Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2012, Article ID 803417, 18 pages
doi:10.1155/2012/803417

Research Article

Existence and Uniqueness of Positive Solutions for a Singular Fractional Three-Point Boundary Value Problem

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Received 12 June 2012; Accepted 29 July 2012

Academic Editor: Juan J. Trujillo

We investigate the existence and uniqueness of positive solutions for the following singular fractional three-point boundary value problem $D_{0^+}^{\alpha}u(t) + f(t, u(t)) = 0, 0 < t < 1$, $u(0) = u'(0) = u''(0) = 0, u''(1) = \beta u''(\eta)$, where $3 < \alpha \le 4$, $D_{0^+}^{\alpha}$ is the standard Riemann-Liouville derivative and $f : (0,1] \times [0,\infty) \to [0,\infty)$ with $\lim_{t\to 0^+} f(t,\cdot) = \infty$ (i.e., f is singular at $t = 0$). Our analysis relies on a fixed point theorem in partially ordered metric spaces.

1. Introduction

Fractional differential equations have been of great interest recently. It is caused both by the intensive development of the theory of fractional calculus itself and by the applications (see, e.g., [1–5]).

Recently, many papers have appeared dealing with the existence of solutions of nonlinear fractional boundary value problems.

In [6], the authors studied the existence and multiplicity of positive solutions for the boundary value problem:

$$\begin{gathered} D_{0^+}^{\alpha}u(t) + f(t, u(t)) = 0, \quad 0 < t < 1, \\ u(0) = u(1) = 0, \end{gathered} \tag{1.1}$$

where $1 < \alpha \le 2$ and $f : [0,1] \times [0,\infty) \to [0,\infty)$ is continuous, by using some fixed point theorem on cones.

In [7], the authors considered the following nonlinear fractional boundary value problem:

$$D_{0^+}^{\alpha}u(t) + f(t, u(t)) = 0, \quad 0 < t < 1,$$
$$u(0) = u'(0) = u''(0) = u''(1) = 0, \tag{1.2}$$

where $3 < \alpha \leq 4$ and $f : [0,1] \times [0,\infty) \to [0,\infty)$ is continuous. They obtained their results by using lower and upper solution method and fixed point theorems.

In [8] the authors investigated the existence and uniqueness of positive and nondecreasing solutions for a class of singular fractional boundary value problems by using a fixed point theorem in partially ordered metric spaces.

Very recently, in [9] the authors studied the existence of solutions of the following three-point boundary value problem:

$$D_{0^+}^{\alpha}u(t) + f(t, u(t)) = 0, \quad 0 < t < 1,$$
$$u(0) = u'(0) = u''(0) = 0, \quad u''(1) = \beta u''(\eta), \tag{1.3}$$

where $3 < \alpha \leq 4$, $0 < \eta < 1$, $0 < \beta\eta^{\alpha-3} < 1$ and $f : [0,1] \times [0,\infty) \to [0,\infty)$ is continuous.

Motivated by [8, 9], in this paper we discuss the existence and uniqueness of positive solutions for Problem (1.3) assuming that $f : (0,1] \times [0,\infty) \to [0,\infty)$ is such that $\lim_{t\to 0^+} f(t,\cdot) = \infty$ (i.e., f is singular at $t = 0$). Our main tool is a fixed point theorem in partially ordered metric spaces which appears in [10].

2. Preliminaries and Basic Facts

For the convenience of the reader, we present some notations and lemmas which will be used in the proof of our results.

Definition 2.1 (see [5]). The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $f : (0,\infty) \to \mathbb{R}$ is given by

$$I_{0^+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)\,ds, \tag{2.1}$$

provided that the right-hand side is pointwise defined on $(0,\infty)$ and where $\Gamma(\alpha)$ denotes the classical gamma function.

Definition 2.2 (see [2]). The Riemann-Liouville fractional derivative of order $\alpha > 0$ of a function $f : (0,\infty) \to \mathbb{R}$ is defined as

$$D_{0^+}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_0^t \frac{f(s)}{(t-s)^{\alpha-n+1}}\,ds, \tag{2.2}$$

where $n = [\alpha] + 1$ and $[\alpha]$ denotes the integer part of α.

The following two lemmas can be found in [2] and they are crucial in finding an integral representation of the boundary value problem (1.3).

Lemma 2.3 (see [2]). *Assume that $u \in C(0,1) \cap L^1(0,1)$ and $\alpha > 0$.*

Then the fractional differential equation

$$D_{0^+}^{\alpha} u(t) = 0 \tag{2.3}$$

has

$$u(t) = c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_n t^{\alpha-n}, \tag{2.4}$$

where $c_i \in \mathbb{R}$, $i = 1, 2, \ldots, n$ and $n = [\alpha] + 1$, as unique solution.

Lemma 2.4 (see [2]). *Assume that $u \in C(0,1) \cap L^1(0,1)$ with fractional derivative of order $\alpha > 0$ that belongs to $C(0,1) \cap L^1(0,1)$. Then*

$$I_{0^+}^{\alpha} D_{0^+}^{\alpha} u(t) = u(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_n t^{\alpha-n}, \tag{2.5}$$

for some $c_i \in \mathbb{R}$, $i = 1, 2, \ldots, n$ and $n = [\alpha] + 1$.

By using Lemma 2.4, in [9] the authors proved the following result.

