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Research Article

Quasimonotone and Almost Increasing Sequences and Their New Applications

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Recently, we have proved a main theorem dealing with the absolute Nörlund summability factors of infinite series by using δ-quasimonotone sequences. In this paper, we prove that result under weaker conditions. A new result has also been obtained.

1. Introduction

A positive sequence (b_n) is said to be almost increasing if there exist a positive increasing sequence (c_n) and two positive constants A and B such that $Ac_n \leq b_n \leq Bc_n$ (see [1]). Obviously, every increasing sequence is almost increasing. However, the converse need not be true as can be seen by taking an example, say $b_n = ne^{(-1)^n}$. A sequence (d_n) is said to be δ-quasimonotone if $d_n > 0$ ultimately and $\Delta d_n = d_n - d_{n+1} \geq -\delta_n$, where $\delta = (\delta_n)$ is a sequence of positive numbers (see [2]). Let $\sum a_n$ be a given infinite series with the sequence of partial sums (s_n) and $w_n = na_n$. By u_n^α and t_n^α, we denote the nth Cesàro means of order α, with $\alpha > -1$, of the sequences (s_n) and (na_n), respectively, that is,

$$u_n^\alpha = \frac{1}{A_n^\alpha} \sum_{v=0}^{n} A_{n-v}^{\alpha-1} s_v, \tag{1.1}$$

$$t_n^\alpha = \frac{1}{A_n^\alpha} \sum_{v=1}^{n} A_{n-v}^{\alpha-1} v a_v, \tag{1.2}$$

where

$$A_n^\alpha = \binom{n+\alpha}{n} = \frac{(\alpha+1)(\alpha+2)\cdots(\alpha+n)}{n!} = O(n^\alpha), \qquad A_{-n}^\alpha = 0 \quad \text{for } n > 0. \tag{1.3}$$

The series $\sum a_n$ is said to be summable $|C,\alpha|_k$, $k \geq 1$, if (see [3])

$$\sum_{n=1}^{\infty} n^{k-1} \left| u_n^{\alpha} - u_{n-1}^{\alpha} \right|^k = \sum_{n=1}^{\infty} \frac{1}{n} |t_n^{\alpha}|^k < \infty. \tag{1.4}$$

If we take $\alpha = 1$, then $|C,\alpha|_k$ summability reduces to $|C,1|_k$ summability.

Let (p_n) be a sequence of constants, real or complex, and let us write

$$P_n = p_0 + p_1 + p_2 + \cdots + p_n \neq 0, \quad (n \geq 0). \tag{1.5}$$

The sequence-to-sequence transformation

$$V_n = \frac{1}{P_n} \sum_{v=0}^{n} p_{n-v} s_v \tag{1.6}$$

defines the sequence (V_n) of the Nörlund mean of the sequence (s_n), generated by the sequence of coefficients (p_n). The series $\sum a_n$ is said to be summable $|N,p_n|_k$, $k \geq 1$, if (see [4])

$$\sum_{n=1}^{\infty} n^{k-1} |V_n - V_{n-1}|^k < \infty. \tag{1.7}$$

In the special case when

$$p_n = \frac{\Gamma(n+\alpha)}{\Gamma(\alpha)\Gamma(n+1)}, \quad \alpha \geq 0, \tag{1.8}$$

the Nörlund mean reduces to the (C,α) mean and $|N,p_n|_k$ summability becomes $|C,\alpha|_k$ summability. For $p_n = 1$, we get the $(C,1)$ mean and then $|N,p_n|_k$ summability becomes $|C,1|_k$ summability. Also, if we take $k = 1$, then we get $|N,p_n|$ summability. For any sequence (λ_n), we write $\Delta\lambda_n = \lambda_n - \lambda_{n+1}$. Quite recently, in [5], we have proved the following theorem dealing with the absolute Nörlund summability factors of infinite series.

