline{x}\|\\&\leq \|x_n - \overline{x}\|^2.\end{aligned} \tag{3.25}$$

Thus $\{f(x_n)\}$ is bounded.

Next, we show that $\lim_{n\to\infty}\|x_n - T(h)x_n\| = 0$, for all $h \geq 0$. From (3.18), we observe that

$$\left\|x_n - \frac{1}{t_n}\int_0^{t_n} T(s)v_n ds\right\| \leq \alpha_n \left\|u + \gamma f(x_n) - (I + \mu A)\frac{1}{t_n}\int_0^{t_n} T(s)v_n ds\right\| + \beta_n \left\|x_n - \frac{1}{t_n}\int_0^{t_n} T(s)v_n ds\right\|. \tag{3.26}$$

It follows that

$$\left\|x_n - \frac{1}{t_n}\int_0^{t_n} T(s)v_n\, ds\right\| \leq \frac{\alpha_n}{1-\beta_n}\left\|u + \gamma f(x_n) - (I + \mu A)\frac{1}{t_n}\int_0^{t_n} T(s)v_n\, ds\right\|. \tag{3.27}$$

By the conditions (C1) and (C2), we obtain

$$\lim_{n\to\infty}\left\|x_n - \frac{1}{t_n}\int_0^{t_n} T(s)v_n\, ds\right\| = 0. \tag{3.28}$$

Let $B = \{\omega \in H : \|\omega - \overline{x}\| \leq \phi^{-1}(\|u + \gamma f(\overline{x}) - (I + \mu A)\overline{x}\|/\gamma)\}$, then B is nonempty bounded closed and convex subset of H, which is $T(h)$-invariant for all $h \geq 0$ and contains $\{x_n\}$. It follows from Lemma 2.4 that

$$\lim_{n\to\infty}\left\|\frac{1}{t_n}\int_0^{t_n} T(s)v_n\, ds - T(h)\left(\frac{1}{t_n}\int_0^{t_n} T(s)v_n\, ds\right)\right\| = 0. \tag{3.29}$$

On the other hand, we note that

$$\begin{aligned}\|x_n - T(h)x_n\| &\leq \left\|x_n - \frac{1}{t_n}\int_0^{t_n} T(s)v_n\, ds\right\| + \left\|\frac{1}{t_n}\int_0^{t_n} T(s)v_n\, ds - T(h)\left(\frac{1}{t_n}\int_0^{t_n} T(s)v_n\, ds\right)\right\| \\ &\quad + \left\|T(h)\left(\frac{1}{t_n}\int_0^{t_n} T(s)v_n\, ds\right) - T(h)x_n\right\| \\ &\leq 2\left\|x_n - \frac{1}{t_n}\int_0^{t_n} T(s)v_n\, ds\right\| + \left\|\frac{1}{t_n}\int_0^{t_n} T(s)v_n\, ds - T(h)\left(\frac{1}{t_n}\int_0^{t_n} T(s)v_n\, ds\right)\right\|.\end{aligned} \tag{3.30}$$

From (3.28) and (3.29), for all $h \geq 0$, we have

$$\lim_{n\to\infty}\|x_n - T(h)x_n\| = 0. \tag{3.31}$$

Next, we show that $\lim_{n\to\infty}\|V_n^k x_n - V_n^{k-1}x_n\| = 0$ for all $k \in \{1,2,\dots,N\}$. Since $V_{r_{k,n}}^{(\Theta_k,\varphi_k)}$ is firmly nonexpansive for all $k \in \{1,2,\dots,N\}$ and $V_n^k = V_{r_{k,n}}^{(\Theta_k,\varphi_k)} V_{r_{k-1,n}}^{(\Theta_{k-1},\varphi_{k-1})} \cdots V_{r_{1,n}}^{(\Theta_1,\varphi_1)}$ for $k \in \{1,2,\dots,N\}$, hence for $\overline{x} \in \Omega$, we have

$$\begin{aligned}
\left\|V_n^k x_n - \overline{x}\right\|^2 &= \left\|V_{r_{k,n}}^{(\Theta_k,\varphi_k)} V_n^{k-1}x_n - V_{r_{k,n}}^{(\Theta_k,\varphi_k)}\overline{x}\right\|^2 \\
&\le \left\langle V_{r_{k,n}}^{(\Theta_k,\varphi_k)} V_n^{k-1}x_n - V_{r_{k,n}}^{(\Theta_k,\varphi_k)}\overline{x}, V_n^{k-1}x_n - \overline{x}\right\rangle \\
&= \left\langle V_n^k x_n - \overline{x}, V_n^{k-1}x_n - \overline{x}\right\rangle \\
&= \frac{1}{2}\left\{\left\|V_n^k x_n - \overline{x}\right\|^2 + \left\|V_n^{k-1}x_n - \overline{x}\right\|^2 - \left\|V_n^k x_n - V_n^{k-1}x_n\right\|^2\right\}.
\end{aligned} \tag{3.32}$$

It follows that

$$\left\|V_n^k x_n - \overline{x}\right\|^2 \le \|x_n - \overline{x}\|^2 - \left\|V_n^k x_n - V_n^{k-1}x_n\right\|^2. \tag{3.33}$$