Lemma 2.5 (see [9]). *Let $0 < \eta < 1$ and $\beta \neq 1/\eta^{\alpha-3}$ and $h \in C[0,1]$.*

Then the boundary value problem

$$\begin{gathered} D_{0^+}^{\alpha} u(t) + h(t) = 0, \quad 0 < t < 1, \\ u(0) = u'(0) = u''(0) = 0, \quad u''(1) = \beta u''(\eta), \end{gathered} \tag{2.6}$$

where $3 < \alpha \leq 4$, has as unique solution

$$u(t) = \int_0^1 G(t,s) h(s) ds + \frac{\beta t^{\alpha-1}}{(\alpha-1)(\alpha-2)(1-\beta\eta^{\alpha-3})} \int_0^1 H(\eta,s) h(s) ds, \tag{2.7}$$

where

$$\begin{gathered} G(t,s) = \begin{cases} \dfrac{t^{\alpha-1}(1-s)^{\alpha-3} - (t-s)^{\alpha-1}}{\Gamma(\alpha)}, & 0 \leq s \leq t \leq 1, \\ \dfrac{t^{\alpha-1}(1-s)^{\alpha-3}}{\Gamma(\alpha)}, & 0 \leq t \leq s \leq 1, \end{cases} \\ H(t,s) = \frac{\partial^2 G(t,s)}{\partial t^2} = \begin{cases} \dfrac{(\alpha-1)(\alpha-2)}{\Gamma(\alpha)} \left[t^{\alpha-3}(1-s)^{\alpha-3} - (t-s)^{\alpha-3} \right], & 0 \leq s \leq t \leq 1, \\ \dfrac{(\alpha-1)(\alpha-2)}{\Gamma(\alpha)} t^{\alpha-3}(1-s)^{\alpha-3}, & 0 \leq t \leq s \leq 1. \end{cases} \end{gathered} \tag{2.8}$$

Remark 2.6. In [9] it is proved that G is a continuous function on $[0,1] \times [0,1]$, $G(t,s) \geq 0$, $G(t,1) = 0$ and

$$\sup_{0 \leq t \leq 1} \int_0^1 G(t,s)ds = \frac{2}{(\alpha - 2)\Gamma(\alpha + 1)}, \qquad \int_0^1 H(\eta, s)ds = \frac{\eta^{\alpha-3}(\alpha - 1)(1 - \eta)}{\Gamma(\alpha)}. \tag{2.9}$$

In the sequel, we present the fixed point theorem which we will use later. Previously, we present the following class of functions.

By $\mathcal{S}$ we denote the class of functions $\beta : [0, \infty) \to [0, 1)$ satisfying the following condition:

$$\beta(t_n) \longrightarrow 1 \text{ implies } t_n \longrightarrow 0. \tag{2.10}$$

Examples of functions belonging to $\mathcal{S}$ are $\beta(t) = kt$ with $0 \leq k < 1$ and $\beta(t) = 1/(1 + t)$.

The fixed point theorem which we will use later appears in [10].

Theorem 2.7 (see [10]). *Let $(X, \leq)$ be a partially ordered set and suppose that there exists a metric d in X such that (X, d) is a complete metric space. Let $T : X \to X$ be a nondecreasing mapping such that there exists an element $x_0 \in X$ with $x_0 \leq Tx_0$. Suppose that there exists $\beta \in \mathcal{S}$ such that*

$$d(Tx, Ty) \leq \beta(d(x, y))d(x, y) \quad \text{for } x, y \in X \text{ with } x \geq y. \tag{2.11}$$

Assume that either T is continuous or X is such that

$$\text{if } (x_n) \text{ is a nondecreasing sequence in } X \text{ such that } x_n \longrightarrow x \text{ then } x_n \leq x \ \forall n \in \mathbb{N}. \tag{2.12}$$

Besides, if

$$\text{for all } x, y \in X \text{ there exists } z \in X \text{ which is comparable to } x,\ y, \tag{2.13}$$

then T has a unique fixed point.

In our considerations, we will work in the Banach space $C[0,1] = \{x : [0,1] \to \mathbb{R}, x \text{ is continuous}\}$ with the classical metric given by $d(x, y) = \sup_{0 \leq t \leq 1} |x(t) - y(t)|$.

Notice that this space can be equipped with a partial order given by

$$x, y \in C[0,1], \quad x \leq y \Longleftrightarrow x(t) \leq y(t) \quad \text{for any } t \in [0,1]. \tag{2.14}$$

In [11] it is proved that $(C[0,1], \leq)$ satisfies condition (2.12) of Theorem 2.7. Moreover, for $x, y \in C[0,1]$, as the function $\max\{x, y\} \in C[0,1]$, $(C[0,1], \leq)$ satisfies condition (2.13).

3. Main Result

Our starting point of this section is the following lemma.

Lemma 3.1. *Suppose that* $0 < \sigma < 1$, $3 < \alpha \le 4$, *and* $F : (0,1] \to \mathbb{R}$ *is a continuous function with* $\lim_{t \to 0^+} F(t) = \infty$. *If* $t^\sigma F(t)$ *is a continuous function on* $[0,1]$ *then the function defined by*

$$L(t) = \int_0^1 G(t,s) F(s) ds \tag{3.1}$$

is continuous on $[0,1]$, *where* $G(t,s)$ *is the Green's function appearing in Lemma 2.5.*

Proof. We divide the proof into three cases.

Case 1 ($t_0 = 0$). It is clear that $L(0) = 0$.

Since $t^\sigma F(t)$ is a continuous function on $[0,1]$, we can find a constant $M > 0$ such that $|t^\sigma F(t)| \le M$ for any $t \in [0,1]$.

Then, we get

$$\begin{aligned}
|L(t) - L(0)| = |L(t)| &= \left| \int_0^1 G(t,s) F(s) ds \right| = \left| \int_0^1 G(t,s) s^{-\sigma} s^{\sigma} F(s) ds \right| \\
&= \left| \int_0^t \frac{t^{\alpha-1}(1-s)^{\alpha-3} - (t-s)^{\alpha-1}}{\Gamma(\alpha)} s^{-\sigma} s^{\sigma} F(s) ds + \int_t^1 \frac{t^{\alpha-1}(1-s)^{\alpha-3}}{\Gamma(\alpha)} s^{-\sigma} s^{\sigma} F(s) ds \right| \\
&= \left| \int_0^1 \frac{t^{\alpha-1}(1-s)^{\alpha-3}}{\Gamma(\alpha)} s^{-\sigma} s^{\sigma} F(s) ds - \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} s^{-\sigma} s^{\sigma} F(s) ds \right| \\
&\le \left| \int_0^1 \frac{t^{\alpha-1}(1-s)^{\alpha-3}}{\Gamma(\alpha)} s^{-\sigma} s^{\sigma} F(s) ds \right| + \left| \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} s^{-\sigma} s^{\sigma} F(s) ds \right| \\
&\le \frac{M t^{\alpha-1}}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-3} s^{-\alpha} ds + \frac{M}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} s^{-\alpha} ds \\
&= \frac{M t^{\alpha-1}}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-3} s^{-\alpha} ds + \frac{M t^{\alpha-1}}{\Gamma(\alpha)} \int_0^t \left(1 - \frac{s}{t}\right)^{\alpha-1} s^{-\alpha} ds \\
&= \frac{M t^{\alpha-1}}{\Gamma(\alpha)} \beta(1-\sigma, \alpha-2) + \frac{M t^{\alpha-1}}{\Gamma(\alpha)} \int_0^t \left(1 - \frac{s}{t}\right)^{\alpha-1} s^{-\alpha} ds.
\end{aligned} \tag{3.2}$$