Theorem A. *Let $p_0 > 0$, $p_n \geq 0$, and (p_n) be a nonincreasing sequence. Let (X_n) be an almost increasing sequence such that $|\Delta X_n| = O(X_n/n)$ and (λ_n) is a sequence such that*

$$|\lambda_n| X_n = O(1) \quad \text{as } n \longrightarrow \infty. \tag{1.9}$$

Suppose also that there exists a sequence of numbers (A_n) such that it is δ-quasimonotone with $\sum n\delta_n X_n < \infty$, $\sum A_n X_n$ is convergent, and $|\Delta\lambda_n| \leq |A_n|$ for all n. If the sequence (w_n^{α}) defined by (see [6])

$$w_n^{\alpha} = \begin{cases} |t_n^{\alpha}|, & \alpha = 1, \\ \max\limits_{1 \leq v \leq n} |t_v^{\alpha}|, & 0 < \alpha < 1, \end{cases} \tag{1.10}$$

satisfies the condition

$$\sum_{n=1}^{m} \frac{(w_n^{\alpha})^k}{n} = O(X_m) \quad \text{as } m \longrightarrow \infty, \tag{1.11}$$

then the series $\sum a_n P_n \lambda_n (n+1)^{-1}$ *is summable* $|N, p_n|_k$, $k \geq 1$.

2. The Main Results

The aim of this paper is to prove Theorem A under weaker conditions. We will prove the following theorems.

Theorem 2.1. *If the sequences* (X_n), (A_n), *and* (λ_n) *are as in Theorem A and if conditions* (1.9) *and*

$$\sum_{n=1}^{m} \frac{(w_n^{\alpha})^k}{n X_n^{k-1}} = O(X_m) \quad \text{as } m \longrightarrow \infty \tag{2.1}$$

are satisfied, then the series $\sum a_n \lambda_n$ *is summable* $|C, \alpha|_k$, $0 < \alpha \leq 1$ *and* $k \geq 1$.

Theorem 2.2. *Let* (p_n) *be as in Theorem A. If the sequences* (X_n), (A_n), *and* (λ_n) *are as in Theorem A and if conditions* (1.9) *and* (2.1) *are satisfied, then the series* $\sum a_n P_n \lambda_n (n+1)^{-1}$ *is summable* $|N, p_n|_k$, $k \geq 1$.

Remark 2.3. The following sequences satisfy the conditions of the theorems:

$$\delta_n = \frac{1}{n^3}, \qquad A_n = \frac{1}{n^2}, \qquad \lambda_n = \frac{1}{n}, \qquad X_n = n^{\epsilon}, \quad 0 < \epsilon < 1. \tag{2.2}$$

Remark 2.4. It should be noted that condition (2.1) is the same as condition (1.11) when $k = 1$. When $k > 1$, condition (2.1) is weaker than condition (1.11), but the converse is not true. In fact, if (1.11) is satisfied, then we get that

$$\sum_{n=1}^{m} \frac{(w_n^{\alpha})^k}{n \ X_n^{k-1}} = O\left(\frac{1}{X_1^{k-1}}\right) \sum_{n=1}^{m} \frac{(w_n^{\alpha})^k}{n} = O(X_m). \tag{2.3}$$

To show that the converse is false when $k > 1$, the following example is sufficient. We can take $X_n = n^{\epsilon}$, $0 < \epsilon < 1$, and then construct a sequence (a_n) such that

$$\frac{(w_n^{\alpha})^k}{n {X_n}^{k-1}} = X_n - X_{n-1}, \tag{2.4}$$

whence

$$\sum_{n=1}^{m} \frac{(w_n^{\alpha})^k}{n {X_n}^{k-1}} = X_m = m^{\epsilon}, \tag{2.5}$$

and so

$$\begin{aligned}
\sum_{n=1}^{m}\frac{(w_n^{\alpha})^k}{n} &= \sum_{n=1}^{m}(X_n - X_{n-1})X_n^{k-1} = \sum_{n=1}^{m}\left(n^{\epsilon} - (n-1)^{\epsilon}\right)n^{\epsilon(k-1)} \\
&\geq \epsilon\sum_{n=1}^{m}(n-1)^{\epsilon-1}n^{\epsilon(k-1)} \\
&= \epsilon\sum_{n=1}^{m}(n-1)^{\epsilon k-1} \sim \frac{m^{\epsilon k}}{k} \quad \text{as } m \longrightarrow \infty.
\end{aligned} \tag{2.6}$$

This is because $v^{\epsilon-1} \geq n^{\epsilon-1}$ for $n-1 \leq v \leq n$.