Now, by Lemma 2.8, we have

$$\begin{aligned}
\|x_n - \overline{x}\|^2 &= \left\|\alpha_n u + \alpha_n(\gamma f(x_n) - (I+\mu A)\overline{x}) + \beta_n\left(x_n - \frac{1}{t_n}\int_0^{t_n} T(s)v_n\,ds\right)\right. \\
&\qquad \left. +(I - \alpha_n(I+\mu A))\left(\frac{1}{t_n}\int_0^{t_n} T(s)v_n\,ds - \overline{x}\right)\right\|^2 \\
&\le \left\|(I - \alpha_n(I+\mu A))\left(\frac{1}{t_n}\int_0^{t_n} T(s)v_n\,ds - \overline{x}\right) + \beta_n\left(x_n - \frac{1}{t_n}\int_0^{t_n} T(s)v_n\,ds\right)\right\|^2 \\
&\quad + 2\alpha_n\langle u + \gamma f(x_n) - (I+\mu A)\overline{x}, x_n - \overline{x}\rangle \\
&\le \left(\left\|(I - \alpha_n(I+\mu A))\left(\frac{1}{t_n}\int_0^{t_n} T(s)v_n\,ds - \overline{x}\right)\right\| + \beta_n\left\|x_n - \frac{1}{t_n}\int_0^{t_n} T(s)v_n\,ds\right\|\right)^2 \\
&\quad + 2\alpha_n\langle u + \gamma f(x_n) - (I+\mu A)\overline{x}, x_n - \overline{x}\rangle \\
&\le \left((1 - \alpha_n - \alpha_n\mu\overline{\gamma})\|v_n - \overline{x}\| + \beta_n\left\|x_n - \frac{1}{t_n}\int_0^{t_n} T(s)v_n\,ds\right\|\right)^2 \\
&\quad + 2\alpha_n\|u + \gamma f(x_n) - (I+\mu A)\overline{x}\|\|x_n - \overline{x}\| \\
&= (1 - \alpha_n - \alpha_n\mu\overline{\gamma})^2\|v_n - \overline{x}\|^2 + c_n,
\end{aligned} \tag{3.34}$$

where

$$c_n := \beta_n^2 \left\| x_n - \frac{1}{t_n}\int_0^{t_n} T(s)v_n\,ds \right\|^2 + 2(1-\alpha_n-\alpha_n\mu\overline{\gamma})\beta_n \|v_n-\overline{x}\| \left\| x_n - \frac{1}{t_n}\int_0^{t_n} T(s)v_n\,ds \right\| \\ + 2\alpha_n \left\| u+\gamma f(x_n)-(I+\mu A)\overline{x} \right\| \|x_n-\overline{x}\|. \tag{3.35}$$

From the condition (C1) and (3.28), we have

$$\lim_{n\to\infty} c_n = 0. \tag{3.36}$$

From (3.20), we observe that

$$\begin{aligned} \|v_n-\overline{x}\| &\le \left\| y_n-\overline{x} \right\| \\ &\le \left\| V_n^N x_n-\overline{x} \right\| \\ &\le \left\| V_n^k x_n-\overline{x} \right\|, \quad \forall k \in \{1,2,\dots,N\}. \end{aligned} \tag{3.37}$$

Substituting (3.33) into (3.34), we have

$$\begin{aligned} \|x_n-\overline{x}\|^2 &\le (1-\alpha_n-\alpha_n\mu\overline{\gamma})^2 \left\{ \|x_n-\overline{x}\|^2 - \left\| V_n^k x_n - V_n^{k-1}x_n \right\|^2 \right\} + c_n \\ &= \left\{ 1-2\alpha_n(1+\mu\overline{\gamma}) + \alpha_n^2(1+\mu\overline{\gamma})^2 \right\} \|x_n-\overline{x}\|^2 \\ &\quad - (1-\alpha_n-\alpha_n\mu\overline{\gamma})^2 \left\| V_n^k x_n - V_n^{k-1}x_n \right\|^2 + c_n, \end{aligned} \tag{3.38}$$

which in turn implies that

$$(1-\alpha_n-\alpha_n\mu\overline{\gamma})^2 \left\| V_n^k x_n - V_n^{k-1}x_n \right\|^2 \le \left\{ 1+\alpha_n^2(1+\mu\overline{\gamma})^2 \right\} \|x_n-\overline{x}\|^2 - \|x_n-\overline{x}\|^2 + c_n. \tag{3.39}$$

From the condition (C1) and from (3.36), we obtain that

$$\lim_{n\to\infty} \left\| V_n^k x_n - V_n^{k-1}x_n \right\| = 0. \tag{3.40}$$

On the other hand, we observe that

$$\begin{aligned}\|x_n - z_n\| &= \left\|x_n - V_n^N x_n\right\| \\ &= \left\|\sum_{k=1}^{N}\left(V_n^k x_n - V_n^{k-1} x_n\right)\right\| \\ &\le \sum_{k=1}^{N}\left\|V_n^k x_n - V_n^{k-1} x_n\right\|.\end{aligned} \tag{3.41}$$

Then, we have

$$\lim_{n\to\infty}\|x_n - z_n\| = 0. \tag{3.42}$$

Moreover, we observe that

$$\left\|z_n - \frac{1}{t_n}\int_0^{t_n} T(s)v_n\, ds\right\| \le \|z_n - x_n\| + \left\|x_n - \frac{1}{t_n}\int_0^{t_n} T(s)v_n\, ds\right\|, \tag{3.43}$$

and hence

$$\lim_{n\to\infty}\left\|z_n - \frac{1}{t_n}\int_0^{t_n} T(s)v_n\, ds\right\| = 0. \tag{3.44}$$