If in the integral $\int_0^t (1 - (s/t))^{\alpha-1} s^{-\sigma} ds$ we use the change of variables $u = s/t$ then we have

$$\int_0^t \left(1 - \frac{s}{t}\right)^{\alpha-1} s^{-\sigma} ds = t^{1-\sigma} \int_0^1 (1-u)^{\alpha-1} u^{-\sigma} du = t^{1-\sigma} \beta(1-\sigma, \alpha). \tag{3.3}$$

This and (3.2) give us

$$|L(t)| \le \frac{M t^{\alpha-1}}{\Gamma(\alpha)} \beta(1-\sigma, \alpha-2) + \frac{M t^{\alpha-\sigma}}{\Gamma(\alpha)} \beta(1-\sigma, \alpha) \tag{3.4}$$

and letting $t \to 0$, we see that $|L(t)| \to 0$.

This proves the continuity of L at $t_0 = 0$.

Case 2 ($t_0 \in (0,1)$). We take $t_n \to t_0$ and we have to prove that $L(t_n) \to L(t_0)$.

Without loss of generality, we can take $t_n > t_0$ (the same argument works for $t_n < t_0$).

In fact,

$$\begin{aligned}
|L(t_n) - L(t_0)| = \Bigg| & \int_0^{t_n} \frac{t_n^{\alpha-1}(1-s)^{\alpha-3} - (t_n-s)^{\alpha-1}}{\Gamma(\alpha)} s^{-\sigma} s^{\sigma} F(s) ds \\
& + \int_{t_n}^1 \frac{t_n^{\alpha-1}(1-s)^{\alpha-3}}{\Gamma(\alpha)} s^{-\sigma} s^{\sigma} F(s) ds \\
& - \int_0^{t_0} \frac{t_0^{\alpha-1}(1-s)^{\alpha-3} - (t_0-s)^{\alpha-1}}{\Gamma(\alpha)} s^{-\sigma} s^{\sigma} F(s) ds \\
& - \int_{t_0}^1 \frac{t_0^{\alpha-1}(1-s)^{\alpha-3}}{\Gamma(\alpha)} s^{-\sigma} s^{\sigma} F(s) ds \Bigg| \\
= \Bigg| & \int_0^1 \frac{t_n^{\alpha-1}(1-s)^{\alpha-3}}{\Gamma(\alpha)} s^{-\sigma} s^{\sigma} F(s) ds - \int_0^{t_n} \frac{(t_n-s)^{\alpha-1}}{\Gamma(\alpha)} s^{-\sigma} s^{\sigma} F(s) ds \\
& - \int_0^1 \frac{t_0^{\alpha-1}(1-s)^{\alpha-3}}{\Gamma(\alpha)} s^{-\sigma} s^{\sigma} F(s) ds + \int_0^{t_0} \frac{(t_0-s)^{\alpha-1}}{\Gamma(\alpha)} s^{-\sigma} s^{\sigma} F(s) ds \Bigg| \\
= \Bigg| & \int_0^1 \frac{\left(t_n^{\alpha-1} - t_0^{\alpha-1}\right)(1-s)^{\alpha-3}}{\Gamma(\alpha)} s^{-\sigma} s^{\sigma} F(s) ds \\
& - \int_0^{t_0} \frac{(t_n-s)^{\alpha-1} - (t_0-s)^{\alpha-1}}{\Gamma(\alpha)} s^{-\sigma} s^{\sigma} F(s) ds \\
& - \int_{t_0}^{t_n} \frac{(t_n-s)^{\alpha-1}}{\Gamma(\alpha)} s^{-\sigma} s^{\sigma} F(s) ds \Bigg| \\
\leq & \frac{M\left(t_n^{\alpha-1} - t_0^{\alpha-1}\right)}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-3} s^{-\sigma} ds \\
& + \frac{M}{\Gamma(\alpha)} \int_0^{t_0} \left((t_n-s)^{\alpha-1} - (t_0-s)^{\alpha-1}\right) s^{-\sigma} ds \\
& + \frac{M}{\Gamma(\alpha)} \int_{t_0}^{t_n} (t_n-s)^{\alpha-1} s^{-\sigma} ds \\
\leq & \frac{M\left(t_n^{\alpha-1} - t_0^{\alpha-1}\right)}{\Gamma(\alpha)} \beta(1-\sigma, \alpha-2) + \frac{M}{\Gamma(\alpha)} I_n^1 + \frac{M}{\Gamma(\alpha)} I_n^2,
\end{aligned} \tag{3.5}$$

where

$$I_n^1 = \int_0^{t_0} \left((t_n - s)^{\alpha-1} - (t_0 - s)^{\alpha-1} \right) s^{-\sigma} ds,$$
$$I_n^2 = \int_{t_0}^{t_n} (t_n - s)^{\alpha-1} s^{-\sigma} ds. \tag{3.6}$$

In the sequel, we will prove that $I_n^1 \to 0$ when $n \to \infty$.

In fact, as $t_n \to t_0$, then

$$\left((t_n - s)^{\alpha-1} - (t_0 - s)^{\alpha-1} \right) s^{-\sigma} \longrightarrow 0 \quad \text{when } n \longrightarrow \infty. \tag{3.7}$$

Moreover,

$$\left| \left((t_n - s)^{\alpha-1} - (t_0 - s)^{\alpha-1} \right) s^{-\sigma} \right| \le \left(|t_n - s|^{\alpha-1} + |t_0 - s|^{\alpha-1} \right) s^{-\sigma} \le 2s^{-\sigma} \tag{3.8}$$

and, as

$$\int_0^1 2s^{-\sigma} ds = 2\left[\frac{s^{-\sigma+1}}{-\sigma+1} \right]_0^1 = \frac{2}{1-\sigma} < \infty, \tag{3.9}$$

we have that the sequence $((t_n-s)^{\alpha-1} - (t_0-s)^{\alpha-1})s^{-\sigma}$ converges pointwise to the zero function and $|((t_n - s)^{\alpha-1} - (t_0 - s)^{\alpha-1})s^{-\sigma}|$ is bounded by a function belonging to $L^1[0,1]$, then by Lebesgue's dominated convergence theorem

$$I_n^1 \longrightarrow 0 \quad \text{when } n \longrightarrow \infty. \tag{3.10}$$

Now, we will prove that $I_n^2 \to 0$ when $n \to \infty$.