This shows that, when $k > 1$, (1.11) implies (2.1) but not conversely.

We need the following lemmas for the proof of our theorem.

Lemma 2.5 (see [7]). *If $0 < \alpha \leq 1$ and $1 \leq v \leq n$, then*

$$\left|\sum_{p=0}^{v}A_{n-p}^{\alpha-1}a_p\right| \leq \max_{1\leq m\leq v}\left|\sum_{p=0}^{m}A_{m-p}^{\alpha-1}a_p\right|. \tag{2.7}$$

Lemma 2.6 (see [8]). *If $-1 < \alpha \leq \beta$, $k > 1$ and the series $\sum a_n$ is summable $|C,\alpha|_k$, then it is also summable $|C,\beta|_k$.*

Lemma 2.7 (see [9]). *Let (X_n) be an almost increasing sequence such that $n|\Delta X_n| = O(X_n)$.*

If (A_n) is a δ-quasimonotone with $\sum n\delta_n X_n < \infty$, $\sum A_n X_n$ is convergent, then

$$\begin{gathered}
nA_nX_n = O(1) \quad \text{as } n \longrightarrow \infty, \\
\sum_{n=1}^{\infty}nX_n|\Delta A_n| < \infty.
\end{gathered} \tag{2.8}$$

Lemma 2.8 (see [10]). *Let $p_0 > 0$, $p_n \geq 0$, and (p_n) be a nonincreasing sequence. If the series $\sum a_n$ is summable $|C,1|_k$, then the series $\sum a_n P_n(n+1)^{-1}$ is summable $|N,p_n|_k$, $k \geq 1$.*

3. Proof of Theorem 2.1

Let (T_n^{α}) be the nth (C,α), with $0 < \alpha \leq 1$, mean of the sequence $(na_n\lambda_n)$. Then, by (1.2), we have

$$T_n^{\alpha} = \frac{1}{A_n^{\alpha}}\sum_{v=1}^{n}A_{n-v}^{\alpha-1}va_v\lambda_v. \tag{3.1}$$

First applying Abel's transformation and then using Lemma 2.5, we have that

$$T_n^{\alpha} = \frac{1}{A_n^{\alpha}} \sum_{v=1}^{n-1} \Delta\lambda_v \sum_{p=1}^{v} A_{n-p}^{\alpha-1} p a_p + \frac{\lambda_n}{A_n^{\alpha}} \sum_{v=1}^{n} A_{n-v}^{\alpha-1} v a_v, \tag{3.2}$$

$$\begin{aligned} |T_n^{\alpha}| &\leq \frac{1}{A_n^{\alpha}} \sum_{v=1}^{n-1} |\Delta\lambda_v| \left| \sum_{p=1}^{v} A_{n-p}^{\alpha-1} p a_p \right| + \frac{|\lambda_n|}{A_n^{\alpha}} \left| \sum_{v=1}^{n} A_{n-v}^{\alpha-1} v a_v \right| \\ &\leq \frac{1}{A_n^{\alpha}} \sum_{v=1}^{n-1} A_v^{\alpha} w_v^{\alpha} |\Delta\lambda_v| + |\lambda_n| w_n^{\alpha} \\ &= T_{n,1}^{\alpha} + T_{n,2}^{\alpha}. \end{aligned} \tag{3.3}$$

To complete the proof of Theorem 2.1, by Minkowski's inequality, it is sufficient to show that

$$\sum_{n=1}^{\infty} n^{-1} \left| T_{n,r}^{\alpha} \right|^k < \infty \quad \text{for } r = 1, 2. \tag{3.4}$$