Next, we show that for all $\overline{x}, \overline{y} \in \Omega$, $\lim_{n\to\infty}\|B_1 y_n - B_1\overline{y}\| = 0$ and $\lim_{n\to\infty}\|B_2 z_n - B_2\overline{x}\| = 0$. By the cocoercivity of the mapping B_1, we have

$$\begin{aligned}\|v_n - \overline{x}\|^2 &= \left\|J_{(M_1,\rho_1)}(y_n - \rho_1 B_1 y_n) - J_{(M_1,\rho_1)}(\overline{y} - \rho_1 B_1\overline{y})\right\|^2 \\ &\le \left\|(y_n - \rho_1 B_1 y_n) - (\overline{y} - \rho_1 B_1\overline{y})\right\|^2 \\ &= \left\|(y_n - \overline{y}) - \rho_1(B_1 y_n - B_1\overline{y})\right\|^2 \\ &= \left\|y_n - \overline{y}\right\|^2 - 2\rho_1\langle B_1 y_n - B_1\overline{y}, y_n - \overline{y}\rangle + \rho_1^2\left\|B_1 y_n - B_1\overline{y}\right\|^2 \\ &\le \|x_n - \overline{x}\|^2 - 2\rho_1\left[-c_1\left\|B_1 y_n - B_1\overline{y}\right\|^2 + d_1\left\|y_n - \overline{y}\right\|^2\right] + \rho_1^2\left\|B_1 y_n - B_1\overline{y}\right\|^2 \\ &\le \|x_n - \overline{x}\|^2 + \left(2\rho_1 c_1 + \rho_1^2 - \frac{2\rho_1 d_1}{L_1^2}\right)\left\|B_1 y_n - B_1\overline{y}\right\|^2.\end{aligned} \tag{3.45}$$

Similarly, we have

$$\begin{aligned}\left\|y_n - \overline{y}\right\|^2 &= \left\|J_{(M_2,\rho_2)}(z_n - \rho_2 B_2 z_n) - J_{(M_2,\rho_2)}(\overline{x} - \rho_2 B_2\overline{x})\right\|^2 \\ &\le \left\|(z_n - \rho_2 B_2 z_n) - (\overline{x} - \rho_2 B_2\overline{x})\right\|^2\end{aligned}$$

$$
\begin{aligned}
&= \left\| (z_n - \overline{x}) - \rho_2 (B_2 z_n - B_2 \overline{x}) \right\|^2 \\
&= \| z_n - \overline{x} \|^2 - 2\rho_2 \langle B_2 z_n - B_2 \overline{x}, z_n - \overline{x} \rangle + \rho_2^2 \| B_2 z_n - B_2 \overline{x} \|^2 \\
&\le \| x_n - \overline{x} \|^2 - 2\rho_2 \left[-c_2 \| B_2 z_n - B_2 \overline{x} \|^2 + d_2 \| z_n - \overline{x} \|^2 \right] + \rho_2^2 \| B_2 z_n - B_2 \overline{x} \|^2 \\
&\le \| x_n - \overline{x} \|^2 + \left(2\rho_2 c_2 + \rho_2^2 - \frac{2\rho_2 d_2}{L_2^2} \right) \| B_2 z_n - B_2 \overline{x} \|^2 .
\end{aligned}
\tag{3.46}
$$

Substituting (3.45) into (3.34), we have

$$
\| x_n - \overline{x} \|^2 \le (1 - \alpha_n - \alpha_n \mu \overline{\gamma})^2 \left\{ \| x_n - \overline{x} \|^2 + \left(2\rho_1 c_1 + \rho_1^2 - \frac{2\rho_1 d_1}{L_1^2} \right) \left\| B_1 y_n - B_1 \overline{y} \right\|^2 \right\} + c_n .
\tag{3.47}
$$

Again, from (3.34), we obtain

$$
\begin{aligned}
\| x_n - \overline{x} \|^2 &\le (1 - \alpha_n - \alpha_n \mu \overline{\gamma})^2 \| v_n - \overline{x} \|^2 + c_n \\
&\le (1 - \alpha_n - \alpha_n \mu \overline{\gamma})^2 \left\| y_n - \overline{y} \right\|^2 + c_n \\
&\le (1 - \alpha_n - \alpha_n \mu \overline{\gamma})^2 \left\{ \| x_n - \overline{x} \|^2 + \left(2\rho_2 c_2 + \rho_2^2 - \frac{2\rho_2 d_2}{L_2^2} \right) \| B_2 z_n - B_2 \overline{x} \|^2 \right\} + c_n .
\end{aligned}
\tag{3.48}
$$