In fact, as

$$I_n^2 = \int_{t_0}^{t_n} (t_n - s)^{\alpha-1} s^{-\sigma} ds \le \int_{t_0}^{t_n} s^{-\sigma} ds = \frac{1}{1-\sigma}\left(t_n^{1-\sigma} - t_0^{1-\sigma} \right) \tag{3.11}$$

and letting $n \to \infty$ and, taking into account that $t_n \to t_0$, from the last expression we get

$$I_n^2 \longrightarrow 0 \quad \text{when } n \longrightarrow \infty. \tag{3.12}$$

Finally, from (3.5), (3.10), and (3.12) we get

$$|L(t_n) - L(t_0)| \longrightarrow 0 \quad \text{when } n \longrightarrow \infty. \tag{3.13}$$

This proves the continuity of L at t_0.

Case 3 ($t_0 = 1$). It is easily checked that $L(1) = 0$.

Now, following the same lines that in the proof of Case 1, we can demonstrate the continuity of L at $t_0 = 1$.

This finishes the proof. □

Lemma 3.2. *Suppose that* $0 < \sigma < 1$, $3 < \alpha \leq 4$, $0 < \beta\eta^{\alpha-3} < 1$, *and* $F : (0,1] \to \mathbb{R}$ *is a continuous function with* $\lim_{t\to 0^+} F(t) = \infty$.

If $t^\sigma F(t)$ *is a continuous function on* $[0,1]$ *then the function defined by*

$$N(t) = \frac{\beta t^{\alpha-1}}{(\alpha-1)(\alpha-2)(1-\beta\eta^{\alpha-3})}\int_0^1 H(\eta,s)F(s)ds \tag{3.14}$$

is continuous on $[0,1]$, *where* $H(t,s)$ *is the function appearing in Lemma 2.5.*

Proof. Since $t^\sigma F(t)$ is continuous on $[0,1]$, there exists a constant $M > 0$ such that $|t^\sigma F(t)| \leq M$ for any $t \in [0,1]$.

Taking into account that

$$|H(t,s)| \leq \frac{2(\alpha-1)(\alpha-2)}{\Gamma(\alpha)} \tag{3.15}$$

we have

$$\begin{aligned}\left|\int_0^1 H(\eta,s)F(s)ds\right| &= \left|\int_0^1 H(\eta,s)s^{-\sigma}s^{\sigma}F(s)ds\right| \\ &\leq \frac{2M(\alpha-1)(\alpha-2)}{\Gamma(\alpha)}\int_0^1 s^{-\sigma}\,ds = \frac{2M(\alpha-1)(\alpha-2)}{\Gamma(\alpha)(1-\alpha)} < \infty,\end{aligned} \tag{3.16}$$

and, consequently, the function N is continuous at any point $t \in [0,1]$. □

Remark 3.3. Notice that the function $H(t,s)$ appearing in Lemma 2.5 which is defined as

$$H(t,s) = \begin{cases} \dfrac{(\alpha-1)(\alpha-2)}{\Gamma(\alpha)}\left[t^{\alpha-3}(1-s)^{\alpha-3} - (t-s)^{\alpha-3}\right], & 0 \leq s \leq t \leq 1, \\ \dfrac{(\alpha-1)(\alpha-2)}{\Gamma(\alpha)}t^{\alpha-3}(1-s)^{\alpha-3}, & 0 \leq t \leq s \leq 1,\end{cases} \tag{3.17}$$

is continuous function on $[0,1]\times[0,1]$ and, moreover, $H(t,s) \geq 0$.

In fact, for $0 \leq t \leq s \leq 1$ it is clear that $H(t,s) \geq 0$.

In the case, $0 \leq s \leq t \leq 1$, we have

$$\begin{aligned} H(t,s) &= \frac{(\alpha-1)(\alpha-2)}{\Gamma(\alpha)}\left[t^{\alpha-3}(1-s)^{\alpha-3} - (t-s)^{\alpha-3}\right] \\ &= \frac{(\alpha-1)(\alpha-2)}{\Gamma(\alpha)}\left[(t-ts)^{\alpha-3} - (t-s)^{\alpha-3}\right] \geq 0.\end{aligned} \tag{3.18}$$

This proves the nonnegative character of the function H on $[0,1]\times[0,1]$.

Lemma 3.4. *Suppose that* $0 < \sigma < 1$. *Then*

$$\sup_{0\le t\le 1}\int_0^1 G(t,s)s^{-\sigma}ds = \frac{1}{\Gamma(\alpha)}\big(\beta(1-\sigma,\alpha-2)-\beta(1-\sigma,\alpha)\big), \tag{3.19}$$

where $G(t,s)$ *is the function appearing in Lemma 2.5.*

Proof. By definition of $G(t,s)$, we have

$$\begin{aligned}\int_0^1 G(t,s)s^{-\sigma}ds &= \int_0^t G(t,s)s^{-\sigma}ds + \int_t^1 G(t,s)s^{-\sigma}ds\\ &= \int_0^t \frac{t^{\alpha-1}(1-s)^{\alpha-3}-(t-s)^{\alpha-1}}{\Gamma(\alpha)}s^{-\sigma}ds + \int_t^1 \frac{t^{\alpha-1}(1-s)^{\alpha-3}}{\Gamma(\alpha)}s^{-\sigma}ds\\ &= \int_0^1 \frac{t^{\alpha-1}(1-s)^{\alpha-3}}{\Gamma(\alpha)}s^{-\sigma}ds - \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}s^{-\alpha}ds\\ &= \frac{t^{\alpha-1}}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-3}s^{-\sigma}ds - \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}s^{-\sigma}ds.\end{aligned} \tag{3.20}$$