Whenever $k > 1$, we can apply Hölder's inequality with indices k and k', where $(1/k) + (1/k') = 1$, we get that

$$\begin{aligned} \sum_{n=2}^{m+1} n^{-1} \left| T_{n,1}^{\alpha} \right|^k &\leq \sum_{n=2}^{m+1} n^{-1} (A_n^{\alpha})^{-k} \left\{ \sum_{v=1}^{m} A_v^{\alpha} w_v^{\alpha} |\Delta\lambda_v| \right\}^k \\ &\leq \sum_{n=2}^{m+1} n^{-1-\alpha k} \left\{ \sum_{v=1}^{n-1} v^{\alpha k} (w_v^{\alpha})^k |A_v|^k \right\} \times \left\{ \sum_{v=1}^{n-1} 1 \right\}^{k-1} \\ &= O(1) \sum_{v=1}^{m} v^{\alpha k} (w_v^{\alpha})^k |A_v| |A_v|^{k-1} \sum_{n=v+1}^{m+1} \frac{1}{n^{2+(\alpha-1)k}} \\ &= O(1) \sum_{v=1}^{m} v^{\alpha k} (w_v^{\alpha})^k |A_v| \frac{1}{(v\ X_v)^{k-1}} \int_v^{\infty} \frac{dx}{x^{2+(\alpha-1)k}} \\ &= O(1) \sum_{v=1}^{m} v |A_v| \frac{(w_v^{\alpha})^k}{v\ X_v^{k-1}} = O(1) \sum_{v=1}^{m-1} \Delta(v|A_v|) \sum_{r=1}^{v} \frac{(w_r^{\alpha})^k}{r\ \ X_r^{k-1}} \\ &\quad + O(1) m |A_m| \sum_{v=1}^{m} \frac{(w_v^{\alpha})^k}{v\ X_v^{k-1}} = O(1) \sum_{v=1}^{m-1} |\Delta(v|A_v|)| X_v \\ &\quad + O(1) m |A_m| X_m = O(1) \sum_{v=1}^{m-1} v |\Delta A_v| X_v + O(1) \sum_{v=1}^{m} |A_v| X_v \\ &\quad + O(1) m |A_m| X_m = O(1) \quad \text{as } m \longrightarrow \infty, \end{aligned} \tag{3.5}$$

by virtue of the hypotheses of Theorem 2.1 and Lemma 2.7. Again, we have that

$$
\begin{aligned}
\sum_{n=1}^{m} n^{-1}\left|T_{n,2}^{\alpha}\right|^{k} &= O(1)\sum_{n=1}^{m}|\lambda_n||\lambda_n|^{k-1}\frac{(w_n^{\alpha})^k}{n} \\
&= O(1)\sum_{n=1}^{m}|\lambda_n|\frac{(w_n^{\alpha})^k}{n\ X_n^{k-1}} = O(1)\sum_{n=1}^{m-1}\Delta|\lambda_n|\sum_{v=1}^{n}\frac{(w_v^{\alpha})^k}{v\ X_v^{k-1}} \\
&\quad + O(1)|\lambda_m|\sum_{n=1}^{m}\frac{(w_n^{\alpha})^k}{n\ \ X_n^{k-1}} = O(1)\sum_{n=1}^{m-1}|\Delta\lambda_n|X_n \\
&\quad + O(1)|\lambda_m|X_m = O(1)\sum_{n=1}^{m-1}|A_n|X_n + O(1)|\lambda_m|X_m \\
&= O(1) \quad \text{as } m \longrightarrow \infty,
\end{aligned}
\tag{3.6}
$$

by virtue of the hypotheses of Theorem 2.1. This completes the proof of Theorem 2.1. If we take $\alpha = 1$, then we get a new result dealing with $|C,1|_k$ summability factors.

Proof of Theorem 2.2. In order to prove Theorem 2.2, we need to consider only the special case in which (N, p_n) is (C, α). Therefore, Theorem 2.2 will then follow by means of Theorem 2.1, Lemma 2.6 (for $\beta = 1$), and Lemma 2.8. If we take $\alpha = 1$, then we get a new result for the absolute Nörlund summability factors of infinite series. □

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