Therefore, from (3.47) and (3.48), we obtain

$$
\begin{aligned}
&(1 - \alpha_n - \alpha_n \mu \overline{\gamma})^2 \left(-2\rho_1 c_1 - \rho_1^2 + \frac{2\rho_1 d_1}{L_1^2} \right) \left\| B_1 y_n - B_1 \overline{y} \right\|^2 \\
&\quad \le (1 - \alpha_n - \alpha_n \mu \overline{\gamma})^2 \| x_n - \overline{x} \|^2 - \| x_n - \overline{x} \|^2 + c_n \\
&\quad \le \left\{ 1 + \alpha_n^2 (1 + \mu \overline{\gamma})^2 \right\} \| x_n - \overline{x} \|^2 - \| x_n - \overline{x} \|^2 + c_n , \\
&(1 - \alpha_n - \alpha_n \mu \overline{\gamma})^2 \left(-2\rho_2 c_2 - \rho_2^2 + \frac{2\rho_2 d_2}{L_2^2} \right) \| B_2 z_n - B_2 \overline{x} \|^2 \\
&\quad \le (1 - \alpha_n - \alpha_n \mu \overline{\gamma})^2 \| x_n - \overline{x} \|^2 - \| x_n - \overline{x} \|^2 + c_n \\
&\quad \le \left\{ 1 + \alpha_n^2 (1 + \mu \overline{\gamma})^2 \right\} \| x_n - \overline{x} \|^2 - \| x_n - \overline{x} \|^2 + c_n .
\end{aligned}
\tag{3.49}
$$

From the condition $(C1)$ and from (3.36), we obtain that

$$
\lim_{n \to \infty} \left\| B_1 y_n - B_1 \overline{y} \right\| = 0, \qquad \lim_{n \to \infty} \| B_2 z_n - B_2 \overline{x} \| = 0.
\tag{3.50}
$$

Next, we show that $\lim_{n\to\infty}\|z_n - v_n\| = 0$ and $\lim_{n\to\infty}\|x_n - v_n\| = 0$. By the nonexpansiveness of $I - \rho_2 B_2$, we have

$$
\begin{aligned}
\|y_n - \overline{y}\|^2 &= \left\| J_{(M_2,\rho_2)}(z_n - \rho_2 B_2 z_n) - J_{(M_2,\rho_2)}(\overline{x} - \rho_2 B_2 \overline{x}) \right\|^2 \\
&\leq \langle (z_n - \rho_2 B_2 z_n) - (\overline{x} - \rho_2 B_2 \overline{x}), y_n - \overline{y} \rangle \\
&= \frac{1}{2}\Big\{ \|(z_n - \rho_2 B_2 z_n) - (\overline{x} - \rho_2 B_2 \overline{x})\|^2 + \|y_n - \overline{y}\|^2 \\
&\quad - \|(z_n - \rho_2 B_2 z_n) - (\overline{x} - \rho_2 B_2 \overline{x}) - (y_n - \overline{y})\|^2 \Big\} \\
&\leq \frac{1}{2}\Big\{ \|z_n - \overline{x}\|^2 + \|y_n - \overline{y}\|^2 - \|(z_n - y_n) - \rho_2(B_2 z_n - B_2\overline{x}) - (\overline{x} - \overline{y})\|^2 \Big\} \\
&= \frac{1}{2}\Big\{ \|z_n - \overline{x}\|^2 + \|y_n - \overline{y}\|^2 - \|(z_n - y_n) - (\overline{x} - \overline{y})\|^2 \\
&\quad + 2\rho_2 \langle (z_n - y_n) - (\overline{x} - \overline{y}), B_2 z_n - B_2 \overline{x} \rangle - \rho_2^2 \|B_2 z_n - B_2 \overline{x}\|^2 \Big\}.
\end{aligned}
\tag{3.51}
$$

So, we obtain

$$
\|y_n - \overline{y}\|^2 \leq \|z_n - \overline{x}\|^2 - \|(z_n - y_n) - (\overline{x} - \overline{y})\|^2 + 2\rho_2 \langle (z_n - y_n) - (\overline{x} - \overline{y}), B_2 z_n - B_2 \overline{x} \rangle.
\tag{3.52}
$$

Substituting (3.52) into (3.34), we have

$$
\begin{aligned}
\|x_n - \overline{x}\|^2 \leq (1 - \alpha_n - \alpha_n \mu \overline{\gamma})^2 \Big\{ &\|z_n - \overline{x}\|^2 - \|(z_n - y_n) - (\overline{x} - \overline{y})\|^2 \\
&+ 2\rho_2 \langle (z_n - y_n) - (\overline{x} - \overline{y}), B_2 z_n - B_2 \overline{x} \rangle \Big\} + c_n,
\end{aligned}
\tag{3.53}
$$

which in turn implies that

$$
\begin{aligned}
(1 - \alpha_n - \alpha_n \mu \overline{\gamma})^2 \|(z_n - y_n) - (\overline{x} - \overline{y})\|^2 &\leq (1 - \alpha_n - \alpha_n \mu \overline{\gamma})^2 \|x_n - \overline{x}\|^2 - \|x_n - \overline{x}\|^2 \\
&\quad + (1 - \alpha_n - \alpha_n \mu \overline{\gamma})^2 2\rho_2 \|(z_n - y_n) - (\overline{x} - \overline{y})\| \\
&\quad \times \|B_2 z_n - B_2 \overline{x}\| + c_n \\
&\leq \Big\{ 1 + \alpha_n^2 (1 + \mu \overline{\gamma})^2 \Big\} \|x_n - \overline{x}\|^2 - \|x_n - \overline{x}\|^2 \\
&\quad + (1 - \alpha_n - \alpha_n \mu \overline{\gamma})^2 2\rho_2 \|(z_n - y_n) - (\overline{x} - \overline{y})\| \\
&\quad \times \|B_2 z_n - B_2 \overline{x}\| + c_n.
\end{aligned}
\tag{3.54}
$$

From the condition (C1) and from (3.36), (3.50), we obtain that

$$
\lim_{n\to\infty} \|(z_n - y_n) - (\overline{x} - \overline{y})\| = 0.
\tag{3.55}
$$