As we saw in Case 1 of Lemma 3.1.

$$\int_0^t (t-s)^{\alpha-1}s^{-\sigma}ds = \frac{t^{\alpha-\sigma}}{\Gamma(\alpha)}\beta(1-\sigma,\alpha) \tag{3.21}$$

and, therefore,

$$\int_0^t G(t,s)s^{-\sigma}ds = \frac{t^{\alpha-1}}{\Gamma(\alpha)}\beta(1-\sigma,\alpha-2) - \frac{t^{\alpha-\sigma}}{\Gamma(\alpha)}\beta(1-\sigma,\alpha). \tag{3.22}$$

Now, using elemental calculus it is easily seen that the function

$$\varphi(t) = \frac{\beta(1-\sigma,\alpha-2)}{\Gamma(\alpha)}t^{\alpha-1} - \frac{\beta(1-\sigma,\alpha)}{\Gamma(\alpha)}t^{\alpha-\sigma} \tag{3.23}$$

is increasing on the interval $[0,1]$ and, therefore,

$$\sup_{0\le t\le 1}\int_0^1 G(t,s)s^{-\sigma}ds = \sup_{0\le t\le 1}\varphi(t) = \varphi(1) = \frac{1}{\Gamma(\alpha)}\big(\beta(1-\sigma,\alpha-2)-\beta(1-\sigma,\alpha)\big). \tag{3.24}$$

□

Lemma 3.5. *Suppose that* $0 < \sigma < 1$ *then*

$$\int_0^1 H(\eta,s)s^{-\sigma}ds = \frac{(\alpha-1)(\alpha-2)}{\Gamma(\alpha)}\left(\eta^{\alpha-3}-\eta^{\alpha-\sigma-2}\right)\beta(1-\sigma,\alpha-2), \tag{3.25}$$

where $H(t,s)$ is the function appearing in Lemma 2.5.

Proof. By definition of $H(t,s)$, we have

$$\begin{aligned}
\int_0^1 H(\eta,s)s^{-\sigma}ds &= \int_0^\eta H(\eta,s)s^{-\sigma}ds + \int_\eta^1 H(\eta,s)s^{-\sigma}ds \\
&= \int_0^\eta \frac{(\alpha-1)(\alpha-2)}{\Gamma(\alpha)}\left[\eta^{\alpha-3}(1-s)^{\alpha-3} - (\eta-s)^{\alpha-3}\right]s^{-\sigma}ds \\
&\quad + \int_\eta^1 \frac{(\alpha-1)(\alpha-2)}{\Gamma(\alpha)}\eta^{\alpha-3}(1-s)^{\alpha-3}s^{-\sigma}ds \\
&= \int_0^1 \frac{(\alpha-1)(\alpha-2)}{\Gamma(\alpha)}\eta^{\alpha-3}(1-s)^{\alpha-3}s^{-\sigma}ds - \int_0^\eta \frac{(\alpha-1)(\alpha-2)}{\Gamma(\alpha)}(\eta-s)^{\alpha-3}s^{-\sigma}ds \\
&= \frac{(\alpha-1)(\alpha-2)}{\Gamma(\alpha)}\eta^{\alpha-3}\int_0^1 (1-s)^{\alpha-3}s^{-\sigma}ds - \frac{(\alpha-1)(\alpha-2)}{\Gamma(\alpha)}\int_0^\eta (\eta-s)^{\alpha-3}s^{-\sigma}ds \\
&= \frac{(\alpha-1)(\alpha-2)}{\Gamma(\alpha)}\eta^{\alpha-3}\beta(1-\sigma,\alpha-2) - \frac{(\alpha-1)(\alpha-2)}{\Gamma(\alpha)}\int_0^\eta (\eta-s)^{\alpha-3}s^{-\sigma}ds.
\end{aligned} \tag{3.26}$$

By a similar argument that the one used in the Case 1 of Lemma 3.1, we have

$$\begin{aligned}
\int_0^1 H(\eta,s)s^{-\sigma}ds &= \frac{(\alpha-1)(\alpha-2)}{\Gamma(\alpha)}\eta^{\alpha-3}\beta(1-\sigma,\alpha-2) - \frac{(\alpha-1)(\alpha-2)}{\Gamma(\alpha)}\eta^{\alpha-\sigma-2}\beta(1-\sigma,\alpha-2) \\
&= \frac{(\alpha-1)(\alpha-2)}{\Gamma(\alpha)}\left(\eta^{\alpha-3} - \eta^{\alpha-\sigma-2}\right)\beta(1-\sigma,\alpha-2).
\end{aligned} \tag{3.27}$$

This finishes the proof. □

By commodity, we denote by K the constant given by

$$K = \frac{1}{\Gamma(\alpha)}\left[\left(1 + \frac{\beta(\eta^{\alpha-3} - \eta^{\alpha-\sigma-2})}{1-\beta\eta^{\alpha-3}}\right)\beta(1-\sigma,\alpha-2) - \beta(1-\sigma,\alpha)\right]. \tag{3.28}$$

Moreover, we introduce the following class of functions which will be used in the main result of the paper. By $\mathcal{A}$ we denote the class of functions $\phi : [0,\infty) \to [0,\infty)$ satisfying the following:

(i) ϕ is nondecreasing.

(ii) $\phi(x) < x$ for any $x > 0$.

(iii) $\beta(x) = \phi(x)/x \in \mathcal{S}$, where $\mathcal{S}$ is the class of functions introduced in Remark 2.6.

Theorem 3.6. *Let $0 < \sigma < 1$, $3 < \alpha \leq 4$, $0 < \eta < 1$, $0 < \beta\eta^{\alpha-3} < 1$, and $f : (0,1] \times [0,\infty) \to [0,\infty)$ is a continuous function with $\lim_{t\to 0^+} f(t,\cdot) = \infty$ and such that $t^\sigma f(t,y)$ is a continuous function on $[0,1] \times [0,\infty)$. Assume that there exists $0 < \lambda \leq 1/K$ such that for $x, y \in [0,\infty)$ with $y \geq x$ and $t \in [0,1]$*

$$0 \leq t^\sigma \left(f(t,y) - f(t,x)\right) \leq \lambda\phi(y-x), \tag{3.29}$$

where $\phi \in \mathcal{A}$.