On the other hand, by Lemma 2.8 and from (2.3), we have,

$$\begin{aligned}
\|(y_n - v_n) + (\overline{x} - \overline{y})\|^2 &= \| (y_n - \rho_1 B_1 y_n) - (\overline{y} - \rho_1 B_1 \overline{y}) \\
&\quad - [J_{(M_1,\rho_1)}(y_n - \rho_1 B_1 y_n) - J_{(M_1,\rho_1)}(\overline{y} - \rho_1 B_1 \overline{y})] + \rho_1 (B_1 y_n - B_1 \overline{y}) \|^2 \\
&\le \| (y_n - \rho_1 B_1 y_n) - (\overline{y} - \rho_1 B_1 \overline{y}) \\
&\quad - [J_{(M_1,\rho_1)}(y_n - \rho_1 B_1 y_n) - J_{(M_1,\rho_1)}(\overline{y} - \rho_1 B_1 \overline{y})] \|^2 \\
&\quad + 2\rho_1 \langle B_1 y_n - B_1 \overline{y}, (y_n - v_n) + (\overline{x} - \overline{y}) \rangle \\
&\le \| (y_n - \rho_1 B_1 y_n) - (\overline{y} - \rho_1 B_1 \overline{y}) \|^2 \\
&\quad - \| J_{(M_1,\rho_1)}(y_n - \rho_1 B_1 y_n) - J_{(M_1,\rho_1)}(\overline{y} - \rho_1 B_1 \overline{y}) \|^2 \\
&\quad + 2\rho_1 \| B_1 y_n - B_1 \overline{y} \| \| (y_n - v_n) + (\overline{x} - \overline{y}) \| \\
&\le \| (y_n - \rho_1 B_1 y_n) - (\overline{y} - \rho_1 B_1 \overline{y}) \|^2 \\
&\quad - \left\| \frac{1}{t_n} \int_0^{t_n} T(s) J_{(M_1,\rho_1)}(y_n - \rho_1 B_1 y_n) ds \right. \\
&\quad \left. - \frac{1}{t_n} \int_0^{t_n} T(s) J_{(M_1,\rho_1)}(\overline{y} - \rho_1 B_1 \overline{y}) ds \right\|^2 \\
&\quad + 2\rho_1 \| B_1 y_n - B_1 \overline{y} \| \| (y_n - v_n) + (\overline{x} - \overline{y}) \| \\
&= \| (y_n - \rho_1 B_1 y_n) - (\overline{y} - \rho_1 B_1 \overline{y}) \|^2 \\
&\quad - \left\| \frac{1}{t_n} \int_0^{t_n} T(s) v_n \, ds - \frac{1}{t_n} \int_0^{t_n} T(s) \, \overline{x} ds \right\|^2 \\
&\quad + 2\rho_1 \| B_1 y_n - B_1 \overline{y} \| \| (y_n - v_n) + (\overline{x} - \overline{y}) \| \\
&\le \left\| (y_n - \rho_1 B_1 y_n) - (\overline{y} - \rho_1 B_1 \overline{y}) - \left(\frac{1}{t_n} \int_0^{t_n} T(s) v_n \, ds - \overline{x} \right) \right\| \\
&\quad \times \left\{ \| (y_n - \rho_1 B_1 y_n) - (\overline{y} - \rho_1 B_1 \overline{y}) \| + \left\| \frac{1}{t_n} \int_0^{t_n} T(s) v_n ds - \overline{x} \right\| \right\} \\
&\quad + 2\rho_1 \| B_1 y_n - B_1 \overline{y} \| \| (y_n - v_n) + (\overline{x} - \overline{y}) \| \\
&= \left\| \left(z_n - \frac{1}{t_n} \int_0^{t_n} T(s) v_n \, ds \right) + (\overline{x} - \overline{y}) - (z_n - y_n) - \rho_1 (B_1 y_n - B_1 \overline{y}) \right\| \\
&\quad \times \left\{ \| (y_n - \rho_1 B_1 y_n) - (\overline{y} - \rho_1 B_1 \overline{y}) \| + \left\| \frac{1}{t_n} \int_0^{t_n} T(s) v_n \, ds - \overline{x} \right\| \right\} \\
&\quad + 2\rho_1 \| B_1 y_n - B_1 \overline{y} \| \| (y_n - v_n) + (\overline{x} - \overline{y}) \|.
\end{aligned} \tag{3.56}$$

we obtain that

$$\lim_{n\to\infty}\left\|\left(y_n - v_n\right) + \left(\overline{x} - \overline{y}\right)\right\| = 0. \tag{3.57}$$

In fact, since

$$\|z_n - v_n\| \le \left\|\left(z_n - y_n\right) - \left(\overline{x} - \overline{y}\right)\right\| + \left\|\left(y_n - v_n\right) + \left(\overline{x} - \overline{y}\right)\right\|, \tag{3.58}$$

so

$$\lim_{n\to\infty}\|z_n - v_n\| = 0. \tag{3.59}$$

And since

$$\|x_n - v_n\| \le \|x_n - z_n\| + \|z_n - v_n\|, \tag{3.60}$$

and so

$$\lim_{n\to\infty}\|x_n - v_n\| = 0. \tag{3.61}$$

Next, we show that $\tilde{x} \in \Omega := F(\mathcal{S}) \cap \bigcap_{k=1}^{N}\text{MEP }(\Theta_k, \varphi_k) \cap F(Q)$.

(i) We first show that $\tilde{x} \in F(\mathcal{S})$. Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j} \rightharpoonup \tilde{x} \in H$ as $j \to \infty$. From (3.31) and Lemma 2.5, we obtain that $\tilde{x} \in F(\mathcal{S})$.

(ii) Now, we show that $\tilde{x} \in \bigcap_{k=1}^{N}\text{MEP }(\Theta_k, \varphi_k)$. Since $V_n^k = V_{r_{k,n}}^{(\Theta_k,\varphi_k)} V_n^{k-1}$ for $k \in \{1, 2, \ldots, N\}$. Hence, for all $x \in H$ and for all $k \in \{1, 2, \ldots, N\}$, we obtain

$$\Theta_k\left(V_n^k x_n, x\right) + \varphi_k(x) - \varphi_k\left(V_n^k x_n\right) + \frac{1}{r_{k,n}}\left\langle K_k'\left(V_n^k x_n\right) - K_k'\left(V_n^{k-1} x_n\right), \eta_k\left(x, V_n^k x_n\right)\right\rangle \ge 0. \tag{3.62}$$

And hence

$$\begin{aligned}\left\langle \frac{K_k'\left(V_n^k x_{n_j}\right) - K_k'\left(V_n^{k-1} x_{n_j}\right)}{r_{k,n}}, \eta_k\left(x, V_n^k x_{n_j}\right)\right\rangle &\ge -\Theta_k\left(V_n^k x_{n_j}, x\right) - \varphi_k(x) \\ &\quad + \varphi_k\left(V_n^k x_{n_j}\right), \quad \forall x \in H.\end{aligned} \tag{3.63}$$

By the assumptions that φ_k is lower semicontinuous and by conditions (H4), (H5), the mapping $x \mapsto (-\Theta_k(x, y))$ is lower semicontinuous. So, they are weakly lower semicontinuous. Since $x_{n_j} \rightharpoonup \tilde{x}$, we have that $V_n^k x_n \rightharpoonup \tilde{x}$ for all $k \in \{1,2,\ldots,N\}$ and from (i)(c), (ii), (3.40). Now, taking lower limit as $j \to \infty$ in (3.63), we obtain that

$$\Theta_k(\tilde{x}, x) + \varphi_k(x) - \varphi_k(\tilde{x}) \geq 0, \quad \forall x \in H,\ \forall k \in \{1,2,\ldots,N\}. \tag{3.64}$$

Therefore $\tilde{x} \in \bigcap_{k=1}^{N} \mathrm{MEP}\ (\Theta_k, \varphi_k)$.

(iii) Now, we show that $\tilde{x} \in F(Q)$, where Q is defined as in Lemma 2.11. Since Q is nonexpansive. Then, we have

$$\begin{aligned}
\|v_n - Qv_n\| &= \left\| J_{(M_1,\rho_1)}(y_n - \rho_1 B_1 y_n) - Qv_n \right\| \\
&= \left\| J_{(M_1,\rho_1)}\left[J_{(M_2,\rho_2)}(z_n - \rho_2 B_2 z_n) - \rho_1 B_1 J_{(M_2,\rho_2)}(z_n - \rho_2 B_2 z_n)\right] - Qv_n \right\| \\
&= \|Qz_n - Qv_n\| \\
&\leq \|z_n - v_n\|.
\end{aligned} \tag{3.65}$$

From (3.59), we have $\lim_{n\to\infty}\|v_n - Qv_n\| = 0$. Since $x_{n_j} \rightharpoonup \tilde{x}$ and from (3.61), we also have $v_{n_j} \rightharpoonup \tilde{x}$. Hence, we obtain by Lemma 2.5 that $\tilde{x} \in F(Q)$.

Next, we show that $\{x_n\}$ is sequentially compact, namely, there is a subsequence $\{x_{n_j}\} \subset \{x_n\}$ that converges strongly to $\tilde{x} \in \Omega$ as $j \to \infty$. From (3.23), replacing $\overline{x}$ by $\tilde{x}$ to obtain

$$\gamma\phi(\|x_n - \tilde{x}\|)(\|x_n - \tilde{x}\|) \leq \langle u + \gamma f(\tilde{x}) - (I + \mu A)\tilde{x}, x_n - \tilde{x}\rangle. \tag{3.66}$$

Now, replacing n with n_j in (3.66) and letting $j \to \infty$, since $x_{n_j} \rightharpoonup \tilde{x}$, we obtain that $x_{n_j} \to \tilde{x}$ as $j \to \infty$.