Then Problem (1.3) *has a unique positive solution (this means that $x(t) > 0$ for $t \in (0,1)$).*

Proof. Consider the cone:

$$P = \{u \in C[0,1] : u(t) \geq 0\}. \tag{3.30}$$

Since P is a closed set of $C[0,1]$, P is a complete metric space with the distance given by $d(u,v) = \sup_{0\leq t\leq 1}|u(t) - v(t)|$, for $u, v \in P$.

It is easily checked that P satisfies conditions (2.12) and (2.13) of Theorem 2.7.

Now, for $u \in P$ we define the operator T by

$$(Tu)(t) = \int_0^1 G(t,s)f(s,u(s))ds + \frac{\beta t^{\alpha-1}}{(\alpha-1)(\alpha-2)\left(1-\beta\eta^{\alpha-3}\right)}\int_0^1 H(\eta,s)f(s,u(s))ds. \tag{3.31}$$

By Lemmas 3.1 and 3.2, for $u \in P$ we have $Tu \in C[0,1]$.

Moreover, in view of the nonnegative character of $G(t,s)$, $H(\eta,s)$, and $f(s,x)$, we have that $Tu \in P$ for $u \in P$.

In what follows, we check that assumptions in Theorem 2.7 are satisfied.

Firstly, we will prove that T is nondecreasing.

In fact, by (3.29), for $u \geq v$ we have

$$\begin{aligned}(Tu)(t) &= \int_0^1 G(t,s) f(s,u(s))ds + \frac{\beta t^{\alpha-1}}{(\alpha-1)(\alpha-2)(1-\beta\eta^{\alpha-3})}\int_0^1 H(\eta,s) f(s,u(s))ds \\ &= \int_0^1 G(t,s)s^{-\sigma}s^{\sigma} f(s,u(s))ds \\ &\quad + \frac{\beta t^{\alpha-1}}{(\alpha-1)(\alpha-2)(1-\beta\eta^{\alpha-3})}\int_0^1 H(\eta,s)s^{-\sigma}s^{\sigma} f(s,u(s))ds \\ &\geq \int_0^1 G(t,s)s^{-\sigma}s^{\sigma} f(s,v(s))ds \\ &\quad + \frac{\beta t^{\alpha-1}}{(\alpha-1)(\alpha-2)(1-\beta\eta^{\alpha-3})}\int_0^1 H(\eta,s)s^{-\sigma}s^{\sigma} f(s,v(s))ds \\ &= \int_0^1 G(t,s) f(s,v(s))ds \\ &\quad + \frac{\beta t^{\alpha-1}}{(\alpha-1)(\alpha-2)(1-\beta\eta^{\alpha-3})}\int_0^1 H(\eta,s) f(s,v(s))ds = (Tv)(t).\end{aligned} \tag{3.32}$$

This proves that T is a nondecreasing operator.

On the other hand, for $u \geq v$ and $u \neq v$, we have

$$\begin{aligned}d(Tu,Tv) &= \sup_{0\leq t\leq 1}|(Tu)(t)-(Tv)(t)| = \sup_{0\leq t\leq 1}((Tu)(t)-(Tv)(t)) \\ &= \sup_{0\leq t\leq 1}\left[\int_0^1 G(t,s)(f(s,u(s))-f(s,v(s)))ds\right. \\ &\qquad \left. + \frac{\beta t^{\alpha-1}}{(\alpha-1)(\alpha-2)(1-\beta\eta^{\alpha-3})}\int_0^1 H(\eta,s)(f(s,u(s))-f(s,v(s)))ds\right] \\ &\leq \sup_{0\leq t\leq 1}\int_0^1 G(t,s)s^{-\sigma}s^{\sigma}(f(s,u(s))-f(s,v(s)))ds \\ &\quad + \frac{\beta}{(\alpha-1)(\alpha-2)(1-\beta\eta^{\alpha-3})}\int_0^1 H(\eta,s)s^{-\sigma}s^{\sigma}(f(s,u(s))-f(s,v(s)))ds \\ &\leq \sup_{0\leq t\leq 1}\int_0^1 G(t,s)s^{-\sigma}\lambda(\phi(u(s)-v(s)))ds \\ &\quad + \frac{\beta}{(\alpha-1)(\alpha-2)(1-\beta\eta^{\alpha-3})}\int_0^1 H(\eta,s)s^{-\sigma}\lambda\,(\phi(u(s)-v(s)))ds.\end{aligned} \tag{3.33}$$

Since ϕ is nondecreasing, the last inequality implies

$$\begin{aligned} d(Tu,Tv) &\leq \lambda\phi(d(u,v))\sup_{0\leq t\leq 0}\int_0^1 G(t,s)s^{-\sigma}ds \\ &\quad+\frac{\beta}{(\alpha-1)(\alpha-2)(1-\beta\eta^{\alpha-3})}\lambda\phi(d(u,v))\int_0^1 H(\eta,s)s^{-\sigma}ds \\ &= \lambda\phi(d(u,v))\left[\sup_{0\leq t\leq 0}\int_0^1 G(t,s)s^{-\sigma}\,ds+\frac{\beta}{(\alpha-1)(\alpha-2)(1-\beta\eta^{\alpha-3})}\int_0^1 H(\eta,s)s^{-\sigma}ds\right]. \end{aligned} \tag{3.34}$$

Now, from Lemmas 3.4 and 3.5 it follows:

$$\begin{aligned} d(Tu,Tv) &\leq \lambda\phi(d(u,v))\left[\frac{1}{\Gamma(\alpha)}(\beta(1-\sigma,\alpha-2)-\beta(1-\sigma,\alpha))+\frac{\beta}{(\alpha-1)(\alpha-2)(1-\beta\eta^{\alpha-3})}\right. \\ &\qquad\left.\times\frac{(\alpha-1)(\alpha-2)}{\Gamma(\alpha)}\left(\eta^{\alpha-3}-\eta^{\alpha-\sigma-2}\right)\beta(1-\sigma,\alpha-2)\right] \\ &= \lambda\phi(d(u,v))\left[\frac{1}{\Gamma(\alpha)}(\beta(1-\sigma,\alpha-2)-\beta(1-\sigma,\alpha))\right. \\ &\qquad\left.+\frac{\beta(\eta^{\alpha-3}-\eta^{\alpha-\sigma-2})}{(1-\beta\eta^{\alpha-3})\Gamma(\alpha)}\beta(1-\sigma,\alpha-2)\right] \\ &= \lambda\phi(d(u,v))\left[\frac{1}{\Gamma(\alpha)}\left[\left(1+\frac{\beta(\eta^{\alpha-3}-\eta^{\alpha-\sigma-2})}{1-\beta\eta^{\alpha-3}}\right)\beta(1-\sigma,\alpha-2)-\beta(1-\sigma,\alpha)\right]\right] \\ &= \lambda\phi(d(u,v))K. \end{aligned} \tag{3.35}$$