Next, we show that $\tilde{x}$ is the unique solution of the variational inequality (3.5). Since

$$x_n = \alpha_n(u + \gamma f(x_n)) + \beta_n x_n + ((1-\beta_n)I - \alpha_n(I + \mu A))\frac{1}{t_n}\int_0^{t_n} T(s)v_n\, ds, \tag{3.67}$$

we derive that

$$\begin{aligned}
(-u + (I + \mu A - \gamma f)x_n) = &-\frac{(1-\beta_n)}{\alpha_n}\left(x_n - \frac{1}{t_n}\int_0^{t_n} T(s)v_n\, ds\right) \\
&+ (I + \mu A)\left(x_n - \frac{1}{t_n}\int_0^{t_n} T(s)v_n\, ds\right).
\end{aligned} \tag{3.68}$$

For all $z \in \Omega$, it follows that

$$\begin{aligned}
&\langle -u + (I + \mu A - \gamma f)x_n, x_n - z \rangle \\
&\quad = -\frac{(1-\beta_n)}{\alpha_n} \left\langle \left(I - \frac{1}{t_n}\int_0^{t_n} T(s)QV_n^N ds \right)x_n - \left(I - \frac{1}{t_n}\int_0^{t_n} T(s)QV_n^N ds \right)z, x_n - z \right\rangle \\
&\qquad + \left\langle (I + \mu A)\left(x_n - \frac{1}{t_n}\int_0^{t_n} T(s)v_n ds \right), x_n - z \right\rangle.
\end{aligned} \tag{3.69}$$

By Lemma 2.10, we obtain that $I - \sigma_t(\cdot)$ is monotone, where $\sigma_t(\cdot) := (1/t)\int_0^t T(s)\,ds$. Then, from (3.69), we have

$$\langle -u + (I + \mu A - \gamma f)x_n, x_n - z \rangle \le \langle (I + \mu A)(x_n - \sigma_t(v_n)), x_n - z \rangle. \tag{3.70}$$

Now, replacing n by n_j in (3.70) and letting $j \to \infty$ and $x_{n_j} \to \tilde{x}$, we notice that

$$x_{n_j} - \frac{1}{t_{n_j}}\int_0^{t_{n_j}} T(s)v_{n_j}\,ds \to 0. \tag{3.71}$$

Then, we have

$$\langle u + (\gamma f - (I + \mu A))\tilde{x}, z - \tilde{x} \rangle \le 0, \quad \forall z \in \Omega. \tag{3.72}$$

That is, $\tilde{x}$ is the solution of variational inequality (3.5).

Finally, we show that $\{x_n\}$ converges strongly to $\tilde{x} \in \Omega$. Suppose that there exists another subsequence $x_{n_k} \to \hat{x}$ as $k \to \infty$. We note Lemma 2.5 that $\hat{x} \in \Omega$ is the solution of the variational inequality (3.5). Hence $\tilde{x} = \hat{x} = x^*$ by uniqueness. In summary, we have shown that $\{x_n\}$ is sequentially compact and each cluster point of the sequence $\{x_n\}$ is equal to x^*. Then, we conclude that $x_n \to x^*$ as $n \to \infty$. This completes the proof. □

From Theorem 3.2, we can deduce the following result.

Theorem 3.3. *Let C be a nonempty closed and convex subset of a real Hilbert space H. Let $\varphi_i : C \to \mathbb{R}$ $(i = 1,2,\ldots,N)$ be a finite family of lower semicontinuous and convex function, $\Theta_i : C \times C \to \mathbb{R}$ $(i = 1,2,\ldots,N)$ be a finite family of bifunctions satisfying (H1)–(H5), $\eta_i : C \times H \to H$ be a finite family of Lipschitz continuous mappings with a constant σ_i $(i = 1,2,\ldots,N)$. Let $\mathcal{S} = \{T(t) : t \in \mathbb{R}^+\}$ be a nonexpansive semigroup from C into H and $B_i : H \to H$ $(i = 1,2)$ be an L_i-Lipschitzian and relaxed (c_i, d_i)-cocoercive mapping with $\rho_i \in (0, 2(d_i - c_iL_i^2)/L_i^2]$ for all $i = 1,2$. Let $\overline{Q} : C \to C$ be a mapping defined by*

$$\overline{Q}x := P_C\left[P_C(x - \rho_2 B_2 x) - \rho_1 B_1 P_C(x - \rho_2 B_2 x)\right]. \tag{3.73}$$

Assume that $\Omega := F(\mathcal{S}) \cap \bigcap_{k=1}^N \mathrm{MEP}(\Theta_k, \varphi_k) \cap F(\overline{Q}) \neq \emptyset$. Let $f : H \to H$ be a ϕ-strongly pseudocontractive mapping with $\lim_{t\to+\infty}\phi(t) = +\infty$ and A be a strongly positive linear bounded operator on H with a coefficient $\overline{\gamma} > 0$. Let $\mu > 0$ and $\gamma > 0$ be two constants such that $0 < \gamma < 1 + \mu\overline{\gamma}$. Let $\{r_{i,n}\}(i = 1, 2, \ldots, N)$ be a finite family of positive real sequence such that $\liminf_{n\to\infty} r_{i,n} > 0$, $\{\alpha_n\}$ and $\{\beta_n\}$ be two sequences in $[0,1]$, and $\{t_n\}$ be a positive real divergent sequence. For any fixed $u \in H$, let $\{x_n\}$ be the sequence defined by