Since $0<\lambda\leq 1/K$, from the last inequality we obtain

$$d(Tu,Tv)\leq\lambda\phi(d(u,v))K\leq\phi(d(u,v)), \tag{3.36}$$

and, since $u\neq v$,

$$d(Tu,Tv)\leq\frac{\phi(d(u,v))}{d(u,v)}d(u,v)=\beta(d(u,v))d(u,v). \tag{3.37}$$

Since this inequality is obviously satisfied for $u=v$, we have

$$d(Tu,Tv)\leq\beta(d(u,v))d(u,v)\quad\text{for any } u,v\in P \text{ with } u\geq v. \tag{3.38}$$

Finally, since the zero function satisfies $0 \leq T0$, Theorem 2.7 says us that the operator T has a unique fixed point in P, or, equivalently, Problem (1.3) has a unique nonnegative solution x in $C[0,1]$.

Now, we will prove that x is a positive solution.

In contrary case, we can find $0 < t^* < 1$ such that $x(t^*) = 0$.

Taking into account that the nonnegative solution x of Problem (1.3) is a fixed point of the operator, we have

$$x(t) = \int_0^1 G(t,s) f(s,x(s)) ds + \frac{\beta t^{\alpha-1}}{(\alpha-1)(\alpha-2)(1-\beta\eta^{\alpha-3})} \int_0^1 H(\eta,s) f(s,x(s)) ds \tag{3.39}$$

and, particularly,

$$x(t^*) = \int_0^1 G(t^*,s) f(s,x(s)) ds + \frac{\beta t^{*\alpha-1}}{(\alpha-1)(\alpha-2)(1-\beta\eta^{\alpha-3})} \int_0^1 H(\eta,s) f(s,x(s)) ds = 0. \tag{3.40}$$

Since both summands in the right hand are nonnegative (see Remarks 2.6 and 3.3) we have

$$\begin{aligned} &\int_0^1 G(t^*,s) f(s,x(s)) ds = 0, \\ &\int_0^1 H(\eta,s) f(s,x(s)) ds = 0. \end{aligned} \tag{3.41}$$

Given the nonnegative character of $G(t,s)$, $H(\eta,s)$, and $f(s,u)$, we have

$$\begin{aligned} &G(t^*,s) f(s,x(s)) = 0 \quad \text{a.e. } (s), \\ &H(\eta,s) f(s,x(s)) = 0 \quad \text{a.e. } (s). \end{aligned} \tag{3.42}$$

Taking into account that $\lim_{t\to 0^+} f(t,0) = \infty$, this means that for $M > 0$ we can find $\delta > 0$ such that for $s \in [0,1] \cap (0,\delta)$ we have $f(s,0) > M$. Notice that $[0,1] \cap (0,\delta) \subset \{s \in [0,1] : f(s,x(s)) > M\}$ and $\mu([0,1] \cap (0,\delta)) > 0$, where μ is the Lebesgue measure on $[0,1]$.

This and (3.42) give us that

$$\begin{aligned} &G(t^*,s) = 0 \quad \text{a.e. } (s), \\ &H(\eta,s) = 0 \quad \text{a.e. } (s), \end{aligned} \tag{3.43}$$

and this is a contradiction since $G(t^*,s)$ and $H(\eta,s)$ are rational functions in the variable s.

Therefore, $x(t) > 0$ for $t \in (0,1)$.

This finishes the proof. □

In order to present an example which illustrates our results, we need to prove some properties about the hyperbolic tangent function.

Previously, we recalled some definitions.

Definition 3.7. A function $f : [0,\infty) \to [0,\infty)$ is said to be subadditive if it satisfies

$$f(x+y) \leq f(x) + f(y) \quad \text{for any } x, y \in [0,\infty). \tag{3.44}$$

An example of subadditive function is the square root function, that is, $f(x) = \sqrt{x}$.

Remark 3.8. Suppose that $f : [0,\infty) \to [0,\infty)$ is subadditive and $y \leq x$ then

$$f(x) - f(y) \leq f(x-y). \tag{3.45}$$

In fact, since

$$f(x) = f(x-y+y) \leq f(x-y) + f(y) \tag{3.46}$$

this inequality implies that

$$f(x) - f(y) \leq f(x-y). \tag{3.47}$$

Recall that a function $f : [0,\infty) \to [0,\infty)$ is concave if for any $x, y \in [0,\infty)$ and $\lambda \in [0,1]$.

$$f(\lambda x + (1-\lambda)y) \geq \lambda f(x) + (1-\lambda) f(y). \tag{3.48}$$

Lemma 3.9. *Let $f : [0,\infty) \to [0,\infty)$ be a concave function with $f(0) = 0$. Then f is subadditive.*

Proof. We take $x, y \in [0,\infty)$.

Since f is concave and $f(0) = 0$, we get

$$\begin{gathered} f(x) = f\left(\frac{y}{x+y}\, 0 + \frac{x}{x+y}(x+y)\right) \geq \frac{y}{x+y} f(0) + \frac{x}{x+y} f(x+y) = \frac{x}{x+y} f(x+y), \\ f(y) = f\left(\frac{x}{x+y}\, 0 + \frac{y}{x+y}(x+y)\right) \geq \frac{x}{x+y} f(0) + \frac{y}{x+y} f(x+y) = \frac{y}{x+y} f(x+y). \end{gathered} \tag{3.49}$$

Adding these inequalities, we have

$$f(x) + f(y) \geq \frac{x}{x+y} f(x+y) + \frac{y}{x+y} f(x+y) = f(x+y). \tag{3.50}$$

This proves the lemma. □

In what follows, we will prove that the function

$$f(x) = \tanh x = \frac{e^{2x}-1}{e^{2x}+1} \tag{3.51}$$

belongs to the class $\mathcal{A}$ previously defined.

Lemma 3.10. *The function* $f : [0,\infty) \to [0,\infty)$ *defined as*

$$f(x) = \tanh x = \frac{e^{2x} - 1}{e^{2x} + 1} \tag{3.52}$$

satisfies:

(a) $f \in \mathcal{A}$.

(b) f *is subadditive.*

Proof. (a) Since $f'(x) = 4e^{2x}/(e^{2x}+1)^2 > 0$ for $x > 0$, f is nondecreasing.

Moreover, the function

$$g(x) = x - \tanh x = x - \frac{e^{2x} - 1}{e^{2x} + 1} \tag{3.53}$$

has as derivative

$$g'(x) = \frac{\left(e^{2x} - 1\right)^2}{\left(e^{2x} + 1\right)^2} > 0 \quad \text{for } x > 0 \tag{3.54}$$

and, consequently, g is strictly nondecreasing on $(0,\infty)$.

Since $g(0) = 0$, we have $0 = g(0) < g(x)$ for $x > 0$ or, equivalently, $f(x) = \tanh x < x$ for $x > 0$.

In order to prove that $\beta(x) = \tanh x / x \in \mathcal{S}$, notice that if $\beta(t_n) \to 1$ then the sequence (t_n) is a bounded sequence.

In fact, in contrary case $t_n \to \infty$ and we have

$$\beta(t_n) = \frac{\tanh t_n}{t_n} \longrightarrow 0 \tag{3.55}$$

which contradicts the fact that $\beta(t_n) \to 1$.

Now, we suppose that $\beta(t_n) \to 1$ and $t_n \nrightarrow 0$.

Then, we can find $\varepsilon > 0$ such that for each $n \in \mathbb{N}$ there exists $\varrho_n \geq n$ with $t_{\varrho_n} \geq \varepsilon$.

Since (t_n) is a bounded sequence (because $\beta(t_n) \to 1$) we can find a subsequence of (t_{ϱ_n}), which we will denote of the same way, such that $t_{\varrho_n} \to a$.

As $\beta(t_n) \to 1$, it follows that

$$\beta(t_{\varrho_n}) = \frac{\tanh t_{\varrho_n}}{t_{\varrho_n}} \longrightarrow \frac{\tanh a}{a} = 1 \tag{3.56}$$

and, as the unique solution of the equation $\tanh x = x$ on $[0,\infty)$ is $x_0 = 0$, we deduce that $a = 0$.

Therefore, $t_{\varrho_n} \to 0$ and this implies that there exists $n_0 \in \mathbb{N}$ such that $t_{\varrho_n} < \varepsilon$ for $n \geq n_0$.

This contradicts the fact that $t_{\varrho_n} \geq \varepsilon$ for any $n \in \mathbb{N}$.

Therefore, $t_n \to 0$.

This proves that $\beta(x) = \tanh x/x \in \mathcal{S}$.
Therefore, $f \in \mathcal{A}$.
(b) Since $\tanh 0 = 0$ and

$$(\tanh x)'' = \frac{8e^{2x}(1-e^{2x})}{(e^{2x}+1)^3} < 0 \quad \text{for } x > 0, \tag{3.57}$$

this means that $f(x) = \tanh x$ is a concave function with $\tanh 0 = 0$ and, by Lemma 3.9, $f(x) = \tanh x$ is subadditive. □

Remark 3.11. By Remark 3.8 and by (b) of Lemma 3.9, for $x, y \in [0, \infty)$ with $y \le x$

$$\tanh x - \tanh y \le \tanh(x-y). \tag{3.58}$$

Now, we present an example which illustrates our result.

Example 3.12. Consider the following singular fractional boundary value problem

$$\begin{gathered} D_{0^+}^{7/2}u(t) + \frac{\lambda(t^2+1)\tanh u(t)}{t^{1/2}} = 0, \quad 0 < t < 1, \\ u(0) = u'(0) = u''(0) = 0, \quad u''(1) = u''\left(\frac{1}{4}\right). \end{gathered} \tag{3.59}$$

In this case, $\sigma = 1/2$, $\eta = 1/4$, $\beta = 1$ and $\alpha = 7/2$.
Moreover, in this case $f(t,u) = \lambda(t^2+1)\tanh u/t^{1/2}$ for $(t,u) \in (0,1] \times [0,\infty)$.
Notice that f is continuous in $(0,1] \times [0,\infty)$ and $\lim_{t\to 0^+} f(t,\cdot) = \infty$.
Now, we check that $f(t,u)$ satisfies assumptions appearing in Theorem 3.6.
It is clear that $t^{1/2}f(t,u) = \lambda(t^2+1)\tanh u$ is a continuous function on $[0,1] \times [0,\infty)$.
Moreover, by Lemma 3.10 and Remark 3.11, for $u \ge v$ and $t \in [0,1]$ we have

$$\begin{aligned} 0 &\le t^{1/2}(f(t,u) - f(t,v)) \\ &= \lambda\left(t^2+1\right)(\tanh u - \tanh v) \\ &\le \lambda\left(t^2+1\right)\tanh(u-v) \le 2\lambda\tanh(u-v), \end{aligned} \tag{3.60}$$

where $f(x) = \tanh x$ is a function belonging to $\mathcal{A}$ (see, Lemma 3.10).
Finally, Theorem 3.6 says us that Problem (3.59) has a unique positive solution for

$$2\lambda \le \frac{1}{K} = \frac{\Gamma(7/2)}{\left(1 + \left((1/4)^{1/2} - 1/4\right)/\left(1-(1/4)^{1/2}\right)\right)\beta(1/2,3/2) \; - \beta(1/2,7/2)} = \frac{30}{7\sqrt{\pi}}. \tag{3.61}$$

Or, equivalently, for $\lambda \le 15/7\sqrt{\pi}$.

Acknowledgment

This research was partially supported by "Universidad de Las Palmas de Gran Canaria", Project ULPGC 2010–006.

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