$$\begin{gathered}
\Theta_1\left(u_n^{(1)}, x\right) + \varphi_1(x) - \varphi_1\left(u_n^{(1)}\right) + \frac{1}{r_{1,n}}\left\langle K_1'\left(u_n^{(1)}\right) - K_1'(x_n), \eta_1\left(x, u_n^{(1)}\right)\right\rangle \geq 0, \quad \forall x \in C, \\
\Theta_2\left(u_n^{(2)}, x\right) + \varphi_2(x) - \varphi_2\left(u_n^{(2)}\right) + \frac{1}{r_{2,n}}\left\langle K_2'\left(u_n^{(2)}\right) - K_2'\left(u_n^{(1)}\right), \eta_1\left(x, u_n^{(2)}\right)\right\rangle \geq 0, \quad \forall x \in C, \\
\vdots \\
\Theta_N\left(u_n^{(N)}, x\right) + \varphi_N(x) - \varphi_N\left(u_n^{(N)}\right) + \frac{1}{r_{N,n}}\left\langle K_N'\left(u_n^{(N)}\right) - K_N'\left(u_n^{(N-1)}\right), \eta_N\left(x, u_n^{(N)}\right)\right\rangle \geq 0, \ \forall x \in C, \\
y_n = P_C\left(u_n^{(N)} - \rho_2 B_2 u_n^{(N)}\right), \\
x_n = \alpha_n(u + \gamma f(x_n)) + \beta_n x_n + ((1-\beta_n)I - \alpha_n(I + \mu A))\frac{1}{t_n}\int_0^{t_n} T(s) P_C(y_n - \rho_1 B_1 y_n)\, ds,
\end{gathered} \tag{3.74}$$

where

$$\begin{aligned}
u_n^{(1)} &= V_{r_{1,n}}^{(\Theta_1,\varphi_1)} x_n, \\
u_n^{(i)} &= V_{r_{i,n}}^{(\Theta_i,\varphi_i)} u_n^{(i-1)} = V_{r_{i,n}}^{(\Theta_i,\varphi_i)} V_{r_{i-1,n}}^{(\Theta_{i-1},\varphi_{i-1})} u_n^{(i-2)} = V_{r_{i,n}}^{(\Theta_i,\varphi_i)} \cdots V_{r_{2,n}}^{(\Theta_2,\varphi_2)} u_n^{(1)} \\
&= V_{r_{i,n}}^{(\Theta_i,\varphi_i)} \cdots V_{r_{2,n}}^{(\Theta_2,\varphi_2)} V_{r_{1,n}}^{(\Theta_1,\varphi_1)} x_n, \quad i = 2, 3, \ldots, N,
\end{aligned} \tag{3.75}$$

and $V_{r_{i,n}}^{(\Theta_i,\varphi_i)} : C \to C$, $i = 1, 2, \ldots, N$ is the mapping defined by (2.8). Assume the following.

(i) $\eta_i : C \times C \to \mathbb{R}$ is Lipschitz continuous with constant $\sigma_i > 0$ $(i = 1, 2, \ldots, N)$ such that

  (a) $\eta_i(x, y) + \eta_i(y, x) = 0$ for all $x, y \in C$,

  (b) $\eta_i(\cdot, \cdot)$ is affine in the first variable,

  (c) for each fixed $y \in C$, $x \mapsto \eta_i(y, x)$ is sequentially continuous from the weak topology to the weak topology.

(ii) $K_i : C \to \mathbb{R}$ is η_i—strongly convex with constant $\mu_i > 0$, and its derivative K_i' is not only continuous from the weak topology to the strong topology but also Lipschitz continuous with constant ν_i such that $\mu_i \geq \sigma_i \nu_i$.

(iii) For all $i = 1, 2, \ldots, N$ and for all $x \in C$, there exists a bounded subset $D_x \subset C$ and $z_x \in C$ such that for all $y \notin D_x$,

$$\Theta_i(y, z_n) + \varphi_i(z_x) - \varphi_i(y) + \frac{1}{r_{i,n}}\langle K_i'(y) - K_i'(x), \eta_i(z_x, y)\rangle < 0. \tag{3.76}$$

If the following conditions are satisfied:

(C1) $\lim_{n\to\infty}\alpha_n = 0$,

(C2) $0 < \liminf_{n\to\infty}\beta_n \le \limsup_{n\to\infty}\beta_n < 1$,

then the sequence $\{x_n\}$ defined by (3.74) converges strongly to $x^* \in \Omega := F(\mathcal{S}) \cap \bigcap_{k=1}^{N}\mathrm{MEP}(\Theta_k, \varphi_k) \cap F(\overline{Q})$, provided $V_{r_{i,n}}^{(\Theta_i,\varphi_i)}$ is firmly nonexpansive, where x^* is the unique solution of the variational inequality

$$\left\langle u + \left(\gamma f - (I + \mu A)\right)x^*, z - x^*\right\rangle \le 0, \quad \forall z \in \Omega, \tag{3.77}$$

or, equivalently, x^* is the unique solution of the optimization problem

$$\min_{x\in\Omega}\frac{\mu}{2}\langle Ax, x\rangle + \frac{1}{2}\|x - u\|^2 - h(x), \tag{3.78}$$

where h is a potential function for γf and (x^*, y^*) is the solution of general system of variational inequality problem

$$\begin{aligned} &\left\langle \rho_1 B_1 y^* + x^* - y^*, x - x^*\right\rangle \ge 0, \quad \forall x \in C, \\ &\left\langle \rho_2 B_2 x^* + y^* - x^*, x - y^*\right\rangle \ge 0, \quad \forall x \in C. \end{aligned} \tag{3.79}$$

Proof. From Theorem 3.2, taking $M_1 = M_2 = \partial\delta_C$, where C is a nonempty closed convex subset of H and $\delta_C : H \to [0, +\infty)$ is the indicator function of C, then, we have $J_{(M_1,\rho_1)} = J_{(M_2,\rho_2)} = P_C$ and the quasivariational inclusion problem (1.17) is equivalent to the classical variational inequality (1.19). Thus, we can get the desired conclusion immediately. □

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